양자장론과 표준모형

QUANTUM FIELD THEORY AND STANDARD MODEL

서민석
한국교원대학교

2022 KIAS 입자물리 여름캠프

Dynamics of Local Quantum Field and Renormalization

LECTURE 2

In QM, time evolution is generated by Hamiltonian H

: In special relativity, H is no longer a scalar, but a time component of 4-vector P^{μ}

Thus, $H\,t$ in $U = e^{-i\,H\,t}$ is not a scalar as well.

In contrast, action S is a scalar

$$S = \int dt L = \int d^4 x \mathcal{L}$$

Indeed, action is a singlet under the symmetry transformations

: equations of motion for fields are covariant under the transformation

+ field ~ representation of the symmetry transformation : covariant

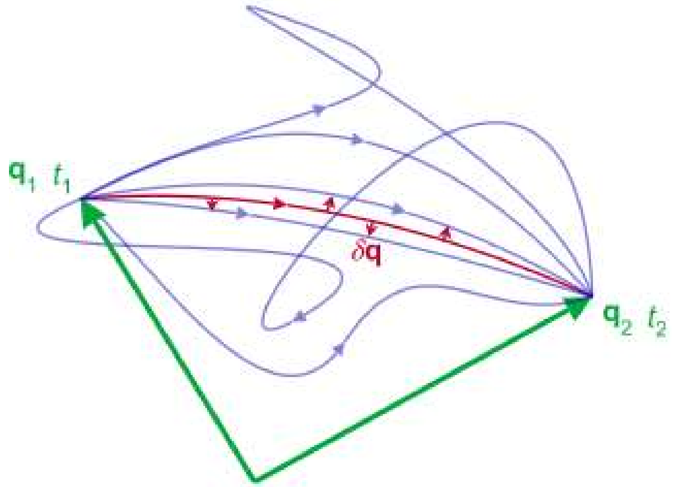
$$\frac{\delta S}{\delta \varphi} = 0$$

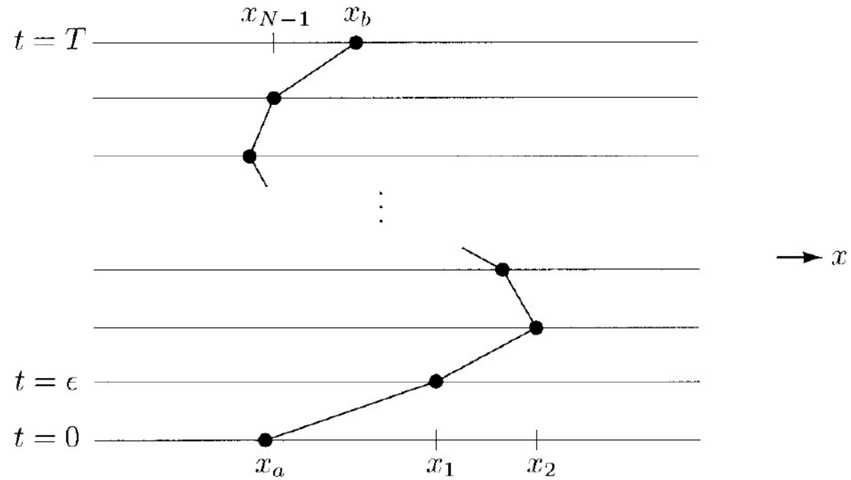
In relativistic theory, it is convenient to consider the action to investigate dynamics

QM treatment : Functional integral (or path integral)

Functional Integral (path integral)

Question : If a particle is located at q_i at t_i, the probability amplitude to find the particle at q_f at t_f?





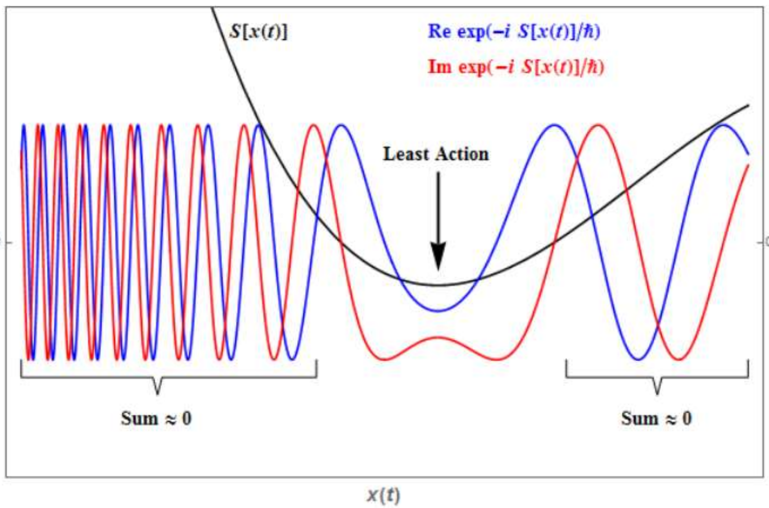
$$\langle q_f, t_f | q_i, t_i \rangle = \langle q_f | e^{-iHt} | q_i \rangle \qquad \epsilon = \frac{t_f - t_i}{N}, \qquad N \to \infty$$

$$= \int q_{N-1} \cdots dq_1 \langle q_f, t_f | q_{N-1}, t_{N-1} \rangle \cdots \langle q_n, t_n | q_{n-1}, t_{n-1} \rangle \cdots \langle q_1, t_1 | q_i, t_i \rangle$$

$$
\begin{aligned}
\langle q_n, t_n | q_{n-1}, t_{n-1}\rangle &= \langle q_n | e^{-\frac{i}{\hbar}\epsilon\left(\frac{\hat{p}^2}{2m}+\hat{V}(q)\right)} | q_{n-1}\rangle \simeq \langle q_n | e^{-\frac{i}{\hbar}\epsilon\frac{\hat{p}^2}{2m}} e^{-\frac{i}{\hbar}\epsilon\hat{V}(q)} | q_{n-1}\rangle \\
&= \langle q_n | e^{-\frac{i}{\hbar}\epsilon\frac{\hat{p}^2}{2m}} | q_{n-1}\rangle e^{-\frac{i}{\hbar}\epsilon V(q_{n-1})} = \int dp \langle q_n | p\rangle\langle p | q_{n-1}\rangle e^{-\frac{i}{\hbar}\epsilon\frac{p^2}{2m}} e^{-\frac{i}{\hbar}\epsilon V(q_{n-1})} \\
&= \int \frac{dp}{2\pi\hbar} e^{\frac{i}{\hbar}p(q_n - q_{n-1}) - \frac{i}{\hbar}\epsilon\frac{p^2}{2m}} e^{-\frac{i}{\hbar}\epsilon V(q_{n-1})} \\
&= \left(\frac{m}{2\pi\hbar i\epsilon}\right)^{1/2} e^{\frac{im}{2\hbar\epsilon}(q_n - q_{n-1})^2 - \frac{i}{\hbar}\epsilon V(q_{n-1})}
\end{aligned}
$$

