Lecture 4: Unity in Scattering

The theories that keep appearing in the S-matrix program are secretly unified!!

Unification in hep-ph

$$
\begin{tikzcd}[column sep=small]
 & SO(10) \arrow[dl] \arrow[dr] & & & \text{high energy} \arrow[dd, Rightarrow] \\
SU(5) \arrow[dr] & & SU(4) \otimes SU(2)^2 \arrow[dl] & & \\
 & SU(3) \otimes SU(2) \otimes U(1) & & & \text{low energy}
\end{tikzcd}
$$

Unification in Amplitudes

$$
\begin{tikzcd}[column sep=small]
 & & \text{gravity} \arrow[dl] \arrow[dr] & & & \text{high spin} \arrow[dd, Rightarrow] \\
 & \text{Yang-Mills} \arrow[dl] \arrow[dr] & & \text{Born-Infeld} \arrow[dl] \arrow[dr] & & \\
\phi^3 \text{ theory} & & \text{nonlinear sigma model} & & \text{special Galileon} & \text{low spin}
\end{tikzcd}
$$

"Gravity is the mother of all theories": given the tree-level S-matrix for graviton + dilaton + two-form, one obtains all the other S-matrices.

Mandelstam Differential Operators

$$A = e_1^{\mu_1} e_2^{\mu_2} \cdots e_n^{\mu_n} A_{\mu_1 \mu_2 \cdots \mu_n}$$

$= \text{function of } p_i p_j \, , \; p_i \epsilon_j \, , \; \epsilon_i \epsilon_j$ where $p_i^2 = p_i \epsilon_i = \epsilon_i^2 = 0$

Next, construct differential operators imposing physical constraints.

- Ward identity: $A\big|_{e_i \to p_i} \equiv W_i A = 0$

$$\Longrightarrow W_i \equiv \sum_{v = p_j, \epsilon_j} (p_i v) \frac{\partial}{\partial (v e_i)}$$

- total momentum conservation: $\mathbb{P}_v A = 0$

$$\Longrightarrow \mathbb{P}_v \equiv \sum_i p_i v$$

$$[\mathbb{P}_v, \mathbb{P}_{v'}] = 0$$

commutation relations :

$$[\mathbb{W}_i, \mathbb{W}_j] = 0$$

annilates A

$$[\mathbb{W}_i, \mathbb{P}_v] = \delta_{v e_i} \mathbb{P}_{p_i} \sim 0$$

Amplitudes Transmutation

Let us construct a "transmutation operator", $\mathbb{T}$, that maps momentum-conserving, gauge invariant functions to new ones, so

(modulo terms that annihilate A)

IF $[\mathbb{W}_i, \mathbb{T}] = [\mathbb{P}_v, \mathbb{T}] = 0$,

THEN $\mathbb{W}_i(\mathbb{T} A) = \mathbb{T}(\mathbb{W}_i A) = 0$

$$\mathbb{P}_v(\underbrace{\mathbb{T} A}_{\text{transmuted object}}) = \mathbb{T}(\mathbb{P}_v A) = 0$$

$$\mathcal{T} = \sum_{i,j} A_{ij}\, \partial_{p_i p_j} + B_{ij}\, \partial_{p_i \epsilon_j} + C_{ij}\, \partial_{\epsilon_i \epsilon_j}$$

(ansatz)

functions of $p_i p_j$, $p_i \epsilon_j$, $\epsilon_i \epsilon_j$
where $A_{ii} = B_{ii} = C_{ii} = 0$

There are many such operators, but a handful send (color-ordered) amplitudes to other amplitudes!!

transmutation operators

$$\mathcal{T}_{ij} = \frac{\partial}{\partial \epsilon_i \epsilon_j}$$ (2 gluon $\rightarrow$ 2 scalar)

$$\mathcal{T}_{ijk} = \frac{\partial}{\partial p_i \epsilon_j} - \frac{\partial}{\partial p_k \epsilon_j}$$ (1 gluon $\rightarrow$ 1 scalar)

$$\mathbb{L}_i = \sum_j p_i p_j \frac{\partial}{\partial p_j \epsilon_i}$$ (1 gluon $\rightarrow$ 1 pion)

$$\mathbb{T}[123\cdots n] \equiv \mathbb{T}_{1,n} \prod_{i=2}^{n-1} \mathbb{T}_{i-1,i,n} \qquad (n \text{ gluon} \to n \text{ scalar})$$

$$\mathbb{L}[123\cdots n] \equiv \mathbb{T}_{1,n} \prod_{i=2}^{n-1} \mathbb{L}_{i} \qquad (n \text{ gluon} \to n \text{ pion})$$

Unifying Relations

(BAS = "biadjoint scalar theory" $\sim \phi^3$

$\mathcal{L}_{BAS} = \frac{1}{2}\partial\phi^{a\tilde{a}}\partial\phi^{a\tilde{a}} + f^{abc} f^{\tilde{a}\tilde{b}\tilde{c}} \phi^{a\tilde{a}}\phi^{b\tilde{b}} f^{c\tilde{c}}$)

$$A_{BAS} = \mathbb{T}[123\cdots n]\, A_{YM}$$

$$A_{NLSM} = \mathbb{L}[123\cdots n]\, A_{YM}$$

(graviton + dilaton + 2-form

$\varepsilon_{\mu\tilde{\nu}} = \varepsilon_{\mu}\tilde{\varepsilon}_{\tilde{\nu}}$)

$$A_{YM} = \mathbb{T}[123\cdots n]\, A_{grav}$$

$$A_{BI} = \mathbb{L}[123\cdots n]\, A_{grav}$$

$$A_{NLSM} = \mathbb{T}[123\cdots n]\, A_{BI}$$

Born-Infeld

$$A_{SG} = \mathbb{L}[123\cdots n]\, A_{BI}$$

Special Galileon

e.g. #1) YM $\longrightarrow \phi^3$

take any representation in terms of $p_i p_j$, $p_i \epsilon_j$, $\epsilon_i \epsilon_j$

$$A_{BAS}(\phi_1 \phi_2 \phi_3 \phi_4) = \mathbb{T}[1234]\, A_{YM}(g_1 g_2 g_3 g_4)$$

$$= \frac{1}{s} + \frac{1}{t}$$

e.g. #2) YM $\longrightarrow$ NLSM

$$A_{NLSM}(\pi_1 \pi_2 \pi_3 \pi_4) = \mathbb{L}[1234]\, A_{YM}(g_1 g_2 g_3 g_4)$$

$$= s+t$$

e.g. #3) YM $\longrightarrow$ SQCD

$$A_{SQCD}(\phi_1 \phi_2, \phi_3 \phi_4) = \mathbb{T}[12]\,\mathbb{T}[34]\, A_{YM}(g_1 g_2 g_3 g_4)$$

$$= \frac{u}{s}$$

Web of Theories

"Gravity as the mother of all theories"

$$
\begin{tikzcd}[column sep=small]
 & & \text{gravity} \arrow[dl, "\mathbb{T}"'] \arrow[dr, "\mathbb{L}"] & & \\
 & \text{Yang-Mills} \arrow[dl, "\mathbb{T}"'] \arrow[dr, "\mathbb{L}"] & & \text{Born-Infeld} \arrow[dl, "\mathbb{T}"'] \arrow[dr, "\mathbb{L}"] & \\
\phi^3 \text{ theory} & & \begin{array}{c}\text{nonlinear sigma}\\ \text{model}\end{array} & & \begin{array}{c}\text{special}\\ \text{Galileon}\end{array}
\end{tikzcd}
$$