양자장론과 표준모형

QUANTUM FIELD THEORY AND STANDARD MODEL

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Notes

1. **We will use "Natural Unit"**

$$c = \hbar = 1$$

$$[\mathrm{length}] = [\mathrm{time}] = [\mathrm{energy}]^{-1} = [\mathrm{mass}]^{-1}$$

2. **Metric convention**

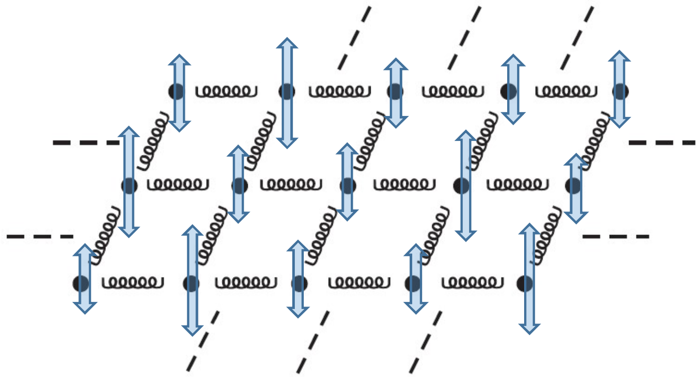
$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

3. **arXiv.org , inspirehep.net**

What is Quantum Field and Why Quantum Field?

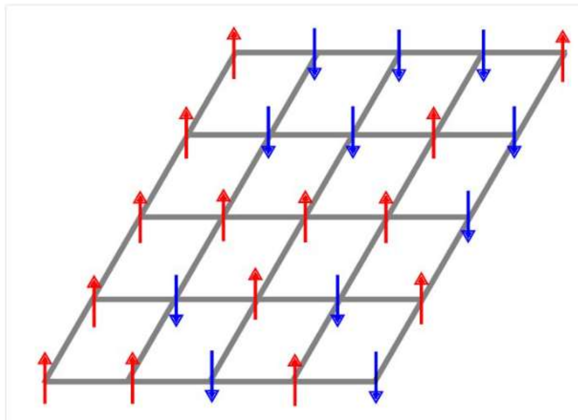
LECTURE 1

Classical Field



(wave in continuum limit)

Quantum Field



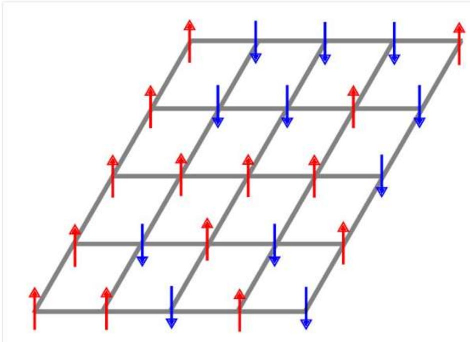
Coherent motion of infinitely large number of degrees of freedom (thermodynamic limit)

Question : Why thermodynamic limit is important?

Example: Nearest neighbor Ising model

(Ref: N. Goldenfeld, Lectures on phase transition and the renormalization group)

$$\mathcal{H} = -H\sum_{i=1}^{N} S_i - J\sum_{\langle ij\rangle} S_i S_j$$



$$Z = \sum_{S_1}\cdots\sum_{S_N} e^{-\beta\mathcal{H}} \equiv e^{-\beta F(H,J,T)} \qquad \beta = \frac{1}{k_B T}, \qquad \sum_{S_i} = \sum_{S_i=\pm 1}$$

magnetic moment :

$$M = \frac{1}{N}\sum_{i=1}^{N}\langle S_i\rangle = \frac{1}{N}k_B T\frac{\partial}{\partial H}\log Z = -\frac{1}{N}\frac{\partial}{\partial H}F$$

$$M = \frac{1}{N}\sum_{i=1}^{N}\langle S_i \rangle = \frac{1}{N}k_B T \frac{\partial}{\partial H}\log Z = -\frac{1}{N}\frac{\partial}{\partial H}F$$

Since

$$F(H,J,T) = F(-H,J,T)$$

$$N \times M = -\frac{\partial}{\partial H}F(H) = -\frac{\partial}{\partial H}F(-H) = \frac{\partial}{\partial(-H)}F(-H) = -N \times M = 0$$

No Magnetization ? : True for finite N

In one-dimensional case,

$$Z = \lambda_1^N + \lambda_2^N$$

$$\lambda_{1,2} = e^{\beta J}\left(\cosh(\beta H) \pm \sqrt{\sinh^2(\beta H) + e^{-\beta J}}\right) \qquad \lambda_1 > \lambda_2$$

$$Z = \lambda_1^N\left(1 + \left(\frac{\lambda_2}{\lambda_1}\right)^N\right) \to \lambda_1^N \qquad N \to \infty$$

$$F \simeq N(-k_B T \log \lambda_1) \equiv N f, \qquad M = -\frac{\partial f}{\partial H}$$

In thermodynamic limit, free energy density which is independent of N is well defined

Details: $h \equiv \beta H, \quad K \equiv \beta J$ + periodic boundary condition $S_1 = S_{N+1}$

$$Z = \sum_{S_1} \cdots \sum_{S_N} e^{h \sum_i S_i + K \sum_i S_i S_{i+1}}$$

$$= \sum_{S_1} \cdots \sum_{S_N} e^{\frac{h}{2}(S_1+S_2)+KS_1S_2} e^{\frac{h}{2}(S_2+S_3)+KS_2S_3} \cdots e^{\frac{h}{2}(S_N+S_1)+KS_NS_1}$$

$$= \sum_{S_1} \cdots \sum_{S_N} T_{S_1S_2} T_{S_2S_3} \cdots T_{S_NS_1}$$

$$= \sum_{S_1} (T^N)_{S_1S_1} = \mathrm{Tr}(T^N)$$

S_i	S_{i+1}	$T_{S_i,S_{i+1}}$
$+1$	$+1$	e^{h+K}
-1	-1	e^{-h+K}
$+1$	-1	e^{-K}
-1	$+1$	e^{-K}