$$
\begin{aligned}
\langle q_f, t_f | q_i, t_i\rangle &= \left(\frac{m}{2\pi\hbar i\epsilon}\right)^{N/2} \int \prod_{n=1}^{N-1} dq_n e^{\frac{i}{\hbar}\sum_n \epsilon\left(\frac{1}{2}\left(\frac{q_n - q_{n-1}}{\epsilon}\right)^2 - V(q_{n-1})\right)} \\
&= \left(\frac{m}{2\pi\hbar i\epsilon}\right)^{N/2} \int \prod_{n=1}^{N-1} dq_n e^{\frac{i}{\hbar}\int dt\left(\frac{1}{2}\dot{q}^2 - V(q)\right)} \\
&\equiv \int \mathcal{D}q e^{\frac{i}{\hbar}S} = \sum_{\text{all possible paths}} e^{\frac{i}{\hbar}S}
\end{aligned}
$$

classical limit : $\hbar \to 0$



1. Free real scalar

$$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

$$S = \int d^4x \mathcal{L}, \qquad \mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2\right) \sim -\frac{1}{2}\phi(\partial_\mu \partial^\mu + m^2)\phi$$

$$\langle \phi_f, t_f | \phi_i, t_i \rangle = \langle \phi_f | e^{-iH(t_f - t_i)} | \phi_i \rangle = \sum_{n,m} \langle \phi_f | n \rangle \langle n | e^{-iH(t_f - t_i)} | m \rangle \langle m | \phi_i \rangle$$

$$\underset{\substack{t_f \to \infty, \\ t_i \to -\infty}}{\longrightarrow} \langle 0 | e^{-iH(t_f - t_i)} | 0 \rangle$$

$$Z_0[J] = \mathcal{N}_0 \int \mathcal{D}\phi e^{i \int d^4x \left[\frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) + J(x)\phi(x) + \frac{i}{2}\epsilon \phi^2\right]} = \langle 0, t = \infty | 0, t = -\infty \rangle_J$$

normalization : $\langle 0, t=\infty | 0, t=-\infty \rangle_{J=0} = 1$ or $Z_0[J=0]=1$

$$Z_0[J] = \mathcal{N}_0 \int \mathcal{D}\phi e^{-i\int d^4x \left[\frac{1}{2}\phi(\partial_\mu\partial^\mu + m^2 - i\epsilon)\phi - J(x)\phi(x)\right]}$$

$$= \frac{\int \mathcal{D}\phi e^{iS[J]}}{\int \mathcal{D}\phi e^{iS[J=0]}} = \frac{\int \mathcal{D}\phi e^{-i\int d^4x \left[\frac{1}{2}\phi(\partial_\mu\partial^\mu + m^2 - i\epsilon)\phi - J(x)\phi(x)\right]}}{\int \mathcal{D}\phi e^{-i\int d^4x \left[\frac{1}{2}\phi(\partial_\mu\partial^\mu + m^2 - i\epsilon)\phi\right]}}$$

Using $\int dx e^{-\alpha x^2 + \beta x} = \left(\frac{\pi}{\alpha}\right)^{1/2} e^{\frac{\beta^2}{4\alpha}}$

$$Z_0[J] = e^{-\frac{i}{2}\int d^4x d^4y J(x)\Delta_F(x-y)J(y)}$$

(Note $Z_0[J=0]=1$)

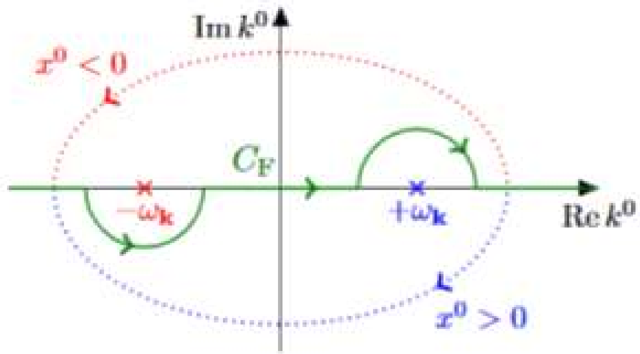
$$Z_0[J] = e^{-\frac{i}{2}\int d^4x d^4y J(x)\Delta_F(x-y)J(y)}$$

$$(\partial_\mu\partial^\mu + m^2 - i\epsilon)\Delta_F(x-y) = -\delta^4(x-y)$$

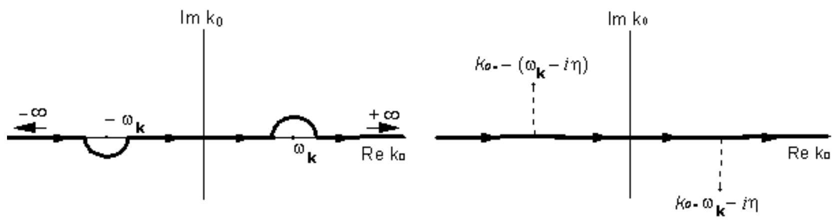
Feynman propagator : inverse of $(\partial_\mu\partial^\mu + m^2 - i\epsilon)$

$$k^2 - m^2 + i\epsilon = k_0^2 - (\mathbf{k}^2 + m^2 - i\epsilon) = k_0^2 - (E(k)^2 - i\epsilon)$$

$$\begin{aligned} i\Delta_F(x-y) &= \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2 + i\epsilon}e^{-ik(x-y)} \\ &= \theta(x^0 - y^0)\Delta_+(x-y;m) + \theta(y^0 - x^0)\Delta_+(y-x;m) \\ &= \langle 0|T(\phi(x)\phi(y))|0\rangle \end{aligned}$$

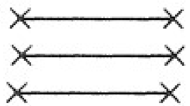


$$\Delta_+(x;m) = \int \frac{d^3k}{(2\pi)^3 2E(k)}e^{-ik(x-y)} = \langle 0|\phi(x)\phi(y)|0\rangle$$



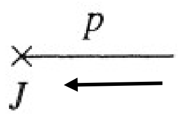
$$Z_0[J] = e^{-\frac{i}{2}\int d^4x d^4y J(x)\Delta_F(x-y)J(y)}$$

$$= \mathcal{N}\left[1 - \frac{i}{2}\int d^4x d^4y J(x)\Delta_F(x-y)J(y) + \frac{1}{2}\left(\frac{-i}{2}\right)^2\left[\int d^4x d^4y J(x)\Delta_F(x-y)J(y)\right]^2 + \cdots\right]$$

$= 1 + \frac{1}{2}$  $+ \frac{1}{2!}\left(\frac{1}{2}\right)^2$  $+ \frac{1}{3!}\left(\frac{1}{2}\right)^3$  $+ \cdots$

Rule :

$$\frac{1}{(2\pi)^4}\frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon},$$

