Lecture I: Outlook

[useful references]

"TASI Lectures on Scattering Amplitudes", Cheung [1708.03872]

"Scattering Amplitudes", Elvang + Huang [1308.1697]

"Calculating Amplitudes Efficiently", Dixon [hep-ph/9601359]

What Defines a Theory?

(modern S-matrix program) $\supset$ particle, string, pheno, collider, math, gravitational waves, cosmo

Q: What Defines a theory???

A: $S = \int d^4x \, \mathcal{L}(x)$

spacetime locality

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!}\phi^4$$

L.I.

$\mathbb{Z}_2$

- couplings <u>not</u> fixed by principles are <u>free</u>
- 19 S.M parameters = unfixed by principles

S-matrix in a Nut-shell

physical principles (L.I., unitarity, locality)

$\Downarrow$

~~QFT~~ (Lagrangian, Hamiltonian, equations of motion, etc....)

$\Downarrow$

observables (amplitudes, correlators, etc..)

Bootstrap in a Nut-shell

fewer constraints

19 - parameter theory ("SM")

1 - parameter theory ("GUT")

more constraints

no solutions !!! ("your principles are mutually inconsistent!")

Why bother with a bootstrap??

- Feynman diagrams are complicated!

  (let's unpack this shortly...)

- lessons from the S-matrix

structure

double copy, amplituhedra, CHY, celestial CFT, ...

applications

collider pheno, SUGRA, gravitational waves, ...

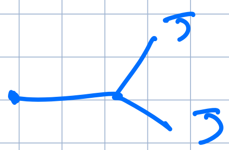
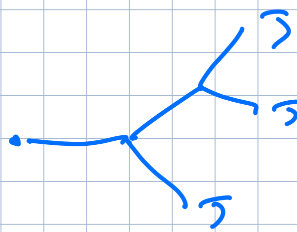
But first, how complicated are Feynman diagrams, really???

Berends - Giele Recursion

$$\mathcal{L} = \frac{1}{2}\partial\phi^2 + \frac{\lambda}{3!}\phi + \phi J \Longrightarrow \Box\phi = J + \frac{\lambda\phi^2}{2}$$

Solve perturbatively: $\phi = \phi_0 + \lambda \phi_1 + \lambda^2 \phi_2 + \lambda^3 \phi_3 + \cdots$

$\phi_0 = \frac{1}{\Box} J =$  $\Rightarrow \quad \Box \phi_1 = \frac{\lambda}{2} \phi_0^2$

$\phi_1 =$  , $\phi_2 =$  , etc...

Counting Feynman Diagrams

"toy" Berends-Giele:

$\Box \to 1$

$\lambda \to 1$

(all propagators and vertices set to 1)

ϕ^3 Theory

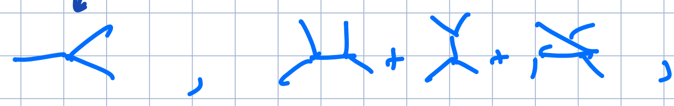
$$\phi = J + \frac{\phi^2}{2}$$

number, not a field

$$\Rightarrow \quad \phi(J) = 1 \pm \sqrt{1 - 2J}$$

(choose branch with $\phi(0) = 0$)

$$= J + (1)\frac{J^2}{2!} + (3)\frac{J^3}{3!} + (15)\frac{J^4}{4!} + (105)\frac{J^5}{5!} + \cdots$$



$C_n =$ # of Feynman diagrams in n-particle amplitude

$$\Rightarrow C_n^{(\phi^3)} = (2n-5)!! \sim 2^n n!$$

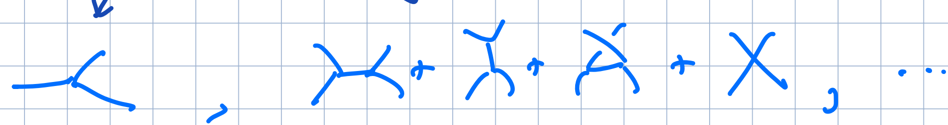
YM Theory

hundreds of diagrams at 6pt scattering !!!