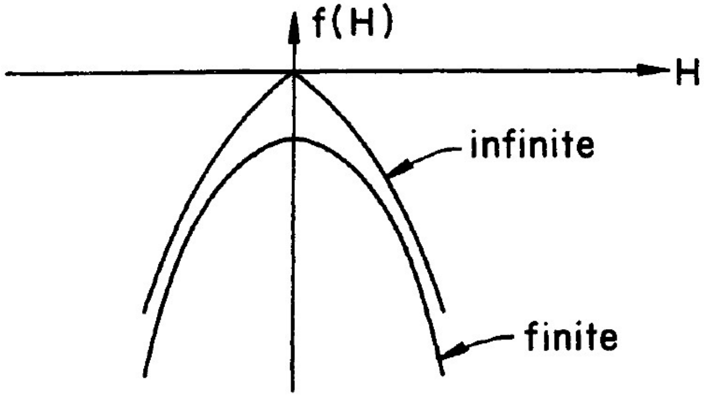
$$T = e^{\frac{h}{2}(S_i+S_{i+1})+KS_iS_{i+1}} = \begin{pmatrix} e^{h+K} & e^{-K} \\ e^{-K} & e^{-h+K} \end{pmatrix}$$

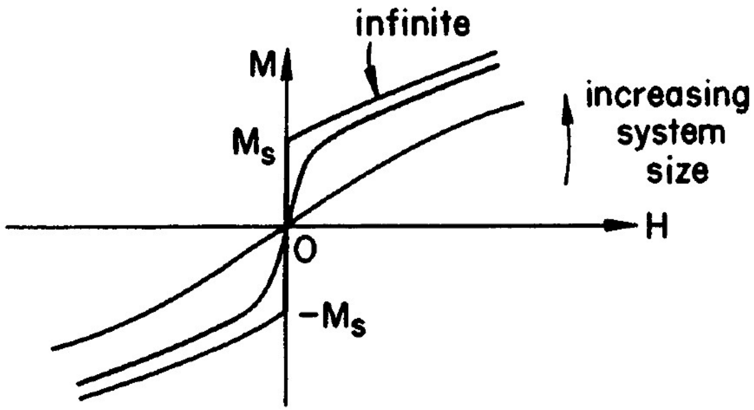
For $T \to 0$ ($\beta \to \infty$),

$$\lambda_1 \simeq e^{\beta H}\big(\cosh(\beta H) + |\sinh(\beta H)|\big) = e^{\beta(J+|H|)}$$

$$f = -k_B T \log(\lambda_1) = -J - |H|$$

$$M = \begin{cases} +1 & H > 0 \\ -1 & H < 0 \end{cases}$$





Continuum limit : Heisenberg Model

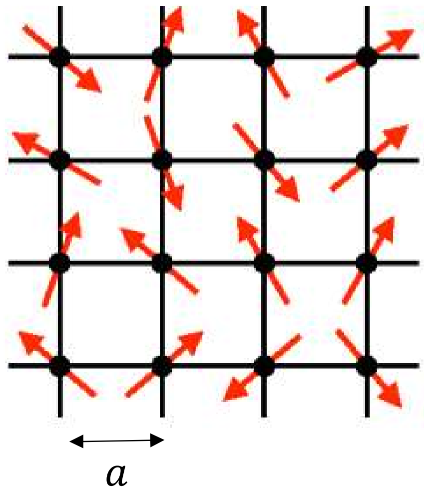
$$S_i = \pm 1 \to \vec{S}(\vec{x}) \text{ with } |\vec{S}| = 1$$

$$\mathcal{H} = -J \sum_{\langle \mathbf{x}, \mathbf{x}' \rangle} S^a(\mathbf{x}) S^a(\mathbf{x}') - H^a \sum_{\mathbf{x}} S^a(\mathbf{x}) + \frac{\lambda}{4} \sum_{\mathbf{x}} \left(S^a(\mathbf{x}) S^a(\mathbf{x}) - 1 \right)^2$$

$$\sum_{\langle \mathbf{x}, \mathbf{x}' \rangle} S^a(\mathbf{x}) S^a(\mathbf{x}') = \sum_{\mathbf{x}} S^a(\mathbf{x}) S^a(\mathbf{x} + \mathbf{a}) = \frac{1}{2} \sum_{\mathbf{x}} S^a(\mathbf{x}) (S^a(\mathbf{x} + \mathbf{a}) + S^a(\mathbf{x} - \mathbf{a}))$$

$$= \sum_{\mathbf{x}} \left(S^a(\mathbf{x}) S^a(\mathbf{x}) + \frac{1}{2} S^a(\mathbf{x}) \nabla^2 S^a(\mathbf{x}) \right)$$

$$\sum_{\mathbf{x}} \to \int \frac{d^d x}{a^d} \qquad \varphi^a = \sqrt{J} a^{\frac{2-d}{2}} S^a$$



$$\mathcal{H} = \int d^d x \Big(-\frac{1}{2}\varphi^a \nabla^2 \varphi^a - (J + \frac{\lambda}{2}) a^{-2} \varphi^a \varphi^a + \frac{1}{4}(\frac{\lambda}{J^2}) a^{d-4} (\varphi^a \varphi^a)^2 \Big)$$
$$= \int d^d x \Big(-\frac{1}{2}\nabla\varphi^a \nabla \varphi^a - (J + \frac{\lambda}{2}) a^{-2} \varphi^a \varphi^a + \frac{1}{4}(\frac{\lambda}{J^2}) a^{d-4} (\varphi^a \varphi^a)^2 \Big)$$