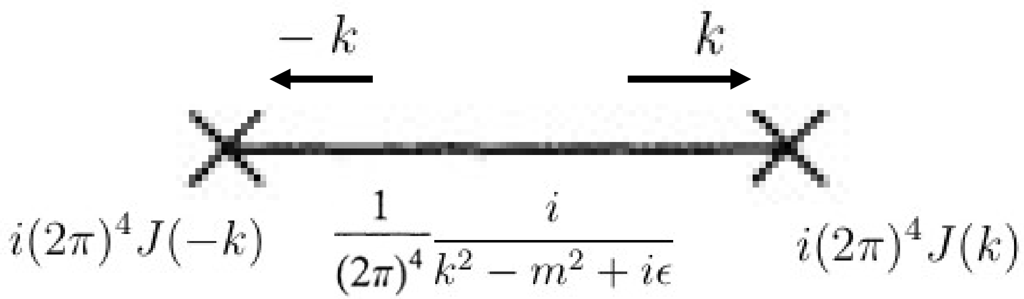


$$\mathrm{i}(2\pi)^4 J(p).$$

Example :

$$-\frac{i}{2}\int d^4x d^4y J(x)\Delta_F(x-y)J(y) = -\frac{i}{2}\int d^4x d^4y d^4p_1 d^4p_2 d^4k J(p_1)\frac{e^{-i(p_1+k)x}e^{-i(p_2-k)y}}{k^2-m^2+i\epsilon}J(p_2)\frac{1}{(2\pi)^4}$$

$$= \frac{1}{2}\int d^4k (iJ(-k))\frac{i}{k^2-m^2+i\epsilon}(iJ(k))(2\pi)^4.$$



Static case :

$$J_{1,2} = \delta^3(x - x_{1,2}) \longrightarrow J_i(k) = \int \frac{d^4x}{(2\pi)^4} e^{ikx} \delta^3(x - x_i) = \int \frac{dt_i}{(2\pi)^4} e^{ik_0 t_i}$$

$$\frac{1}{2}\int d^4k (iJ(-k)) \frac{i}{k^2 - m^2 + i\epsilon} (iJ(k)) (2\pi)^4$$

$$= -\boxed{2} \times \frac{i}{2} \int dk^0 d^3k dx^0 dy^0 \frac{e^{-ik^0(x^0 - y^0)}}{(2\pi)^4} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{1}{k^2 - m^2 + i\epsilon}$$

$\boxed{J_1 J_2, J_2 J_1}$

$$= i \int dx^0 \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{\mathbf{k}^2 + m^2} \equiv -iVt$$

$$V = -\frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} e^{-m|\mathbf{x}-\mathbf{y}|}$$

Yukawa potential

As m becomes heavier, interaction becomes short-range (negligible for $r > 1/m$)

Correlation functions

First, observe that $$\langle 0|T(\phi(x)\phi(y))|0\rangle = i\Delta_F(x-y) = \frac{1}{i^2}\frac{\delta^2}{\delta J(x)\delta J(y)}Z_0[J]\Big|_{J=0}$$

Indeed,

$$\begin{aligned}\langle 0|T(\phi(x)\phi(y))|0\rangle &\overset{x^0>y^0}{=} \langle 0|\phi_H(x)\phi_H(y)|0\rangle \\ &= \langle 0|e^{-iH(T-x^0)}\phi(0,\mathbf{x})e^{-H(x^0-y^0)}\phi(0,\mathbf{y})e^{-iH(y^0+T)}|0\rangle \\ &= \frac{\int \mathcal{D}\phi\phi(x)\phi(y)e^{iS[J]}\Big|_{J=0}}{\int \mathcal{D}\phi e^{iS[J=0]}} = \frac{1}{i^2}\frac{\delta^2}{\delta J(x)\delta J(y)}Z_0[J]\Big|_{J=0}\end{aligned}$$

considering the case of $x^0 < y^0$ in the same way, one finds

$$\langle 0|T(\phi(x)\phi(y))|0\rangle = \frac{\int \mathcal{D}\phi\phi(x)\phi(y)e^{iS[J]}\Big|_{J=0}}{\int \mathcal{D}\phi e^{iS[J=0]}} = \frac{1}{i^2}\frac{\delta^2}{\delta J(x)\delta J(y)}Z_0[J]\Big|_{J=0}$$

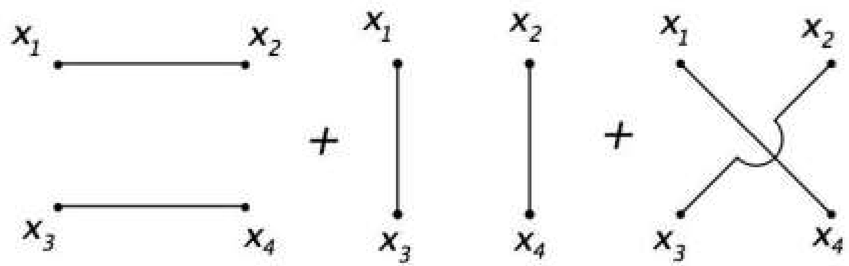
(in the presence of interaction, 2-point correlator defined in the same way is not just a Feynman propagator)

In general,

$$\langle 0|T(\phi(x_1)\cdots\phi(x_n))|0\rangle = \frac{1}{i^n}\frac{\delta^n}{\delta J(x_1)\cdots\delta J(x_n)}Z_0[J]\Big|_{J=0} = G^{(n)}(x_1,\cdots x_n)$$

Diagrammatically, for the free field, this tells us all the possible ways that particles pass through
In the presence of interaction, this would include the connected part.

Example: $\boldsymbol{n = 4}$



1. Self-interacting real scalar (ϕ^4 theory)

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}} = \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2\right] - \frac{\lambda}{4!}\phi^4$$

$$Z[J] = \frac{\exp\left[\mathrm{i}\int \mathscr{L}_{\text{int}}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(z)}\right)\mathrm{d}z\right]\exp\left[-\frac{\mathrm{i}}{2}\int J(x)\Delta_{\mathrm{F}}(x-y)J(y)\,\mathrm{d}x\,\mathrm{d}y\right]}{\left\{\exp\left[\mathrm{i}\int \mathscr{L}_{\text{int}}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(z)}\right)\mathrm{d}z\right]\exp\left[-\frac{\mathrm{i}}{2}\int J(x)\Delta_{\mathrm{F}}(x-y)J(y)\,\mathrm{d}x\,\mathrm{d}y\right]\right\}\Big|_{J=0}}.$$

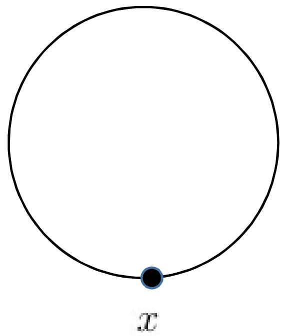
$$e^{i\int d^4x\mathcal{L}_{\text{int}}\left(\frac{\delta}{i\delta J}\right)}Z_0[J] = \left[1 - \frac{i\lambda}{4!}\int d^4z\left(\frac{\delta}{i\delta J(z)}\right)^4 + \mathcal{O}(\lambda^2)\right]e^{-\frac{i}{2}\int d^4x J(x)\Delta_F(x-y)J(y)}$$

Perturbation theory ($\boldsymbol{\lambda \ll 1}$)

$$e^{i\int d^4x\mathcal{L}_{\rm int}\left(\frac{\delta}{i\delta J}\right)}Z_0[J] = \left[1-\frac{i\lambda}{4!}\int d^4z\left(\frac{\delta}{i\delta J(z)}\right)^4+\mathcal{O}(\lambda^2)\right]e^{-\frac{i}{2}\int d^4xJ(x)\Delta_F(x-y)J(y)}$$

