EXERCISES: A GUIDE TO NORMAL COMPLEX SURFACE SINGULARITIES

Ex1. Let $N = \mathbb{Z}^2$ and $M = N^\vee$. Consider the cone $\sigma \subset N_\mathbb{R}$ bounded by v_1 and v_2 in N. For the following cases, find a (finite) generating set of $S_\sigma := \sigma^\vee \cap M$ and describe the associated affine toric variety.

(a) $v_1 = (1, 0)$, $v_2 = (4, -3)$.

(b) $v_1 = (1, 0)$, $v_2 = (5, -2)$.

Ex2. Let $N = \mathbb{Z}^2$, and let $v_1 = (1, 0)$ and $v_2 = (1, \sqrt{2})$ be vectors in $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$. Consider the convex polyhedral cone

$$\sigma = \mathbb{R}_{\geq 0} \cdot v_1 + \mathbb{R}_{\geq 0} \cdot v_2.$$

Prove that the semigroup $S_\sigma = \sigma^\vee \cap M$ is not finitely generated.

Ex3. Let $m > q > 0$ be relative prime integers, $N' = \mathbb{Z}^2$ and $N = \mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{m}(1, q)$. Describe the dual lattice $M := \mathrm{Hom}_\mathbb{Z}(N, \mathbb{Z})$ as a sublattice of $M' = \mathrm{Hom}_\mathbb{Z}(N', \mathbb{Z})$.

Ex4. Let $\sigma = \mathbb{R}_{\geq 0} \cdot (1, 0) + \mathbb{R}_{\geq 0} \cdot (0, 1)$ be the first quadrant, but regarded as a cone in $N_\mathbb{R}$ where $N = \mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{m}(1, q)$ (here, m and $q < m$ are positive integers that are relatively prime). Prove that the affine toric variety X_σ associated to σ is the space $\mathbb{C}^2/G$ where

(a) $G = \mathbb{Z}/m\mathbb{Z}$ is the cyclic group generated by $\zeta = \exp\left(\frac{2\pi\sqrt{-1}}{m}\right)$, and

(b) the action of G on $\mathbb{C}^2$ is given by

$$\zeta \cdot (x, y) = (\zeta x, \zeta^q y).$$

Ex5. Assume $q = 1$ in **Ex4**.

(a) Describe the affine coordinate ring of $X := \mathbb{C}^2/G$.

(b) Compute the resolution of $0 \in X$, and the associated Hirzebruch-Jung continued fraction.

Repeat (a) and (b) for $q = m - 1$.

Divisors and linear equivalences. Let $N = \mathbb{Z}^2$, and $M = \mathrm{Hom}_\mathbb{Z}(N, \mathbb{Z})$, and let Σ be a complete fan in $N_\mathbb{R}$ with one-dimensional rays $\rho_1, \ldots, \rho_d$. Let $u_i \in N$ be the primitive vector in the ray ρ_i. Then, for each $m \in M$, the associated divisor

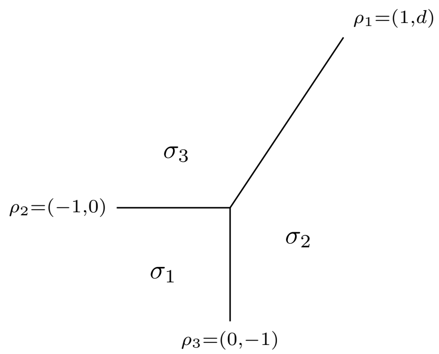
$$\mathrm{div}(\chi^m) := \sum_{i=1}^{d} \langle m, u_i \rangle D_{\rho_i}$$

is linearly equivalent to zero.

If the toric variety X_Σ is singular, then D_{ρ_i} are not necessarily Cartier. A divisor $D = \sum a_i D_{\rho_i} \in \mathrm{Cl}\, X_\Sigma$ is Cartier if and only if D is locally principal at each maximal cone, or equivalently, for each maximal cone σ, there exists m_σ such that $\langle m_\sigma, u_i \rangle = a_i$ whenever $u_i \in \sigma$.

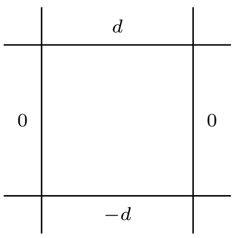
Ex6. Consider the complete fan Σ spanned by primitive vectors $(0,1)$, $(1,0)$, and $(-1,-1)$ in $N = \mathbb{Z}^2$. Prove that these rays define a unique divisor up to linear equivalences.

Ex7. Consider the fan Σ in $N = \mathbb{Z}^2$ given as follows.



(a) Prove that D_{ρ_3} is Cartier, but others are not. Find the smallest integers such that $a_i D_{\rho_i}$ becomes Cartier.

(b) Let Σ' be the fan obtained by subdividing σ_3 by $\rho_0 := (0,1)$. The toric variety X' associated to X' is the Hirzebruch surface $\mathbb{F}_d$. The following figure displays the curves D_{ρ_i} decorated with the self-intersection numbers:



Identify $D_{\rho_0}, \ldots, D_{\rho_3}$ in this figure.