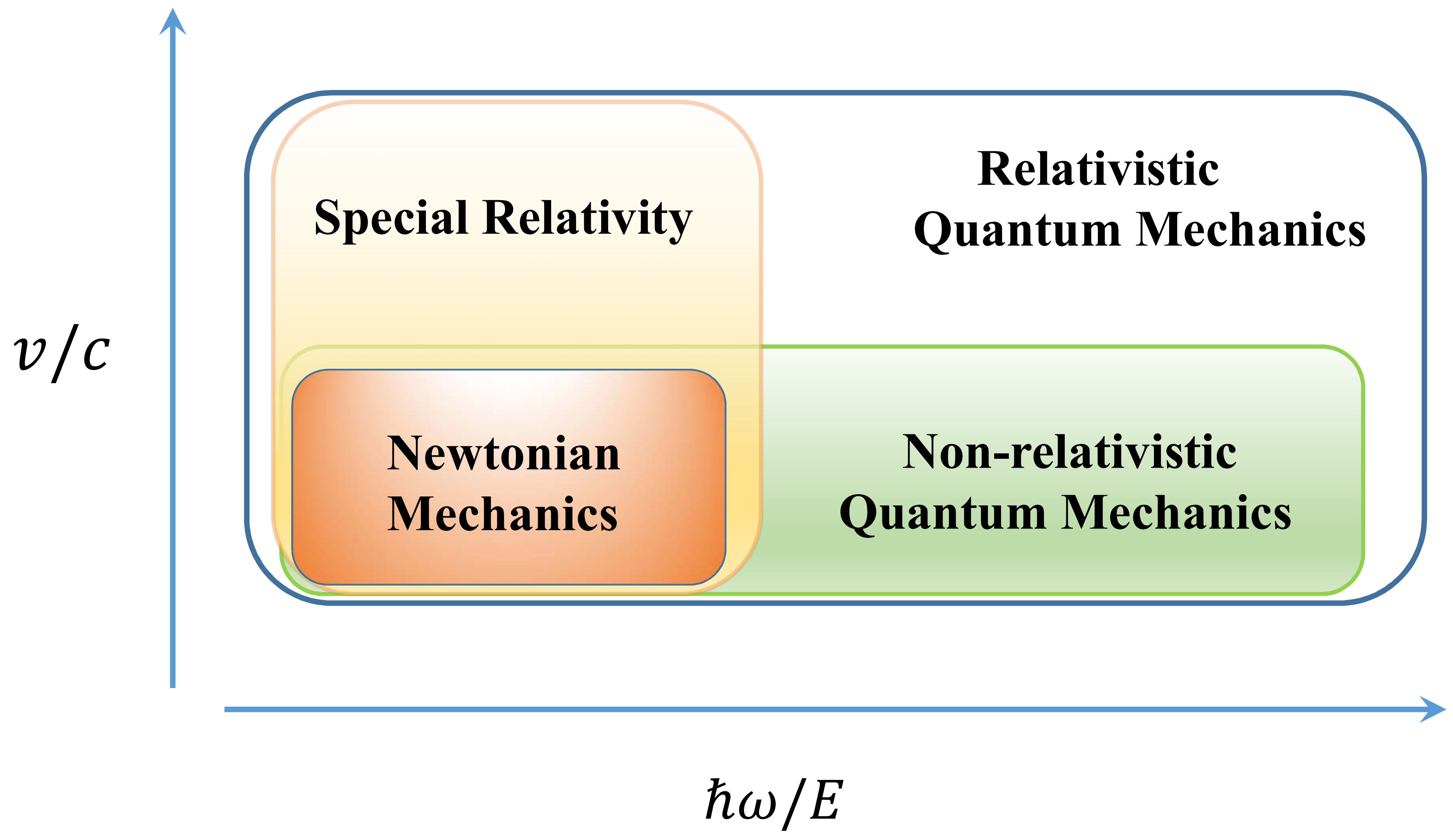
$$Z = \sum_{S(\mathbf{x}_1)} \cdots \sum_{S(\mathbf{x}_N)} e^{-\beta\mathcal{H}} = \prod_{\mathbf{x}} \sum_{S(\mathbf{x})} e^{-\beta\mathcal{H}} \to \prod_{\mathbf{x}} \int d\varphi^a e^{-\beta\mathcal{H}}$$

Functional integral (a.k.a. path integral)

For particle physics,

Quantum Mechanics + **Special Relativity**

= **Relativistic Quantum Mechanics**



Special Relativity
Relativistic
Quantum Mechanics
v/c
Newtonian
Mechanics
Non-relativistic
Quantum Mechanics
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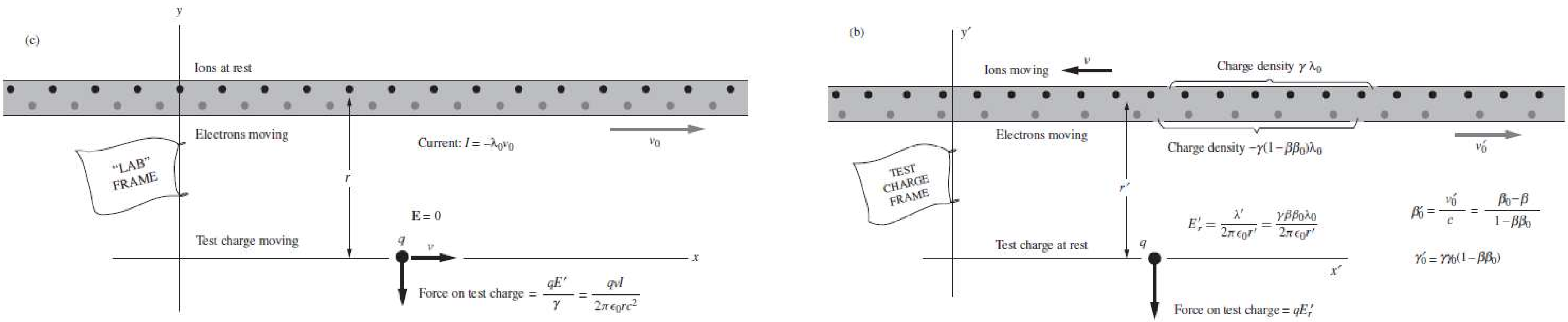
Why Relativistic QM is in the form of Quantum Field Theory (QFT) ?

Hint: Electrodynamics already contains special relativity

Example : Coulomb’s law + special relativity = Ampere’s law + Lorentz force



de Broglies' particle-wave duality :

If photon energy momentum is given by $(\boldsymbol{E}, \boldsymbol{p}) = (\hbar\boldsymbol{\omega}, \hbar\boldsymbol{k})$,

Why not generic particle's energy momentum is given by $(\boldsymbol{E}, \boldsymbol{p}) = (\hbar\boldsymbol{\omega}, \hbar\boldsymbol{k})$?

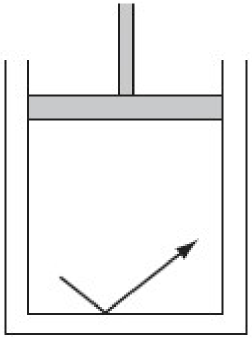
If the particle-wave duality of photon is obtained from the quantization of electromagnetic field operator,

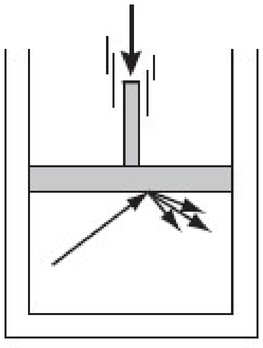
Why not the particle-wave duality of generic particle is obtained from quantization of field operator ?

Then how are the states given?