Diagrammatically, $x \;\text{———}\; y \rightarrow \Delta_{\mathrm{F}}(x-y)$

For example,



$$\Delta_F(x-x)=\Delta_F(0)$$

$$\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(z)}\right)^4 \exp\left(-\frac{\mathrm{i}}{2}\int J\Delta_F J\right) = \{-3 \bigcirc\!\!\bigcirc + 6\mathrm{i} \times\!\!\!-\!\!\overset{\bigcirc}{-}\!\!-\!\!\!\times + \times\!\!\!\!\times \} \exp\left(-\frac{\mathrm{i}}{2}\int J\Delta_F J\right)$$

$$\left[\exp\left(\mathrm{i}\int \mathscr{L}_{\mathrm{int}}\right)\exp\left(-\frac{\mathrm{i}}{2}\int J\Delta_F J\right)\right]\Bigg|_{J=0} = 1 - \frac{\mathrm{i}\lambda}{4!}\int (-3 \bigcirc\!\!\bigcirc)\,\mathrm{d}z$$

$$Z[J] = \frac{\left[1 - \frac{\mathrm{i}\lambda}{4!}\int\left(\cancel{-3 \bigcirc\!\!\bigcirc} + 6\mathrm{i} \times\!\!\!-\!\!\overset{\bigcirc}{-}\!\!-\!\!\!\times + \times\!\!\!\!\times\right)\mathrm{d}z\right]\exp\left(-\frac{\mathrm{i}}{2}\int J\Delta_F J\right)}{1 - \frac{\mathrm{i}g}{4!}\int \cancel{(-3 \bigcirc\!\!\bigcirc)}\,\mathrm{d}z}$$

$$= \left[1 - \frac{\mathrm{i}\lambda}{4!}\int\left(6\mathrm{i} \times\!\!\!-\!\!\overset{\bigcirc}{-}\!\!-\!\!\!\times + \times\!\!\!\!\times\right)\mathrm{d}z\right]\exp\left(-\frac{\mathrm{i}}{2}\int J\Delta_F J\right),$$

Correlation functions

A. 2-point correlator

$$\langle 0|T(\phi(x_1)\phi(x_2))|0\rangle = \frac{1}{i^2}\frac{\delta^2}{\delta J(x_1)\delta J(x_2)} Z_0[J]\Big|_{J=0}$$

$$= \frac{1}{i^2}\frac{\delta^2}{\delta J(x_1)\delta J(x_2)}\left(1 - \frac{i\lambda}{4!}\int d^4 z\right.$$  $$\left.\right) e^{-\frac{i}{2}J\cdot\Delta_F\cdot J}$$

$$= i\Delta_F(x_1 - x_2) - \frac{\lambda}{2}\Delta_F(0)\int d^4 z \Delta_F(z - x_1)\Delta_F(z - x_2) + \mathcal{O}(\lambda^2)$$

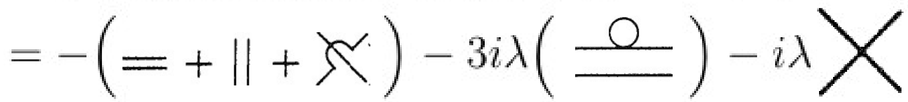
$$= i$$ $$-\frac{\lambda}{2}$$  $$+ \cdots$$

$$
\begin{aligned}
&= \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ik(x-y)}}{k^2 - m^2 + i\epsilon} - \frac{\lambda}{2}\Delta_F(0) \int d^4z \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{e^{-ip(x_1-z)}}{p^2 - m^2 + i\epsilon} \frac{e^{-ip(x_2-z)}}{q^2 - m^2 + i\epsilon} + \cdots \\
&= \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ik(x-y)}}{k^2 - m^2 + i\epsilon} \left(1 + \frac{i\lambda}{2} \frac{\Delta_F(0)}{k^2 - m^2 + i\epsilon}\right) + \cdots \\
&= \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ik(x-y)}}{k^2 - m^2 + i\epsilon} \left(1 - \frac{i\lambda}{2} \frac{\Delta_F(0)}{k^2 - m^2 + i\epsilon}\right)^{-1} + \cdots \\
&= \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ik(x-y)}}{k^2 - m_r^2 + i\epsilon}
\end{aligned}
$$

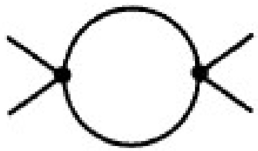
$$m_r^2 = m^2 + \delta m^2 = m^2 + \frac{i\lambda}{2}\Delta_F(0)$$ **: mass renormalization**

B. 4-point correlator

$$\langle 0|T(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4))|0\rangle = \frac{1}{i^4}\frac{\delta^2}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} Z_0[J]\Big|_{J=0}$$



coupling constant λ is also renormalized (gets quantum corrections) by higher order contributions, e.g.,



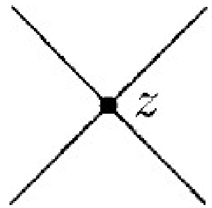
Each term in the perturbative expansion corresponds to the ‘Feynman diagram’ :

‘Feynman rule’:

1. For each propagator,

$$= \Delta_F(x-y)$$

2. For each vertex,



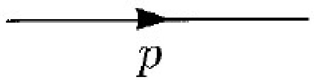
$$= -i\lambda \int d^4z$$

3. For each external point,

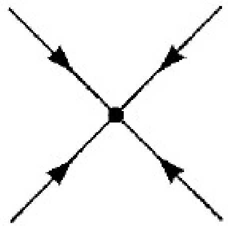
$$= 1$$

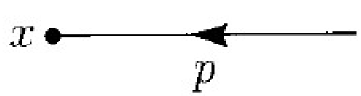
4. Divide by the symmetry factor.

Momentum space Feynman rule :

1. For each propagator, 

$$= \frac{i}{p^2 - m^2 + i\epsilon};$$

2. For each vertex, 
$$= -i\lambda;$$

3. For each external point, 

$$= e^{-ip\cdot x};$$

4. Impose momentum conservation at each vertex;

5. Integrate over each undetermined momentum: $\displaystyle\int \frac{d^4p}{(2\pi)^4};$

6. Divide by the symmetry factor.

More on 2-point correlator

: How time-ordered correlators are connected to the S-matrix elements?

Green's function $$\langle 0|T(\phi(x_1)\cdots\phi(x_n))|0\rangle = \frac{1}{i^n}\frac{\delta^n}{\delta J(x_1)\cdots\delta J(x_n)}Z_0[J]\Big|_{J=0} = G^{(n)}(x_1,\cdots x_n)$$

$$Z[J] = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int d^4x_1\cdots d^4x_n G^n(x_1,\cdots x_n)J(x_1)\cdots J(x_n)$$

'$Z[J]$ is a generator of Green's functions'

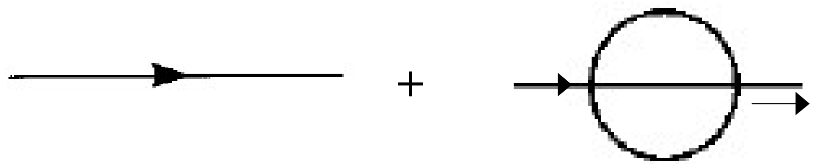
$$S = \lim_{T\to\infty} U(T,-T)$$

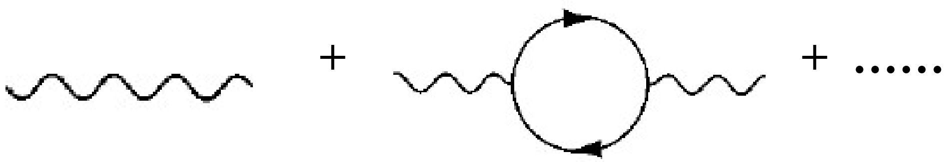
Time ordering of Greens' function looks similar to the S-matrix element

$$\langle k_1'\cdots,k_M'|S|k_1,\cdots,k_M\rangle = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int d^4x_1\cdots d^4x_n\langle k_1'\cdots,k_M'|T(\mathcal{H}_{\rm int}(x_1)\cdots\mathcal{H}_{\rm int}(x_n))|k_1,\cdots,k_M\rangle$$

However, in the presence of interaction, obtaining S-matrix from $Z[J]$ is not straightforward. Then how time-ordered correlators are connected to the S-matrix elements?

