Lecture 2: Basic Bootstrap

The principles of the S-matrix bootstrap:

- dimensional analysis: $[A_n] = d + n - dn/2 \overset{d=4}{=} 4 - n$,

  $[g_{YM}] \overset{d=4}{=} 0$, $[G_N] \overset{d=4}{=} -2$

- Lorentz invariance: functions of $(p_i p_j)$, $\langle ij\rangle [ij]$

- factorization: singularities of amplitudes yield products of sub-amplitudes

we have assumed ghost-free $\Box + m^2 = 0$ free propagation!!

$$\lim_{s \to m^2} (s - m^2) A = \lim_{s \to m^2} (s - m^2) \left(A \right) = \lim_{s \to m^2} (s - m^2) \left(A_L \frac{1}{s - m^2} A_R \right)$$

$$= A_L A_R$$

(Sometimes a weaker assumption of "locality" is assumed.

$A_5 \sim \frac{1}{s_{12} s_{45}}$ ✓✓ but $A_5 \sim \frac{1}{s_{12} s_{23}}$ ✗✗)

On-shell Kinematics

- In specific dimension $D = 3, 4, 6, 10$, the exist spinor helicity; there is a "pure kinematic" form of amplitudes, ie No polarizations, ϵ_μ.

- In $\underbrace{\text{general dimension } D}_{\text{today's focus}}$, we must deal with ϵ_μ. No problem.

Consider on-shell kinematics of a massless spin 1 particle.

p_i^μ = momentum of leg i

ϵ_i^μ = polarization of leg i

where $p_i^2 = p_i \epsilon_i = \epsilon_i^2 = 0$

non-minimal kinematic basis : $\{p_i \cdot p_j\}_{\frac{n(n-1)}{2}}$, $\{p_i \cdot \epsilon_j\}_{n(n-1)}$, $\{\epsilon_i \cdot \epsilon_j\}_{\frac{n(n-1)}{2}}$

$1 \leq i, j \leq n$ s.t. $i \neq j$

To build a minimal kinematic basis,

1) Eliminate $p_n = -(p_1 + \cdots + p_{n-1})$ via momentum conservation.

This means drop $\underbrace{p_n p_i}_{n-1}$ and $\underbrace{p_n \epsilon_i}_{n-1}$ from basis.

2) Impose remaining on-shell conditions for p_n, e_n, so

$$0 = p_n^2 = \sum_i^{n-1} \sum_j^{n-1} p_i p_j \Rightarrow \text{e.g. drop } \overbrace{p_{n-2} \cdot p_{n-1}}^{1}$$

$$0 = p_n e_n = \sum_i^{n-1} p_i e_n \Rightarrow \text{e.g. drop } \overbrace{p_{n-1} \cdot \epsilon_n}^{1}$$

non-minimal kinematic basis : $\{p_i \cdot p_j\}_{\frac{n(n-1)}{2}}$, $\{p_i \cdot \epsilon_j\}_{n(n-1)}$, $\{\epsilon_i \cdot \epsilon_j\}_{\frac{n(n-1)}{2}}$

$\downarrow (-n)$ $\qquad \downarrow (-n)$

minimal kinematic basis : $\{p_i \cdot p_j\}_{\frac{n(n-3)}{2}}$, $\{p_i \cdot \epsilon_j\}_{n(n-2)}$, $\{\epsilon_i \cdot \epsilon_j\}_{\frac{n(n-1)}{2}}$

Note that there exists a nice cyclic basis,

$$\{p_i \, p_j\}_{\frac{n(n-3)}{2}} = \{ s_{12}, s_{123}, s_{1234}, \cdots s_{123\cdots \frac{n-1}{2}}, \text{ and cyclic perms} \}$$

$$s_{ijk\cdots} = (p_i + p_j + p_k + \cdots)^2$$

We ignore D-dependent Gram determinant zeros, e.g. $p_1^{[\mu_1} p_2^{\mu_2} p_3^{\mu_3} p_4^{\mu_4} p_5^{\mu_5]} = 0$

$$\left(\text{analogous to } \lambda_1^{[\alpha_1} \lambda_2^{\alpha_2} \lambda_3^{\alpha_3]} = 0 \right)$$