In special relativity, interactions between particles are inelastic

: particle species as well as the number of particles are not fixed

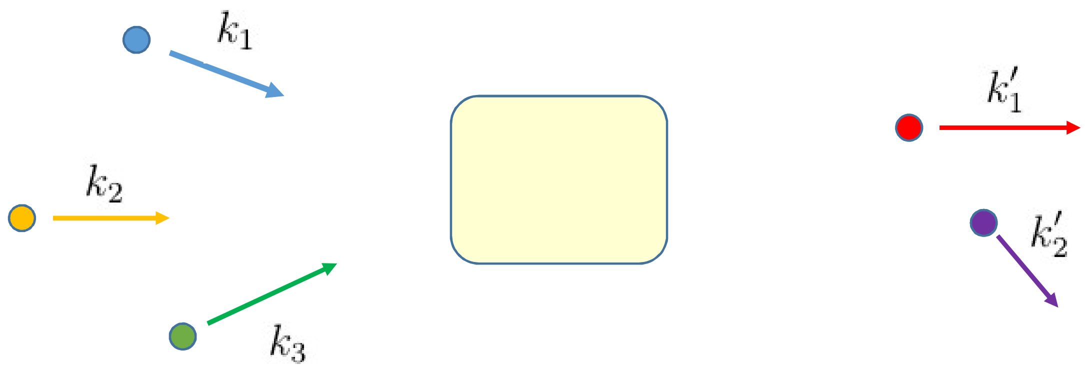




$$\Delta x \Delta p \geq \frac{\hbar}{2} \qquad \Delta x \to 0 \qquad \Delta p \to \infty$$

Photons with energy $E > 2\, m_e$ can create the electron-positron pair

Elementary process :



$|\alpha\rangle_{\rm in} = |k_1, s_1; k_2, s_2; k_3, s_3\rangle$ $\qquad$ $|\beta\rangle_{\rm out} = |k'_1, s'_1; k'_2, s'_2\rangle$

Probability amplitude : ${}_{\rm out}\langle\beta|\alpha\rangle_{\rm in} \equiv \langle\beta|S|\alpha\rangle$ $\qquad$ $S = \lim_{\rm T\to\infty} U(T, -T)$ **S-matrix**

Fock space $\quad \mathcal{H} = \{|k_1, s_1; k_2, s_2; \cdots\rangle\}$ $\qquad$ **appropriate for many-body physics**

Question :

In order to describe the special relativistic S-matrix (or equivalently, probability amplitude), how the Hamiltonian should be given? That is, how the fundamental framework of relativistic quantum mechanics should be given?

Brief summary on Special Relativity

In two inertial frames, the interval between two events read

$$\text{Frame } O \; : \quad (dt, d\mathbf{x})$$

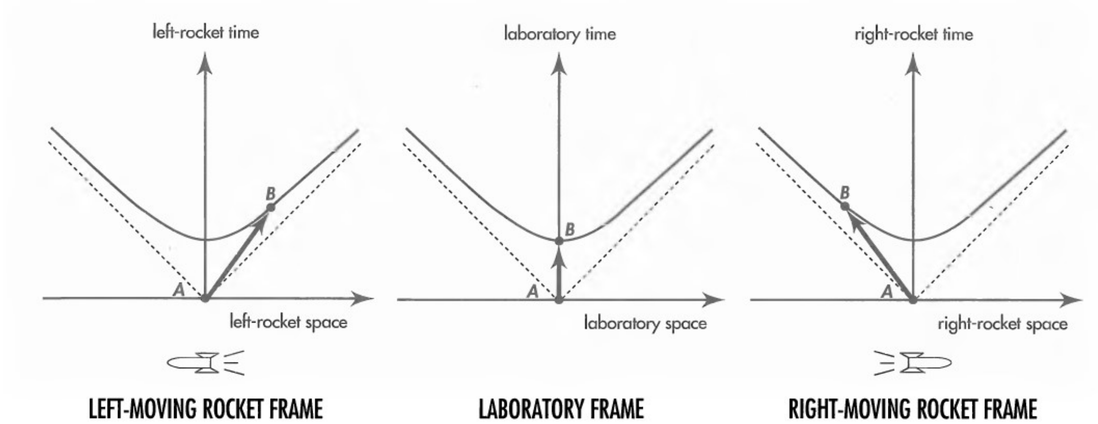
$$\text{Frame } O' \; : \quad (dt', d\mathbf{x}')$$

$$d\tau^2 \equiv dt^2 - d\mathbf{x}^2 = dt'^2 - d\mathbf{x}'^2$$

(Lorentz invariance)

$$m_0^2 = \left(m_0 \frac{dt}{d\tau}\right)^2 - \left(m_0 \frac{d\mathbf{x}}{d\tau}\right)^2 \equiv E^2 - \mathbf{p}^2$$

"on (the mass-)shell"



$$m_0^2 = E^2 - \mathbf{p}^2$$

In QM, $\boldsymbol{E} \rightarrow \boldsymbol{i\partial/\partial t}, \boldsymbol{p} \rightarrow \boldsymbol{i\nabla}$ **:**

$$m_0^2 = -\frac{\partial^2}{\partial t^2} + \nabla^2$$

(Klein-Gordon)

$$\partial_\mu \partial^\mu + m^2 = 0$$

In addition, translation invariance is still imposed:

Lorentz invariance + Translation invariance = Poincare invariance

Poincare group : $(\boldsymbol{\Lambda}, \boldsymbol{a})$

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \qquad (\Lambda_1, a_1)(\Lambda_2, a_2) = (\Lambda_1\Lambda_2, a_1 + \Lambda_1 a_2)$$

$$dx' \cdot dy' \equiv \eta_{\mu\nu} dx'^\mu dy'^\nu = dx \cdot dy \longrightarrow \eta_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$$

"Boost" :

$$\begin{pmatrix} \cosh(\omega) & 0 & 0 & -\sinh(\omega) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh(\omega) & 0 & 0 & \cosh(\omega) \end{pmatrix}$$

"rotation" :

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) & 0 \\ 0 & -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\cosh(\omega) = \frac{1}{\sqrt{1-v^2}} = \gamma$$

$$\sinh(\omega) = \frac{v}{\sqrt{1-v^2}} = v\gamma$$

1. Lorentz invariant integral measure

$$\int d^4k\theta(k_0)\delta(k^2 - m^2) = \int \frac{d^3k}{2E(k)}, \qquad k^2 \equiv k \cdot k, \quad E(k) = \sqrt{k^2 + m^2}$$

2. Fock space normalization

1) covariant normalization

$$\langle k'|k\rangle = 2E(k)\delta^3(\mathbf{k} - \mathbf{k}') \qquad U(\Lambda)|k\rangle = |\Lambda k\rangle, \qquad \langle k'|k\rangle = \langle \Lambda k'|\Lambda k\rangle$$