: particle not just propagates like 

but 

For photon, 

Particle does not propagate gently, but can be converted into various intermediate quantum states

($\Delta E \, \Delta t \geq \hbar/2$)

Kallen-Lehmann spectral decomposition:

$$\langle 0|\phi(x)\phi(0)|0\rangle = \int d^4p \sum_\alpha \langle 0|\phi(x)|\alpha\rangle\langle\alpha|\phi(0)|0\rangle = \int d^4p \sum_\alpha e^{-iP_\alpha x}|\langle 0|\phi(0)|\alpha\rangle|^2 \quad \longleftarrow \quad \phi(x) = e^{iPx}\phi(0)e^{-iPx}$$

$$= \int d^4p e^{-ipx}\sum_\alpha \delta^4(p-p_\alpha)|\langle 0|\phi(0)|\alpha\rangle|^2 \quad \longleftarrow \quad \sum_\alpha \delta^4(p-p_\alpha)|\langle 0|\phi(0)|\alpha\rangle|^2 \equiv \frac{1}{(2\pi)^3}\theta(p_0)\rho(p^2)$$

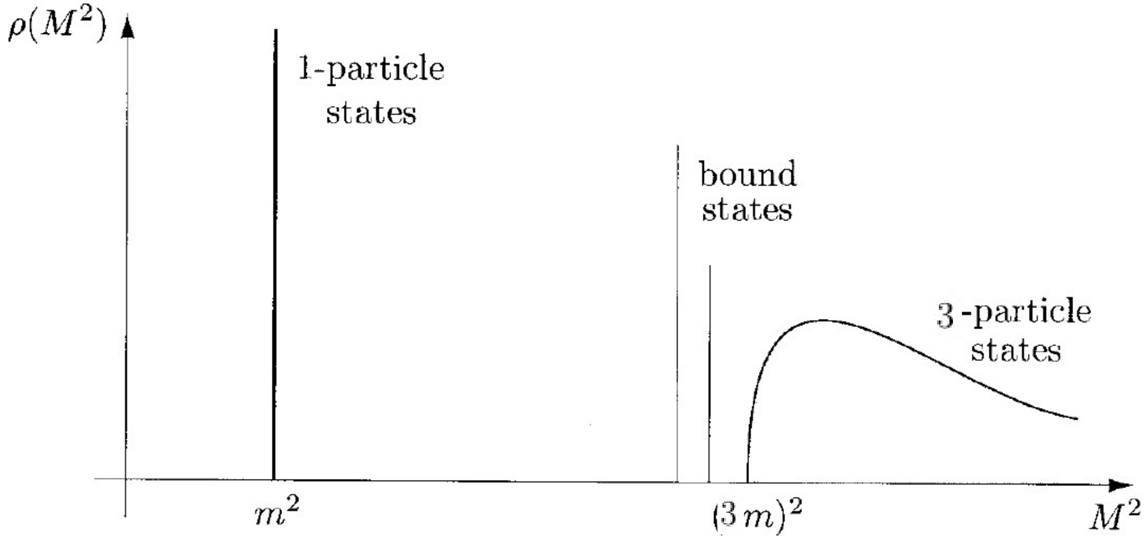
$$\equiv \int_0^\infty d\mu^2 \rho(\mu^2)\Delta_+(x;\mu) \quad \longleftarrow \quad \Delta_+(x;\mu) = \int \frac{d^4p}{(2\pi)^4}\theta(p_0)\delta(p^2-\mu^2)e^{-ipx} = \int \frac{d^3p}{(2\pi)^3 2E(p)}e^{-ipx}$$

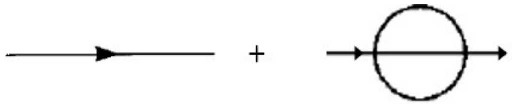
single particle part :

$$\frac{1}{(2\pi)^3}\theta(p_0)\rho_{1-\mathrm{part}}(p^2) = \int d^3k|\langle 0|\phi(0)|k\rangle|^2\delta^4(p-k) = \frac{Z}{(2\pi)^3 2E(p)}\delta(p_0-E(p))\theta(p_0)$$

$Z = |\langle 0|\phi(0)|k\rangle|^2$ **: probability for ϕ to excite 1-particle state from vacuum**

(In fact, generic quantum operator can excite the state of particle(s) from vacuum e.g., hadron)





$$\int d^4x e^{ikx}\langle 0|\phi(x)\phi(0)|0\rangle = \frac{i\,Z}{k^2-m^2+i\epsilon} + \int_{(3m)^2}^{\infty} d\mu^2\rho(\mu^2)\frac{i}{k^2-\mu^2+i\epsilon}$$

Lehmann-Symanzik-Zimmermann (LSZ) reduction

:Relation between Green's functions and S-matrix elements

$$\int d^4x e^{ipx}\langle 0|T(\phi(x)\phi(y_1)\cdots)|0\rangle = \int_{x^0>T_+} d^4x e^{ipx}\langle 0|\phi(x)T(\phi(y_1)\cdots)|0\rangle$$

$x^0 > T_+ \equiv \max(y_i^0)$

ϕ excites 'out' state

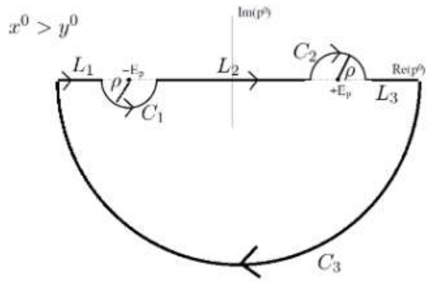
$$= \int\sum_\alpha \int_{x^0>T_+} d^4x e^{ipx}\langle 0|\phi(x)|\alpha\rangle\langle\alpha|T(\phi(y_1)\cdots)|0\rangle$$

$\langle 0|\phi(x)|\alpha\rangle = e^{-ip_\alpha x}\langle 0|\phi(0)|\alpha\rangle$

$$= \int_{x^0>T_+} d^4x \sum_\alpha \int \frac{d^3p}{(2\pi)^3 2E_\alpha} e^{i(p-p_\alpha)x}\langle 0|\phi(0)|\alpha\rangle\langle\alpha|T(\phi(y_1)\cdots)|0\rangle$$

$$= \sum_\alpha \int \frac{d^3p}{(2\pi)^3 2E_\alpha}(2\pi)^3\delta^3(p-p_\alpha)\frac{ie^{i(p_0-E_\alpha)T_+}}{p_0-E_\alpha+i\epsilon}\langle 0|\phi(0)|\alpha\rangle\langle\alpha|T(\phi(y_1)\cdots)|0\rangle$$

$$= \sum_\alpha \frac{i}{p^2-\mu_\alpha^2+i\epsilon}\sqrt{Z_\alpha}\langle\alpha|T(\phi(y_1)\cdots)|0\rangle$$



Green's function is not directly connected to the S-matrix element as it excites various states α from vacuum : we need a projection to one particle state