3pt Kinematics

$$\left\{ \begin{matrix} \varepsilon_1 \varepsilon_2 & \varepsilon_1 \varepsilon_3 & \varepsilon_2 \varepsilon_3 \\ p_1 \varepsilon_2 & p_1 \varepsilon_3 & p_2 \varepsilon_1 \end{matrix} \right\}.$$

$\Longrightarrow$ note that: $p_i \, p_j = 0$

4pt Kinematics

$$\left\{ \begin{matrix} \varepsilon_1 \varepsilon_2 & \varepsilon_1 \varepsilon_3 & \varepsilon_1 \varepsilon_4 & \varepsilon_2 \varepsilon_3 & \varepsilon_2 \varepsilon_4 & \varepsilon_3 \varepsilon_4 \\ p_1 \varepsilon_2 & p_1 \varepsilon_3 & p_1 \varepsilon_4 & p_2 \varepsilon_1 & p_2 \varepsilon_3 & p_2 \varepsilon_4 \\ p_3 \varepsilon_1 & p_3 \varepsilon_2 & p_1 p_2 & p_1 p_3 & & \end{matrix} \right\}$$

$\longleftarrow s, u$

Scalar Bootstrap

3pt Scalar

$$A_3 = \text{constant} \quad (\text{since } p_i p_j = 0)$$

$$= \lambda \quad \Leftarrow \text{amplitude for } \phi^3 \text{ theory}$$

But what about higher derivative cubic scalar theories??

distinct flavors

$$\text{e.g.} \quad \mathcal{L} = \frac{1}{2}\sum_i \partial_\mu \phi_i \partial^\mu \phi_i + k\, \partial_\mu \phi_1 \partial^\mu \phi_2 \phi_3$$

We can see that this theory is trivial at 3pt since:

- The operator can be eliminated by a F.R.

$$k\, \partial_\mu \phi_1 \partial^\mu \phi_2 \phi_3 = k\left(\frac{1}{2}\Box(\phi_1\phi_2) - \Box\phi_1 \phi_2 - \phi_1 \Box \phi_2\right)\phi_3$$

- The amplitude is trivial.

$$A_3(\phi_1\phi_2\phi_3) \sim k\, p_1 \cdot p_2 = 0.$$

4pt Scalar (assume single flavor)

Consider "contact vertices" i.e. local functions of ϕ_i first.

(only trivial for flavorless bosons, unlike NLSM, $\lambda^{ijkl}\partial\pi_i\partial\pi_j\pi_k\pi_l$)

$$\phi^4 : \quad A_4^{(cont)} = 1$$

$$(\partial\phi)^2\phi^2 : \quad A_4^{(cont)} = s+t+u = 0$$

$$(\partial\phi)^4 : \quad A_4^{(cont)} = s^2+t^2+u^2$$

$$(\partial\partial\phi)^2(\partial\phi)^2 : \quad A_4^{(cont)} = s^3+t^3+u^3 \sim stu$$

$$\vdots$$

$A_4^{(cont)}$ = function of $s^2+t^2+u^2$ and stu

= all possible scalar 4pt contact

By definition, the "contact vertices" have no factorization channels,

$$\lim_{s\to 0} s\cdot A_4^{(cont)} = 0$$

If $A_3 \neq 0$, then factorization implies that another piece,

$$\lim_{s \to 0} s \cdot A_4^{(\text{fact})} \sim A_3 \cdot A_3 = \lambda^2$$

$$\Longrightarrow \quad A_4^{(\text{fact})} = \lambda^2 \left(\frac{1}{s} + \frac{1}{t} + \frac{1}{u} \right)$$

This approach makes the vanishing S-matrix from earlier easy to see. Recall the theory:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi \; \lambda(\phi) \quad \text{where} \quad \lambda(\phi) = 1 + \lambda_1 \phi + \frac{\lambda_2}{2!} \phi^2 + \frac{\lambda_3}{3!} \phi^3 + \cdots$$