2) non-covariant normalization

$$\langle k'|k\rangle = \delta^3(\mathbf{k} - \mathbf{k}') \qquad U(\Lambda)|k\rangle = \sqrt{\frac{E(\Lambda k)}{E(k)}}|\Lambda k\rangle$$

Translation is generated by the free Hamiltonian (caution!)

$$P^{(0)\mu} = (H_0, \mathbf{P}^{(\mathbf{0})})$$

$$H_0|k_1, \cdots k_n\rangle = \sum_{i=1}^{n} E(k_i)|k_1, \cdots k_n\rangle, \qquad \mathbf{P}^{(0)}|k_1, \cdots k_n\rangle = \sum_{i=1}^{n} \mathbf{k}_i|k_1, \cdots k_n\rangle$$

In covariant normalization

$$\begin{aligned}\langle k'|U^\dagger(\Lambda)P^{(0)\mu}U(\Lambda)|k\rangle &= (\Lambda^\mu{}_\nu k^\nu)2E(\Lambda k)\delta^3(\Lambda\mathbf{k}' - \Lambda\mathbf{k}) = (\Lambda^\mu{}_\nu k^\nu)2E(k)\delta^3(\mathbf{k}' - \mathbf{k}) \\ &= \Lambda^\mu{}_\nu \langle k'|P^{(0)\nu}|k\rangle\end{aligned}$$

$$U^\dagger(\Lambda)P^{(0)\mu}U(\Lambda) = \Lambda^\mu{}_\nu P^{(0)\nu}$$

Time evolution

$$U(t,t_0) = e^{-iH(t-t_0)}, \qquad H = H_0 + V, \quad V = \int d^3x \mathcal{H}_{\text{int}}$$

Interaction picture:

$$\mathcal{O}_{\text{ip}} = e^{iH_0 t}\mathcal{O}_S e^{-iH_0 t}, \qquad |\alpha;t\rangle_{\text{ip}} = e^{iH_0 t}e^{-iHt}|\alpha\rangle$$

$$U_{\text{ip}}(t,t_0) = e^{iH_0 t}e^{-i(H-H_0)t}e^{-iH_0 t}, \qquad |\alpha;t\rangle_{\text{ip}} = U_{\text{ip}}(t,t_0)|\alpha;t_0\rangle_{\text{ip}}$$

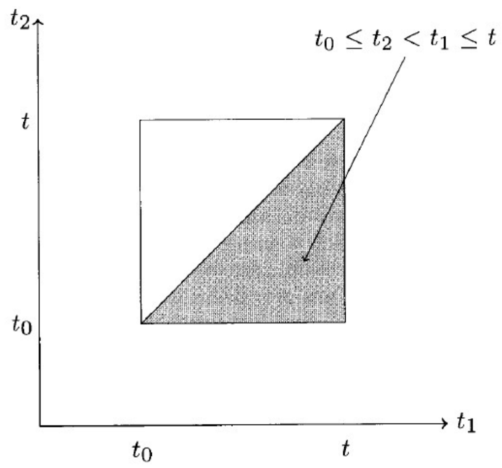
$$i\frac{d}{dt}U_{\text{ip}}(t,t_0) = V_{\text{ip}}U_{\text{ip}}(t,t_0), \qquad V_{\text{ip}} = e^{iH_0 t}Ve^{-iH_0 t}$$

$$U(t,t_0) = \sum_{n=0}^{\infty}(-i)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \ldots \int_{t_0}^{t_{n-1}} dt_n V_{\mathbf{ip}}(t_1) V_{\mathbf{ip}}(t_2) \ldots . V_{\mathrm{ip}}(t_n)$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 dt_2 .. dt_n T\{V_{\mathrm{ip}}(t_1) \ldots . V_{\mathrm{ip}}(t_n)\}$$

Time ordering :

$$T(\mathcal{O}(t_1)\mathcal{O}(t_2)) = \begin{cases} \mathcal{O}(t_1)\mathcal{O}(t_2) & \text{for } t_1 > t_2 \\ \mathcal{O}(t_2)\mathcal{O}(t_2) & \text{for } t_1 < t_2 \end{cases}$$



Requirement :

1., 2. Probability amplitude is invariant under the action of Poincare group

$$[U(a), S] = 0, \qquad [U(\Lambda), S] = 0 \longrightarrow [U(a), V_{\rm ip}] = 0, \qquad [U(\Lambda), V_{\rm ip}] = 0 \qquad U(a) = e^{ia\cdot P^{(0)}}$$

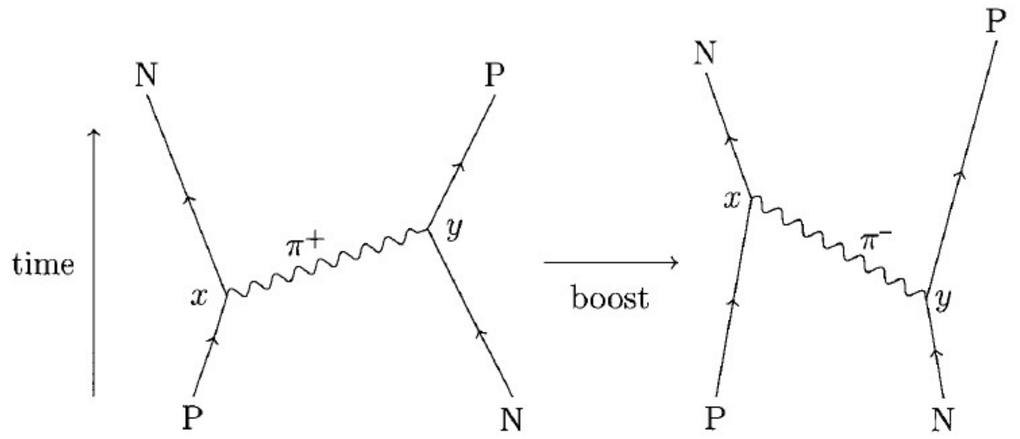
Then we can impose

$$i[P^{(0)}_\mu, \mathcal{H}_{\rm int}] = \partial_\mu \mathcal{H}_{\rm int}, \qquad U(\Lambda)\mathcal{H}_{\rm int}(x)U^\dagger(\Lambda) = \mathcal{H}_{\rm int}(\Lambda x)$$