Ex8. Let X be a complete toric surface whose fan contains a cone σ bounded by the primitive vectors v_1, v_2 in $N = \mathbb{Z}^2$. It defines a singularity $\frac{1}{m}(1,q)$ for $m > 0$ and $q > 0$ relatively prime to m.

(a) Let $v_1 = (a,b)$ and $v_2 = (c,d)$, and let A be the 2×2 matrix of integers:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Prove that $m = |\det A|$.

(b) Let D_i be the divisor associated to the ray $\rho_i := \mathbb{R}_{\geq 0} \cdot v_i$. Prove that

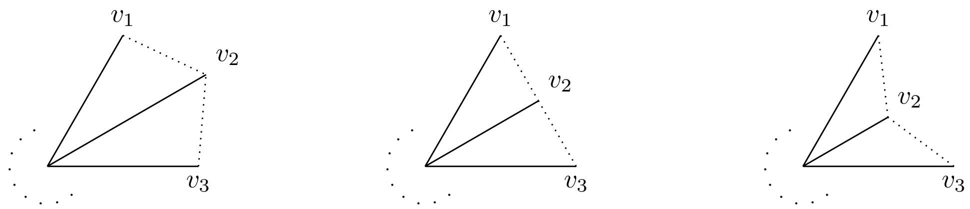
$$(D_1 \cdot D_2) = \frac{1}{m}.$$

(Hint: observe that it is harmless to take any complete toric surface which contains σ)

Ex9. Let X be a complete toric variety whose fan contains three consecutive rays, say $\rho_1, \rho_2, \rho_3 \subset N_{\mathbb{R}}$. Let v_i be the primitive vector in N which generates ρ_i.

There are three cases: v_1, v_2, v_3 are **convex**(left) / **flat**(middle) / **concave**(right).

Let C be the curve corresponding to the ray ρ_2. Prove that



(a) $(K_X \cdot C) < 0 \iff v_1, v_2, v_3$ are convex;

(b) $(K_X \cdot C) = 0 \iff v_1, v_2, v_3$ are flat;

(c) $(K_X \cdot C) > 0 \iff v_1, v_2, v_3$ are concave.

Ex10. For integers $b_1, \dots, b_r \geq 2$, define

$$A(b_1, \dots, b_r) = \det \begin{pmatrix} b_1 & -1 & 0 & \dots & 0 & 0 \\ -1 & b_2 & -1 & \dots & 0 & 0 \\ 0 & -1 & b_3 & \dots & 0 & 0 \\ & \vdots & & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & b_{r-1} & -1 \\ 0 & 0 & 0 & \dots & -1 & b_r \end{pmatrix}.$$

(a) Show that for each $k \leq r - 1$,

$$A(b_k, \dots, b_r) = b_k A(b_{k+1}, \dots, b_r) - A(b_{k+2}, \dots, b_k),$$

where we put $A(\varnothing) = 1$ for convenience.

(b) Fix $\frac{m}{q} = [b_1, \dots, b_r]$. Let $\{\alpha_k\}$ be a sequence of integers such that

$$\frac{\alpha_k}{\alpha_{k+1}} = [b_k, \dots, b_r].$$

Prove that $\alpha_k = b_k \alpha_{k+1} - \alpha_{k+2}$ for each $k \leq r - 1$ (put $\alpha_{r+1} = 1$).

Conclude that $m = A(b_1, \dots, b_r)$ and $q = A(b_2, \dots, b_r)$.

Ex11. Let A be a ring, $I \subset A$ an ideal. The *Rees algebra* is the graded A-algebra $S = \bigoplus_{d \geq 0} S_d$ such that

1. $S_0 = A$;
2. $S_d \simeq I^d$ for $d > 0$;
3. the product $S_d \otimes S_e \to S_{d+e}$ is induced by $I^d \otimes I^e \to I^{d+e}$.

The associated projective variety $\operatorname{Proj} S$ over A is the blow up of A along I.

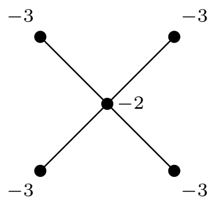
(a) Let $A = \mathbb{C}[x, y]$ and $I = (x, y)$. Prove that the Rees algebra S is isomorphic to

$$A[X, Y]/(xY - yX),$$

showing that $\operatorname{Proj} S = \{((x, y), [X : Y]) \in \mathbb{A}^2 \times \mathbb{P}^1 \mid xY = yX\}$ is the blow up of $\mathbb{C}^2$ at the origin.

(b) Let $A = \mathbb{C}[x, y]$ and $I = (x^2, y^2)$. Compute the blow up of A along I, and check that it contains a non-normal singularity.

Ex12. Consider the configuration Γ of smooth rational curves with the following dual intersection graph



(a) Prove that the intersection matrix is negative definite.

(b) Let S be a smooth projective surface containing the Γ and let $S \to X$ be the contraction of Γ. Compute the discrepancies of the irreducible components of Γ with respect to the singularity in X.

(c) Find the fundamental cycle and determine the rationality of the singularity.