Projection to one-particle state:

$$\left(\frac{p^2-m^2+i\epsilon}{i}\right)\sum_{\alpha}\frac{i}{p^2-\mu_\alpha^2+i\epsilon} = \left(\frac{p^2-m^2+i\epsilon}{i}\right)\left[\frac{i}{p^2-m^2+i\epsilon} + \frac{i}{p^2-\mu_\alpha^2+i\epsilon}\bigg|_{\mu_\alpha^2>(3m)^2} + \cdots\right]$$

$$\xrightarrow[p^2=m^2]{} \quad = \left[1+0\right]$$

Thus, LSZ reduction shows that

$$\langle 0|T(\phi(x_1)\cdots)|0\rangle \xrightarrow{} \sqrt{Z}\langle p|\cdots|0\rangle$$

$$\uparrow$$

$$\frac{p^2-m^2+i\epsilon}{i}$$

Green's function

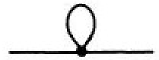
S-matrix element

$$\langle p|\cdots|0\rangle$$

Wavefunction renormalization : $\phi_r(x) = \sqrt{Z}\phi(x)$

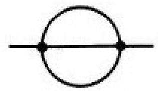
Mass renormalization

$$\Gamma_b^{(2)}(0) = \begin{cases} -m_0^2 \\ -\dfrac{1}{2}\dfrac{g_0}{(4\pi)^2}\left[\Lambda^2 - m_0^2 \ln\dfrac{\Lambda^2}{m_0^2} - m_0^2(1-\gamma)\right] \end{cases}$$



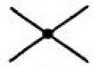
Wavefunction renormalization

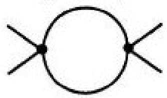
$$\frac{d\Gamma_b^{(2)}}{dp^2}(0) = \begin{cases} -1 \\ -\dfrac{1}{12}\dfrac{g_0^2}{(4\pi)^4}\left(\ln\dfrac{\Lambda^2}{m_0^2} + \text{constant}\right) \end{cases}$$

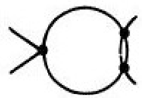


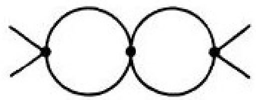
Coupling renormalization

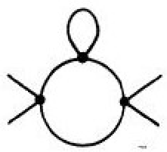
$$\Gamma_b^{(4)}(0) = \begin{cases} -g_0 \\ +\dfrac{3}{2}\dfrac{g_0^2}{(4\pi)^2}\left(\ln\dfrac{\Lambda^2}{m_0^2} - \gamma - 1 - \ln 2\right) \\ -3\dfrac{g_0^3}{(4\pi)^4}\left[\dfrac{1}{2}\left(\ln\dfrac{\Lambda^2}{m_0^2}\right)^2 - \ln\dfrac{\Lambda^2}{m_0^2}(\gamma + \ln 2) + \text{constant}\right] \\ -\dfrac{3}{4}\dfrac{g_0^3}{(4\pi)^4}\left[\left(\ln\dfrac{\Lambda^2}{m_0^2}\right)^2 - 2(\gamma + 1 + \ln 2)\ln\dfrac{\Lambda^2}{m_0^2} \right. \\ \left. + (\gamma + 1 + \ln 2)^2\right] \\ -\dfrac{3}{4}\dfrac{g_0^3}{(4\pi)^4}\left(\dfrac{\Lambda^2}{m_0^2} - \ln\dfrac{\Lambda^2}{m_0^2} + \gamma - 1\right) \end{cases}$$











We have seen that physical quantities we can measure is a result of summing over all possible quantum corrections.

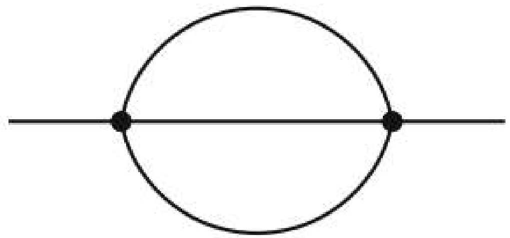
Quantum nature of corrections :

Loop number = the number of independent internal momenta (all possible intermediate states : will be integrated out) after imposing delta function (energy-momentum conservation at each vertex)

I : the number of internal lines

V: the number of vertices

L : the number of loops



Then $L = I - (V - 1)$

Example: $I = 3, V = 2, L = 2$

On the other hand,

$$\frac{S}{\hbar} = \int d^4x \Big[-\frac{1}{2}\phi \underbrace{\frac{\partial_\mu \partial^\mu + m^2}{\hbar}}_{\textbf{propagator } \sim\hbar} \phi - \frac{1}{4!} \underbrace{\frac{\lambda}{\hbar}}_{\textbf{vertex } \sim 1/\hbar} \phi^2 \Big]$$

Thus,

$$(\text{the power of } \hbar) = I - V = L + 1$$

Structure of $Z[J]$

Recommended readings : S. Coleman, E. Weinberg, Phys Rev. D7 (1973) 1888

E. Weinberg, hep-th/0507214

$$Z[J] = \int \mathcal{D}\phi e^{iS+i\int d^4x J(x)\phi(x)} \equiv e^{iW[J]} \qquad W = -i\log Z$$

"partition function"

generating functional of Green's function

$$Z = \sum_n \frac{(-i)^n}{n!} G^{(n)} J^n$$

"free energy"

generating functional of connected Green's function

$$W = \sum_n \frac{(-i)^n}{n!} G_c^{(n)} J^n$$

$$i\frac{\delta W[J]}{\delta J} = \frac{1}{Z[J]}\frac{\delta Z[J]}{\delta J} = \frac{\int \mathcal{D}\phi \phi e^{iS[J]}}{\int \mathcal{D}\phi e^{iS[J]}} = \langle \phi \rangle_J \equiv \phi_c$$

: expectation value of ϕ over all possible quantum fluctuations

Describing dynamics in terms of ϕ_c , instead of J ?

: Legendre transformation

$$\Gamma[\phi_c] = iW[J] - \int d^4x \phi_c(x) J(x)$$ **quantum effective action**

$$\frac{\delta \Gamma}{\delta \phi} = i\cancel{\frac{\delta J}{\delta \phi_c}\frac{\delta W}{\delta J}} - \left(J + \cancel{\phi_c\frac{\delta J}{\delta \phi_c}}\right) = -J$$

$$i\frac{\delta W}{\delta J} = \phi_c$$

Let

$$\Gamma[\phi_c] = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \cdots d^4x_n \Gamma^{(n)}(x_1, \cdots x_n) \phi_c(x_1) \cdots \phi_c(x_n)$$

The meaning of $\Gamma^{(n)}(x_1, \cdots x_n)$ **?**

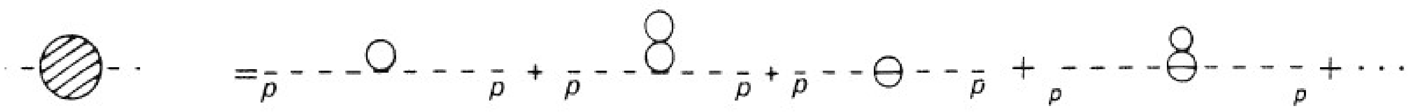
Example :

$$\begin{aligned}
\delta^4(x-y) &= \frac{\delta\phi_c(x)}{\delta\phi_c(y)} = \frac{\delta}{\delta\phi_c} i \frac{\delta W[J]}{\delta J} \\
&= i \int d^4z \frac{\delta J(z)}{\delta\phi_c(y)} \frac{\delta^2 W[J]}{\delta J(z)\delta J(x)} = -i \int d^4z \Big(\frac{\delta^2\Gamma}{\delta\phi_c(y)\delta\phi_c(z)}\Big)\Big(\frac{\delta^2 W}{\delta J(z)\delta J(x)}\Big) \\
&= -i \int d^4z \Gamma^{(2)}(y,z) G_c^{(2)}(z,x)
\end{aligned}$$