3. Causality of operator (NOT of state)

$$[\mathcal{H}(x_1), \mathcal{H}(x_2)] = 0 \quad \text{if } (x_1 - x_2)^2 < 0$$

4. (consequence of 2)   Existence of antiparticle of the same mass



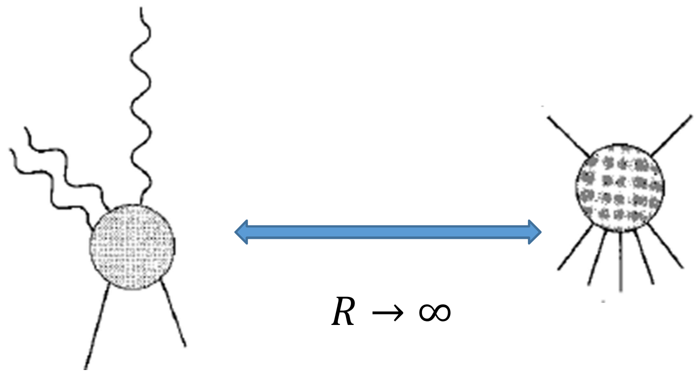
: The probability amplitude must be the same   (CPT theorem)

Note that 'charge conjugation' (particle ↔ antiparticle) is the spacetime symmetry (defined even in the absence of electric charge)

See, Sec. 2.13 of S. Weinberg, Gravitation and cosmology

5. Cluster decomposition

: for far-separated processes, the probability amplitude factorizes



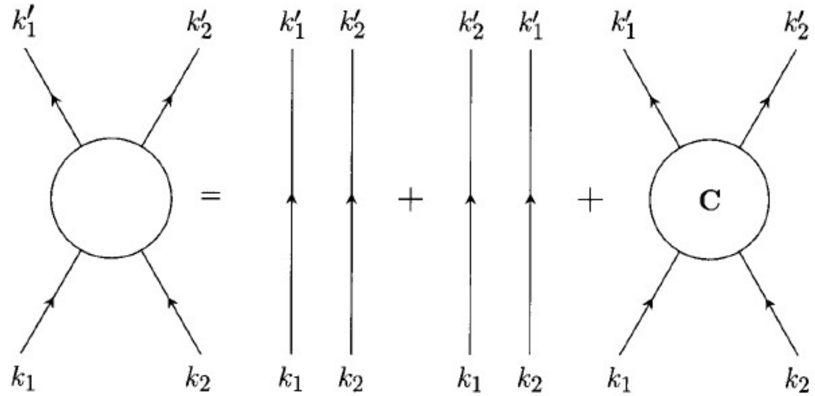
$$\mathcal{M} \simeq \mathcal{M}_1 \times \mathcal{M}_2$$

(see, e.g., Ch. 4, S. Weinberg, The Quantum Theory of Fields Vol. 1,

Ch. 6, A. Duncan, The Conceptual Framework of Quantum Field Theory)

We assume the process of stable spin-0 particle and start with 5. Cluster decomposition

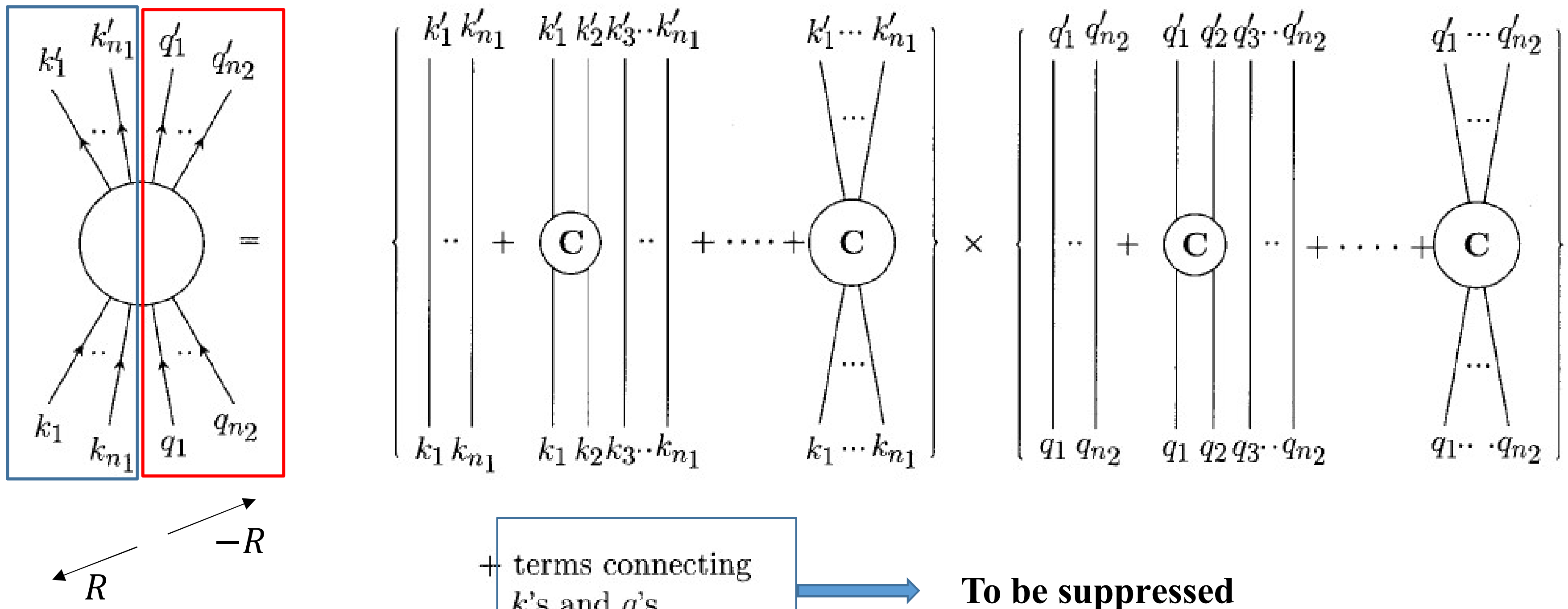
For this purpose, we define 'connected interaction'



$$S_{k_1'k_2',k_1k_2} = S^c_{k_1',k_1}S^c_{k_2',k_2} + S^c_{k_1',k_2}S^c_{k_2',k_1} + S^c_{k_1'k_2',k_1k_2}$$

$$S_{\vec{k}',\vec{k}} = \delta^3(\vec{k}' - \vec{k}) \equiv S^c_{\vec{k}',\vec{k}}$$