$A_3 = 0 \leftarrow$ direct computation

A_4 has <u>no</u> poles, since $A_3 = 0$. Thus

$A_4 = \sum_{i,j} p_i \cdot p_j = 0$. Repeat for $A_5, A_6, \cdots$ etc

Vector Bootstrap

3pt Vector

The amplitude for vectors must be multilinear in polarizations.

assume lowest order in p !!

$$\underbrace{A_3}_{\text{ansatz}} = c_1 (\epsilon_1 \epsilon_3)(p_1 \epsilon_2) + c_2 (\epsilon_1 \epsilon_2)(p_1 \epsilon_3) + c_3 (\epsilon_2 \epsilon_3)(p_2 \epsilon_1)$$

ansatz where $c_{1,2,3}$ = free coefficients

imposing that transverse polarizations are physical

$$\left\{ \begin{aligned} A_3\Big|_{\epsilon_1 = p_1} &= (c_1 + c_2)(p_1 \epsilon_2)(p_1 \epsilon_3) \\ A_3\Big|_{\epsilon_2 = p_2} &= (c_2 - c_3)(p_1 \epsilon_3)(p_2 \epsilon_1) \\ A_3\Big|_{\epsilon_3 = p_3} &= -(c_1 + c_3)(p_1 \epsilon_2)(p_2 \epsilon_1) \end{aligned} \right.$$

$$\Longrightarrow \quad c_2 = c_3 = -c_1$$

$$\Longrightarrow \quad A_3 = (\varepsilon_1\varepsilon_2)(p_1-p_2)\cdot\varepsilon_3 + (\varepsilon_2\varepsilon_3)(p_2-p_3)\varepsilon_1 + (\varepsilon_3\varepsilon_1)(p_3-p_1)\varepsilon_2$$

antisymmetric under exchange !!!

To maintain bosonic properties, we introduce antisymmetric f^{abc}.

$$A_3 = f^{abc}\left[(\varepsilon_1\varepsilon_2)(p_1-p_2)\cdot\varepsilon_3 + (\varepsilon_2\varepsilon_3)(p_2-p_3)\varepsilon_1 + (\varepsilon_3\varepsilon_1)(p_3-p_1)\varepsilon_2\right]$$

$$= A_3^{(YM)} \quad \checkmark\checkmark\checkmark$$

We can consider even higher powers of p.

$$A_3 = \varepsilon\varepsilon\varepsilon p \longrightarrow A_3 = A_3^{(YM)}$$

$A_3 = \varepsilon\varepsilon\varepsilon p p \longrightarrow$ no Lorentz invariant object

$$A_3 = \varepsilon\varepsilon\varepsilon p p p \longrightarrow A_3 = A_3^{(F^3)}$$

where $A_3^{(F^3)} = f^{abc} F_{1\,\nu}^{\mu} F_{2\,\rho}^{\nu} F_{3\,\mu}^{\rho}$, (arises from $\mathcal{L} \supset tr(F^3)$)

and $F_i^{\mu\nu} = p_i^{\mu} \epsilon_i^{\nu} - p_i^{\nu} \epsilon_i^{\mu}$

4pt Vector

- First, consider "contact vertices":

$$A_4^{(cont)} = d_1(\epsilon_1\epsilon_4)(\epsilon_2\epsilon_3) + d_2(\epsilon_1\epsilon_3)(\epsilon_2\epsilon_4) + d_3(\epsilon_1\epsilon_2)(\epsilon_3\epsilon_4)$$

- Second, for the "factorization" pieces, define

$$A_3 = \epsilon_1^{\mu} \epsilon_2^{\nu} \epsilon_3^{\rho} f^{abc} \underbrace{A_{\mu\nu\rho}}_{\text{precisely the YM 3pt vertex!}}$$

$$A_4^{(fact)} = \epsilon_1^{\mu} \epsilon_2^{\nu} \epsilon_3^{\rho} \epsilon_4^{\sigma} \left\{ A_{\mu\nu\alpha} \frac{\eta^{\alpha\beta}}{(p_1+p_2)^2} A_{\rho\sigma\beta} f^{abe} f^{ecd} + (t) + (u) \right\}$$