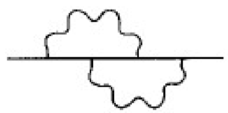
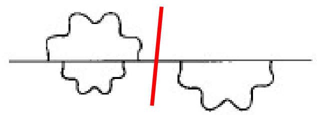
for $\boldsymbol{\phi_c, J = 0}$

$$\Gamma^{(2)}(p) G_c^{(2)}(p) = i$$

$G_c^{(2)}(p) =$ 

Here, 

one-particle-irreducible (1PI) : diagram that cannot be split in two by removing a single line

 is 1PI, while  is not.

$$G_c^{(2)}(p) = G_0(p) + G_0(p)\frac{\Sigma(p)}{\mathrm{i}}G_0(p) + G_0(p)\frac{\Sigma(p)}{\mathrm{i}}G_0(p)\frac{\Sigma(p)}{\mathrm{i}}G_0(p) + \cdots$$

$$= \frac{\mathrm{i}}{p^2 - m^2 - \Sigma(p)}$$

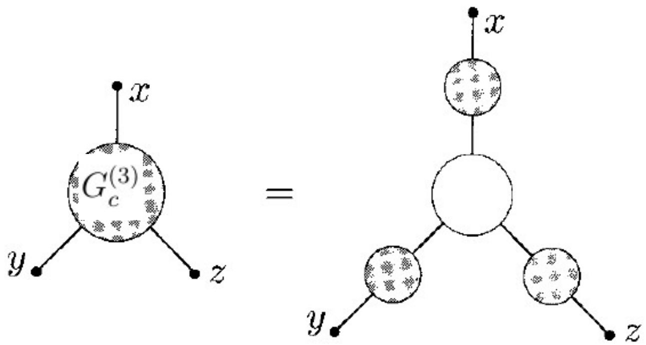
Comparing two results, one finds that

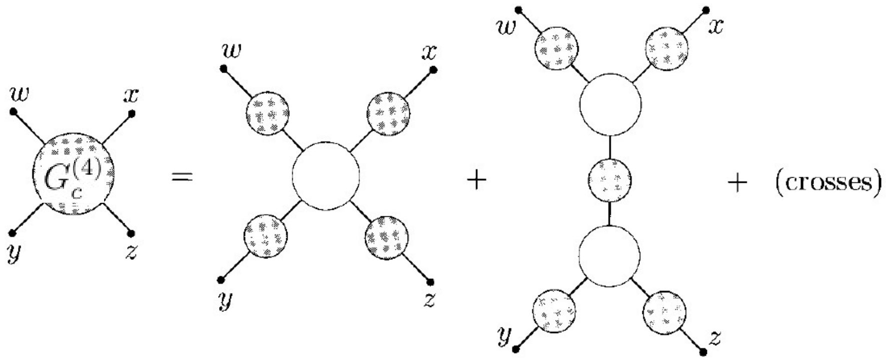
$$\Gamma^{(2)}(p) = p^2 - m^2 - \Gamma(p)$$

In the same way, $\Gamma^{(n)}(x_1, \cdots x_n)$ **is in fact the sum of 1PI diagrams**

i.e., $\Gamma[\phi_c]$ **: generating functional of 1PIs.**

1PI is interpreted as "quantum corrected vertex" of connected Green's function :





Wilsonian picture

: when some part of the system is not visible, how the invisible sector affects dynamics of the visible sector?

Let $\boldsymbol{\phi_V}$ **: visible sector field e.g., light degrees of freedom** $(\boldsymbol{\phi_{|k|<\Lambda}})$

$\boldsymbol{\phi_I}$ **: invisible sector field e.g., heavy degrees of freedom** $(\boldsymbol{\phi_{|k|>\Lambda}})$

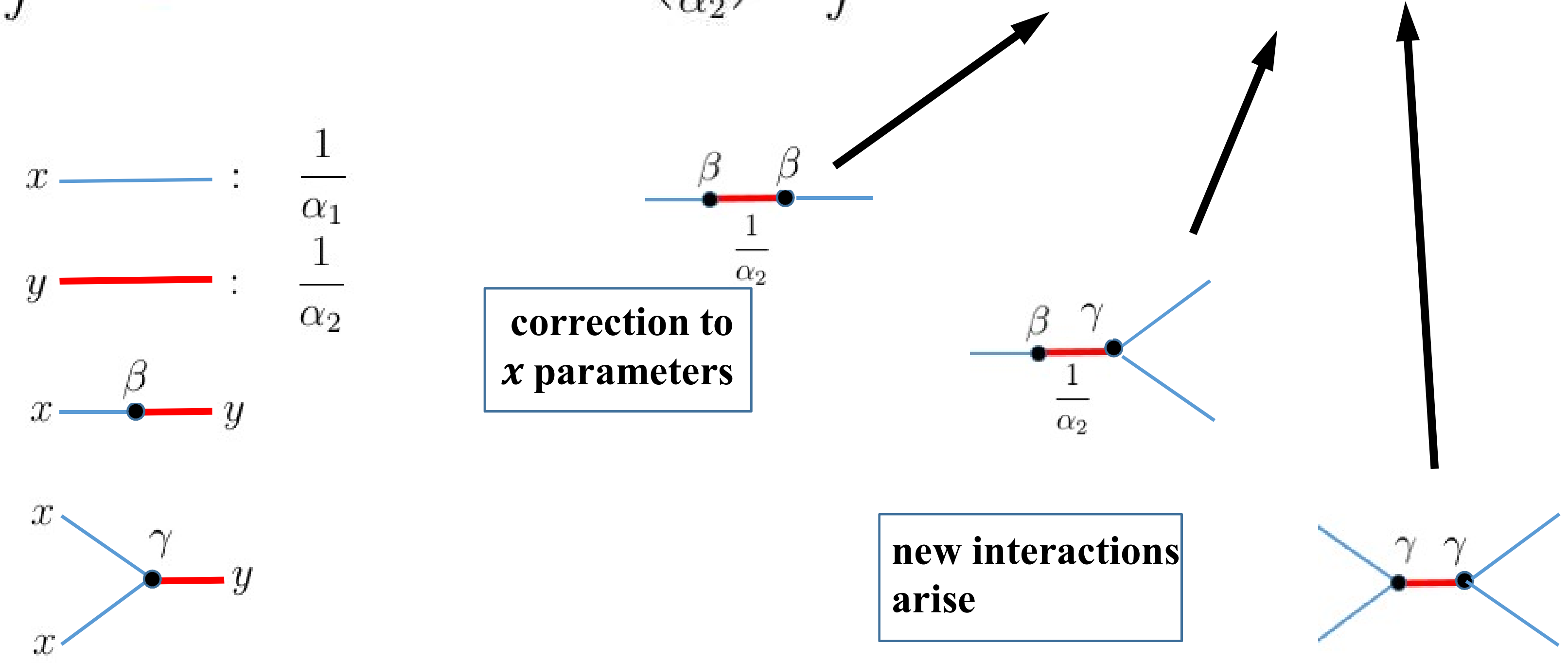
$$
\begin{aligned}
Z[J] &= \int \mathcal{D}\phi_V \mathcal{D}\phi_I e^{iS[\phi_V,\phi_I]+i\int d^4x \phi_V J} \\
&= \int \mathcal{D}\phi_V e^{iS_{\text{eff}}[\phi_V]+i\int d^4x \phi_V J}
\end{aligned}
$$

integrating out the invisible sector fields $\boldsymbol{\phi_I}$

$\boldsymbol{S_{eff}}$ **: Wilsonian effective action**

Illustration :

$$\int dx\,dy\,e^{-\alpha_1 x^2-\alpha_2 y^2+\beta x y+\gamma x^2 y} = \left(\frac{\pi}{\alpha_2}\right)^{1/2} \int dx\,e^{-\left(\alpha_1+\underline{\frac{\beta^2}{4\alpha_2}}\right)x^2+\underline{\frac{\beta\gamma}{2\alpha_2}}x^3+\underline{\frac{\gamma^2}{4\alpha_2}}x^4}$$