The Cluster decomposition imposes that



+ terms connecting k's and q's → **To be suppressed**

For this requirement to be fulfilled, S-matrix must be in the form of

$$S^c = \sum \delta(k_i - k_f) \times (\text{smooth function of } k)$$

each term for the connected part must contain a single delta function only
: suppose S-matrix contains a term containing two delta functions then…

to see what the problem is, we replace the place wave with the wave packet

$$e^{i\mathbf{k}\cdot\mathbf{x}} \to \int d^3\tilde{k}\, g(k, \tilde{k}) e^{i\tilde{\mathbf{k}}\cdot\mathbf{x}}$$

Then the amplitude for the connected part is proportional to

$$\int \prod_i d^3\tilde{k}_i \prod_j d^3\tilde{q}_j \prod_i d^3\tilde{k}'_i \prod_i d^3\tilde{q}'_i e^{-i\left(\sum_i \tilde{k}_i(R+\xi_i)+\sum_i \tilde{q}_j(-R+\eta_i)+\sum_i \tilde{k}'_i(R+\xi'_i)+\sum_i \tilde{q}'_i(-R+\eta'_i)\right)} S^c_{\tilde{k}'_i\tilde{q}'_j\tilde{k}_i\tilde{q}_i}$$

This contains $e^{iR\left(\sum_i \tilde{k}'_i-\sum_j \tilde{q}'-\sum_i \tilde{k}_i+\sum_j \tilde{q}_j\right)}$

When S^c contains a single delta function only $\delta(\sum_i \tilde{k}'_i + \sum_j \tilde{q}'_j - \sum_i \tilde{k}_i - \sum_j \tilde{q}_j)$

it becomes $e^{2iR\left(\sum_j \tilde{q}_j-\sum_j \tilde{q}'_j\right)}$ **: vanishes in the limit $R \to \infty$ after integrating over momenta**

: consistent with the cluster decomposition.

If S^c contains additional delta function $\delta(\sum_j \tilde{q}'_j - \sum_j \tilde{q}_j)$

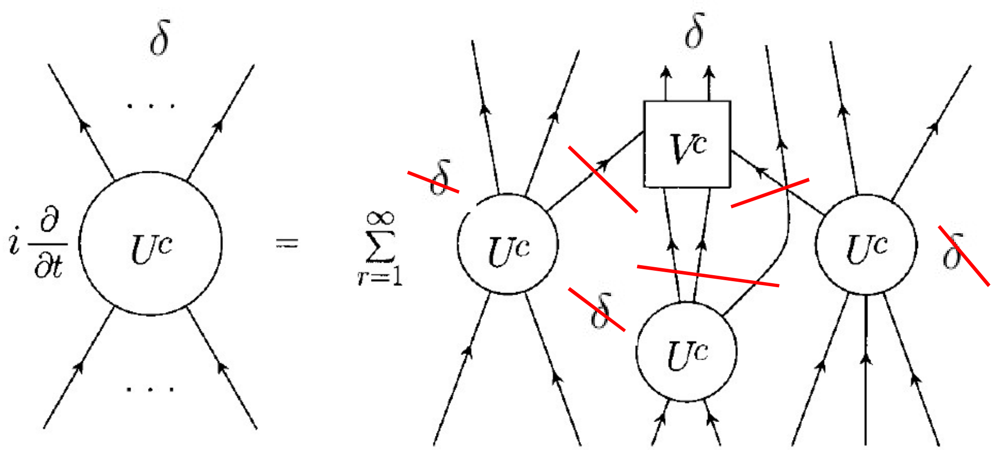
it does not vanish in the limit $R \to \infty$

: cluster decomposition is violated!

The connected part of S-matrix contains a single delta function

: The connected part of H_{int} contains a single delta function

$$i\frac{d}{dt}U_{\text{ip}}(t,t_0) = V_{\text{ip}}U_{\text{ip}}(t,t_0)$$



Delta functions are eliminated in the summation over the intermediate states, so only overall energy-momentum conservation delta function remains

Indeed, interaction Hamiltonian contains at least single delta function (total energy-momentum conservation) due to the translation invariance

$$V_{\rm ip}(x_1, x_2, x_1', x_2') = V(0, X_2, X_1', X_2') \qquad X_2 = x_2 - x_1,\ X_1' = x_1' - x_1,\ X_2' = x_2' - x_1$$

$$\begin{aligned} H^c_{k_1',k_2',k_1,k_2} &= \int d^3x_1' d^3x_2' d^3x_1 d^3x_2 V_{\rm ip}(x_1, x_2, x_1', x_2') e^{-i(k_1'x_1' + k_2'x_2' - k_1x_1 - k_2x_2)} \\ &= \int d^3X_1' d^3X_2' d^3x_1 d^3X_2 V_{\rm ip}(0, X_2, X_1', X_2') e^{-i(k_1' + k_2' - k_1 - k_2)x} e^{-i(k_1'X_1' + k_2'X_2' - k_2X_2)} \\ &= (2\pi)^3 \delta(k_1' + k_2' - k_1 - k_2) \times (\text{smooth function of momenta}) \end{aligned}$$

The cluster decomposition tells us that no more delta function appears.

This is guaranteed for local interaction :

$$\int d^4x \varphi_1(x) \cdots \varphi_n(x) = \int d^4x \int d^4k_1 \cdots d^4k_n e^{ik_1x} \varphi_1(k_1) \cdots e^{ik_nx} \varphi_n(k_n)$$

$$= \int d^4k_1 \cdots d^4k_n (2\pi)^4 \delta(k_1 + \cdots + k_n) \varphi_1(k_1) \cdots \varphi_n(k_n)$$

Then how to construct the Hamiltonian consistent with the cluster decomposition?