$\Rightarrow$ define $A_4 = A_4^{(cont)} + A_4^{(fact)}$

imposing that transverse polarizations are physical

$$\begin{cases} A_4\big|_{\epsilon_1=p_1} = A_4\big|_{\epsilon_2=p_2} \\ \quad \parallel \\ A_4\big|_{\epsilon_3=p_3} = A_4\big|_{\epsilon_4=p_4} = 0 \end{cases}$$

$\Rightarrow$ NO solutions unless $f^{abe} f^{ecd} + f^{bce} f^{ead} + f^{cae} f^{ebd} = 0$

(Jacobi identity is an OUTPUT)

Meanwhile, d_1, d_2, d_3 are fixed s.t. $A_4 = A_4^{(YM)}$ ✓✓

Higher Spin Bootstrap

Here we represent higher spin S particles with a symmetric traceless polarization tensor,

$$\epsilon^{\mu_1 \cdots \mu_s} \quad \text{where} \quad p_{\mu_1} \epsilon^{\mu_1 \cdots \mu_s} = \epsilon_{\mu_1}{}^{\mu_1 \mu_2 \cdots \mu_s} = 0$$

$\parallel$ (WLOG consider tensor product of vectors)

$$\epsilon^{\mu_1} \epsilon^{\mu_2} \cdots \epsilon^{\mu_s}$$

3pt Tensor

quadratic in $\epsilon_1, \epsilon_2, \epsilon_3$

$$A_3 = c_1 (\epsilon_1 \epsilon_3)^2 (p_1 e_2)^2 + c_2 (\epsilon_1 \epsilon_2)(\epsilon_1 \epsilon_3)(p_1 \epsilon_2)(p_1 \epsilon_2)$$
$$+ \ldots$$

Imposing $A_3 \big|_{e_i = p_i} = 0 \implies A_3 = A_3^{(GR)}$ ✓✓

Remarkably, one finds that,

$$\left.\begin{aligned} \Big[A_3^{(YM)} \Big]^2 = A_3^{(GR)} \end{aligned}\right\} \text{“double copy”}$$

dropping f^{abc} factor

3pt Higher Spin

There is a convenient gauge invariant basis for higher spin ansatze.

$$\Theta^{(YM)} = (\epsilon_1 \epsilon_2)(p_1 - p_2)\cdot \epsilon_3 + (\epsilon_2 \epsilon_3)(p_2 - p_3)\epsilon_1 + (\epsilon_3 \epsilon_1)(p_3 - p_1)\epsilon_2$$

$$\Theta^{(F^3)} = \left(p_1^{[\mu} \epsilon_1^{\nu]} \right)\left(p_2^{[\nu} \epsilon_2^{\rho]} \right)\left(p_3^{[\rho} \epsilon_3^{\mu]} \right)$$

$$\mathcal{O}_1^{(SQCD)} = (p_2 - p_3) \cdot \varepsilon_1$$

$$\mathcal{O}_2^{(SQCD)} = (p_2 - p_1) \cdot \varepsilon_2$$

$$\mathcal{O}_3^{(SQCD)} = (p_1 - p_2) \cdot \varepsilon_3$$

scalar - scalar - vector amplitudes

For spin $s = 2$, we have

$$\left[\mathcal{O}^{(YM)}\right]^2 = A_3^{(GR)}$$

$$\left[\mathcal{O}^{(F^3)}\right]^2 = A_3^{(R^3)} \quad \leftarrow \quad \mathcal{L} \supset \sqrt{-g}\, R^3_{\mu\nu\rho\sigma}$$

$$\left[\mathcal{O}^{(YM)}\right]\left[\mathcal{O}^{(F^3)}\right] = A_3^{(GB)} \quad \leftarrow \quad \mathcal{L} \supset \sqrt{-g}\, \mathcal{O}_{\text{Gauss-Bonnet}}$$

For spin $s > 2$, a similar construction holds.