$$A = J + \frac{A^2}{2} + \frac{A^3}{3!}$$

(choose sol'n with $A(0)=0$)

$$\Rightarrow A(J) = J + (1)\frac{J^2}{2!} + (4)\frac{J^3}{3!} + (25)\frac{J^4}{4!} + (220)\frac{J^5}{5!} + \cdots$$



$$\Rightarrow C_n^{(YM)} \sim (\#)^n n!$$

multiplicity dominated by cubic diagrams

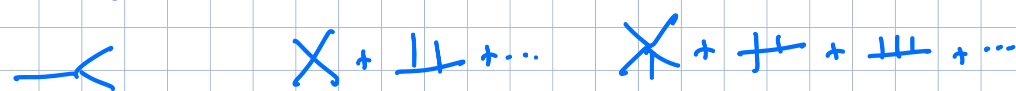
GR

toy version of $\mathcal{L} = \sqrt{-g}\,R(g)$, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$$h = J + \frac{h^2}{2!} + \frac{h^3}{3!} + \frac{h^4}{4!} + \dots = J + e^h - 1 - h$$

$$\Rightarrow h(J) = \frac{J-1}{2} - W\left(-e^{\frac{J-1}{2}}/2\right)$$

$$= J + (1)\frac{J^2}{2!} + (4)\frac{J^3}{3!} + (26)\frac{J^4}{4!} + \dots$$



$$\Rightarrow C_n^{(GR)} \sim (\#)^n\, n!$$

What are the lessons from this analysis?

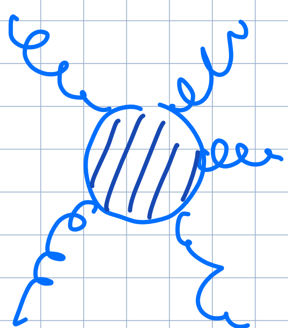
- # of Feynman diagrams approx same between theories!
  (dominated by cubic topologies)

- more / fewer Feynman diagrams does not mean complex / simple
  (e.g. GR is actually simpler than $\mathcal{L} \supset \sum \frac{\lambda_n}{n!}\phi^n$ theory!)

• cf., $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ SUGRA are truly simple

Hidden Simplicity of Scattering

Gauge Theory



$$= A(1^{h_1}2^{h_2}3^{h_3}4^{h_4}5^{h_5})$$

$$\parallel$$

$$\frac{(e_1\cdot e_5)(p_1\cdot e_2)(p_1\cdot e_3)(p_3\cdot e_4)}{(p_1+p_2)^2(p_4+p_5^2)} f^{a_1a_2b} f^{ba_3c} f^{ca_4a_5} +$$ hundreds of terms !!!

$$\Rightarrow A(1^+2^+3^+4^+5^+) = A(1^-2^+3^+4^+5^+) = 0$$

$$A(1^-2^+3^-4^+5^+) = \frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$ (color-stripped amplitude)

where spinor helicity variables satisfy $s_{ij} = (p_i+p_j)^2 = \langle ij\rangle[ij]$.

Parke + Taylor: 8 pg eq. for 6pt $\longrightarrow$ "simple analytic form, ... a theorist's delight"

$$A_n(\cdots i^- \cdots j^- \cdots) = \underbrace{\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}}$$

hidden structures: a) VP Nair and WZW
b) Strominger's celestial CFT
c) Witten's twistor string
⋮

The "amplitudes first" approach looks for new structure via the S-matrix

Gravity

Despite vastly more complicated Feynman vertices,

$$M(1^- 2^- 3^+) : \text{dozens of terms} = \frac{\langle 12\rangle^6}{\langle 13\rangle^2\langle 32\rangle^2}$$

$$M(1^- 2^- 3^+ 4^+) = \text{100 kb file of terms} = \frac{\langle 12\rangle^4\,[34]^4}{stu}$$

$$M(1^- 2^- 3^+ \cdots n^+) \sim \det\left(\frac{[ij]}{\langle ij\rangle}\right)$$ (Hodges 2012)

Origin of Redundancy

Gauge Redundancy

(to manifest L.I. and locality at the same time)

$$\Downarrow$$

4 d.o.f.