: Introduce the creation / annihilation operator

$$[a(k), a^\dagger(k')]_\mp = \delta^3(\mathbf{k}-\mathbf{k}') \quad [a(k), a(k')]_\mp = [a^\dagger(k), a^\dagger(k')]_\mp = 0$$

$$a(k)|k_1, k_2, ..k_N\rangle \equiv \sum_{r=1}^{N} (\pm)^{r-1} \delta^3(k-k_r)|k_1, ..k_{r-1}, k_{r+1}, ..k_N\rangle$$

$$a^\dagger(k)|k_1, k_2, ..k_N\rangle \equiv |k, k_1, k_2, ..k_N\rangle$$

In terms of creation / annihilation operator

$$H = \sum_{M,M'} \frac{1}{M!M'!} \int \prod_i^M d^3k_i \prod_i^{M'} d^3k'_i \delta^3\big(\sum_i k'_i - \sum_i k_i\big) f_{M'M}(k_i, k'_i) a^\dagger(k'_1) \cdots a^\dagger(k'_{M'}) a(k_1) \cdots a(k_M)$$

$$\begin{aligned}
\langle k'|H|k\rangle &= \int d^3k'_1 d^3k_1 \delta^3(k'_1 - k_1) f_{11} \langle k'|a^\dagger(k'_1) a(k_1)|k\rangle \\
&= \int d^3k'_1 d^3k_1 \delta^3(k'_1 - k_1) f_{11} \delta^3(k' - k'_1) \delta^3(k_1 - k) = \delta^3(k' - k) f_{11}(k'k) = H^c_{k'k} \\
\langle k'_1 k'_2|H|k_1 k_2\rangle &= \Big(H^c_{k'_1 k_1} \delta^3(k'_2 - k_2) + \text{Perms.}\Big) + H^c_{k'_1 k'_2 k_1 k_2}
\end{aligned}$$

each H^c contains a single delta function.

Since creation/annihilation operator creates/annihilates a single particle, one can imagine the 'particle field operator**' for a single particle which is linear in creation/annihilation operator.**

Hamiltonian is regarded as a product of particle field operators in a local way :

$$H = \int d^3x \mathcal{H}(x) \qquad \mathcal{H}(x) = (\text{derivative operator})^{2n} \phi_2(x)\phi_2(x)\cdots\phi_n(x)$$

Sum over all possible positions where interaction takes place (d^3x)

Hamiltonian density ($\mathcal{H}(x)$)

Product of fields at the same position (local interaction)

Causality : $[\mathcal{H}(x_1), \mathcal{H}(x_2)] = 0$ **if** $(\boldsymbol{x_1} - \boldsymbol{x_2})^2 < 0$

Consider

$$\phi^{(+)}(x) = \int d^3k f(x;k) a(k)$$

Condition 1 :

From

$$i[P_\mu^{(0)}, \phi^{(+)}(x)] = \partial_\mu \phi^{(+)}(x)$$

one finds

$$[P_\mu^{(0)}, a^\dagger(k)] = k_\mu a^\dagger(k), \qquad [P_\mu^{(0)}, a(k)] = -k_\mu a(k)$$

$$\partial_\mu f(x;k) = -ik_\mu f(x;k) \qquad f(x;k) = f(k) e^{-ik\cdot x}, \quad k^\mu = (E(k), \mathbf{k})$$

Condition 2 : $$U(\Lambda)\phi^{(+)}U^{\dagger}(\Lambda) = \phi^{(+)}(\Lambda x)$$

From $$U(\Lambda)a^{(\dagger)}(k)H^{\dagger}(\Lambda) = \sqrt{\frac{E(\Lambda k)}{E(k)}}a^{(\dagger)}(\Lambda k)$$ **(for non-covariant basis)**

one finds $$\phi^{(+)} = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2E(k)}}e^{-ik\cdot x}a(k)$$

(for covariant basis, $\phi^{(+)} = \int \frac{d^3k}{(2\pi)^3 2E(k)}e^{-ik\cdot x}a(k)$ **)**

Condition 4 :

annihilation/creation of particle $(a(k), a(k)^\dagger)$

annihilation/creation of antiparticle $(a_c(k), a_c(k)^\dagger)$

(cf. some particles are their own antiparticles, e.g., neutral pion while some neutral particles are not their own antiparticles, e.g., neutral Kaon)

SO(2) or U(1) symmetry:

$$\phi_1 \to \cos\theta\phi_1 + \sin\theta\phi_2$$
$$\phi_2 \to -\sin\theta\phi_1 + \cos\theta\phi_2$$

for real scalars

$$\phi = \phi_1 + i\phi^2 \to e^{-i\theta}\phi$$

single complex scalar

Let

$$\phi(x) = \phi^{(+)} + \phi_c^{(+)\dagger} \equiv \phi^{(+)} + \phi^{(-)}$$
$$= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{E(k)}} \left(a(k)e^{-ik\cdot x} + a_c^\dagger(k)e^{ik\cdot x}\right)$$

(natural as particle and antiparticle have the opposite quantum number like charge)

Condition 4 : Causality

Naturally, $[\phi(x), \phi(y)] = [\phi^\dagger(x), \phi^\dagger(y)] = 0$

Then, what about $[\phi(x), \phi^\dagger(y)]$ **if** $(x-y)^2 < 0$ **?**

From $[a(k), a^\dagger(k')] = \delta^3(\mathbf{k} - \mathbf{k}')$

$$[\phi^{(+)}(x), \phi^{(+)\dagger}(y)] = \int \frac{d^3k}{(2\pi)^3 2E(k)} e^{-ik(x-y)} \equiv \Delta_+(x-y;m)$$

The same relation holds for $\boldsymbol{\phi}^{(-)}$

$$[\phi(x), \phi^\dagger(y)] = \int \frac{d^3k}{(2\pi)^3 2E(k)} \left(e^{-ik(x-y)} - e^{ik(x-y)}\right) = \Delta_+(x-y;m) - \Delta_+(x-y;m_c)$$

(particle) **(antiparticle)**

$$[\phi(x), \phi^\dagger(y)] = \int \frac{d^3k}{(2\pi)^3 2E(k)} \left(e^{-ik(x-y)} - e^{ik(x-y)}\right) = \Delta_+(x-y;m) - \Delta_+(x-y;m_c)$$

This is a Lorentz scalar.
When $(x-y)^2 < 0$, one can always find a frame at which $x^0 = y^0$ such that

$$\int \frac{d^3k}{(2\pi)^3} \left(\frac{e^{+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{2E(k)} - \frac{e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{2E_c(k)}\right) = 0$$

for $m = m_c$

That is, when the particle and antiparticle have the same mass,

$$[\phi(x), \phi^\dagger(y)] = 0 \qquad \text{for } (x-y)^2 < 0$$

Local quantum field for spin-0 particle :

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2E(k)}}\left(a(k)e^{-ikx} + a_c^\dagger(k)e^{ikx}\right)$$

annihilation operator for a particle

creation operator for an antiparticle

Plane wave as a solution to Klein-Gordon equation $(\partial_\mu\partial^\mu + m^2)\phi = 0 \quad E(k) = \sqrt{k^2+m^2}$

(for higher spins, solution to equations of motion containing spin structure :

spin-1/2 : Dirac equation, spin-1 : Maxwell equation, spin-3/2 : Rarita-Schwinger equation

spin-2 :Einstein equation)