$$S_{YM} = \int d^D x \, \mathcal{L}(A_\mu)$$

10 d.o.f

$$S_{GR} = \int d^D x \, \mathcal{L}(h_{\mu\nu})$$

to remove excess d.o.f. :

$$A_\mu \to A_\mu + \partial_\mu \Theta$$

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \Theta_\nu + \partial_\nu \Theta_\mu + \cdots$$

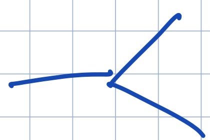
gauge redundancy $\Longrightarrow$ Ward identities $\Longrightarrow$ redundancy of Feynman diagrams

Field Basis Redundancy

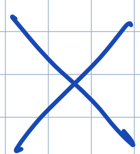
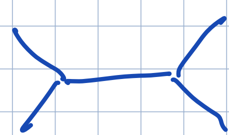
Consider a naively generic scalar EFT,

arbitrary!!

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi\;\lambda(\phi) \quad \text{where} \quad \lambda(\phi) = 1 + \lambda_1\phi + \frac{\lambda_2}{2!}\phi^2 + \frac{\lambda_3}{3!}\phi^3 + \dots$$

A_3 = 

$$\sim \lambda_1(\underbrace{p_1\cdot p_2}_{=\frac{1}{2}(p_1+p_2)^2 = \frac{1}{2}p_3^2} + p_2\cdot p_3 + p_3\cdot p_1) \sim \lambda_1(p_3^2 + p_1^2 + p_2^2) = 0$$

A_4 =  + ( + perms)

$$= \#\lambda_2(s+t+u) + \left(\#\lambda_1^2\, \cancel{s}\,\frac{1}{\cancel{s}} + \text{perms}\right)$$

$$\sim s+t+u = 0$$

$$A_n = 0 \quad (\text{on-shell})$$

huh???

This is secretly a free theory. The S-matrix is invariant under field redefinitions (F.R.),

$$\phi \xrightarrow{F.R.} f(\phi) = \phi + \delta\phi$$

any non linear function of ϕ

$$\mathcal{L}_{free} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \xrightarrow{F.R.} \frac{1}{2}f'(\phi)^2\partial_\mu\phi\partial^\mu\phi$$

$\parallel$ choose $f'(\phi) \coloneqq \lambda(\phi)$

$$\frac{1}{2}\lambda(\phi)\partial_\mu\phi\partial^\mu\phi$$

Q: How can the S-matrix be invariant under a non-symmetry of the action?

A: i) Physical consequences don't depend on coordinates, which is QFT analog of $(x,y,z) \xrightarrow{F.R.} (r,\theta,\phi)$.

Cartesian   Polar

ii) Path integral is still an integral, which means we can change coords.

$$\Rightarrow \quad Z[J] = \int [d\phi]\, e^{iS[\phi] + i\int J\phi}$$

$$\overset{\phi = f(\phi')}{=} \int [d\phi'] \underbrace{\left|\frac{\delta f}{\delta \phi'}\right|}_{\text{should compute Jacobian for loops}} e^{\,iS[\underbrace{f(\phi')}_{\text{action has changed}}] + i\int \underbrace{J f(\phi')}_{J(\phi + \cdots)}}$$

$J(\phi + \underbrace{\cdots}_{\text{killed by LSZ, which extracts 1-particle poles}})$

iii) Technical point: at L.O. in $1/N$ or small g, a F.R. looks like plugging in EOM.

$$S[\phi] \xrightarrow{F.R.} S[\phi] + \int \delta\phi\, \underbrace{(EOM)}_{\Box\phi + \cdots} + \underbrace{\int \delta\phi^2 (\cdots)}_{\text{NLO and higher}} + \cdots$$

Thus, operators of the form $\underbrace{\delta\phi}_{\text{arbitrary!}}\Box\phi$ can be removed by a F.R. modulo NLO terms.

The upshot of gauge and field basis redundancy:

$$\begin{tikzcd} \mathcal{L}_1 \arrow[drr] & \mathcal{L}_2 \arrow[dr] & \cdots \arrow[d] & \mathcal{L}_\infty \arrow[dl] \\ & & A & \end{tikzcd}$$