If y is so heavy that high energy is required for excitation, red line looks like a point

Integrating out heavy fields in ϕ^4 theory

(following Sec.12.1 of M. E. Peskin, D. V. Schroeder, An introduction to quantum field theory see also N. Goldenfeld, Lectures on phase transition and the renormalization group)

$$Z = \int [\mathcal{D}\phi]_\Lambda \, \exp\Big(-\int d^dx \Big[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\Big]\Big)$$

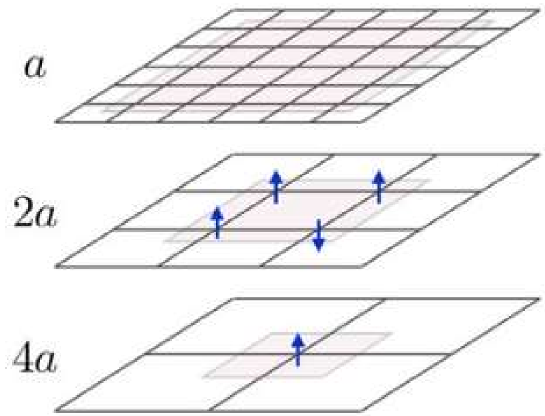
$$[\mathcal{D}\phi]_\Lambda = \prod_{|k|<\Lambda} d\phi(k)$$

$$= \int \mathcal{D}\phi \int \mathcal{D}\hat{\phi} \, \exp\Big(-\int d^dx \Big[\frac{1}{2}(\partial_\mu\phi + \partial_\mu\hat{\phi})^2 + \frac{1}{2}m^2(\phi+\hat{\phi})^2 + \frac{\lambda}{4!}(\phi+\hat{\phi})^4\Big]\Big)$$

$$= \int [\mathcal{D}\phi]_\Lambda \, \exp\Big(-\int d^dx \, \mathcal{L}_{\text{eff}}\Big)$$

Now let us lower the cutoff from Λ to bΛ ($b < 1$) : additional integrating out for modes between Λ and bΛ

After rescaling $k' = k/b, \qquad x' = xb,$



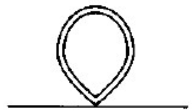
coarse-graining

$$\int d^d x\, \mathcal{L}_{\mathrm{eff}} = \int d^d x'\, b^{-d} \Big[\frac{1}{2}(1+\Delta Z) b^2 (\partial'_\mu \phi)^2 + \frac{1}{2}(m^2 + \Delta m^2)\phi^2 + \frac{1}{4}(\lambda + \Delta\lambda)\phi^4 + \Delta C b^4 (\partial'_\mu \phi)^4 + \Delta D \phi^6 + \cdots \Big].$$

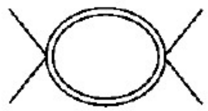
$$= \int d^d x' \Big[\frac{1}{2}(\partial'_\mu \phi')^2 + \frac{1}{2} m'^2 \phi'^2 + \frac{1}{4}\lambda' \phi'^4 + C' (\partial'_\mu \phi')^4 + D' \phi'^6 + \cdots \Big].$$

$$\phi' = \left[b^{2-d}(1+\Delta Z) \right]^{1/2} \phi$$

$$m'^2 = (m^2 + \Delta m^2)(1+\Delta Z)^{-1} b^{-2},$$

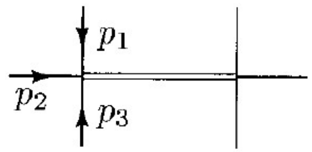


$$\lambda' = (\lambda + \Delta\lambda)(1+\Delta Z)^{-2} b^{d-4},$$



$$C' = (C + \Delta C)(1+\Delta Z)^{-2} b^{d},$$

$$D' = (D + \Delta D)(1+\Delta Z)^{-3} b^{2d-6},$$



From the rescaling, we can compare parameters in effective Lagrangians of the same form, which are defined in the cutoff scale Λ and $\Lambda - d\Lambda$, respectively :

e.g., comparing the correlation length 'in unit of 1/(cutoff)'

Regardless of the actual 'fundamental' cutoff scale (we will call Λ), parameters are fixed by measurement at 'probing' cutoff (we will call μ): once the parameter is fixed, physics is insensitive to details of UV completion.

(different initial conditions, different cutoff, but the same result)

This is a result of tuning between quantum corrections to the parameter.

insensitivity to UV completion :

$$\Lambda \frac{d}{d\Lambda} Z[J] = 0 \qquad \text{or} \qquad \Lambda \frac{d}{d\Lambda} W[J] = 0$$

$$\Lambda \frac{d}{d\Lambda} \langle 0|\phi(x_1)\cdots\phi(x_n)|0\rangle_c = 0$$

$$\Downarrow \qquad \phi_r(x) = \sqrt{Z}\phi(x)$$

$$\Lambda \frac{d}{d\Lambda}\left[Z^{n/2}\langle 0|\phi_r(x_1)\cdots\phi_r(x_n)|0\rangle_c\right] = \Lambda \frac{d}{d\Lambda}\left[Z^{n/2} G_{c,R}^{(n)}(x_1,\cdots x_n;\Lambda,\lambda(\Lambda))\right] = 0$$

$$0 = \left(n\frac{\Lambda}{2Z}\frac{dZ}{d\Lambda} + \Lambda\frac{\partial}{\partial\Lambda} + \Lambda\frac{d\lambda}{d\Lambda}\frac{\partial}{\partial\lambda}\right) G_{c,R}^{(n)}(x_1,\cdots x_n;\Lambda,\lambda(\Lambda))$$

$$\gamma \equiv -\frac{\Lambda}{2Z}\frac{dZ}{d\Lambda} \qquad \beta \equiv -\Lambda\frac{d\lambda}{d\Lambda} \qquad \Lambda\frac{d}{d\Lambda} = \frac{d}{d\log(\Lambda/\mu)}$$

μ : probing scale
varying Λ, μ fixed =
varying μ, Λ fixed

$$\left[\Lambda\frac{\partial}{\partial\Lambda} - \beta\frac{\partial}{\partial\lambda} - n\gamma\right]\frac{\partial}{\partial\lambda} G_{c,R}^{(n)}(x_1,\cdots x_n;\Lambda,\lambda(\Lambda)) = 0$$

Callan-Symanzik equation
(renormalization group equation)
RGE

Remarks:

1. Old-fashioned renormalization : UV cutoff $\rightarrow \infty$ then some artificial scale comes in.

RGE can be obtained from the condition that physics is independent of this artificial scale

See, e.g., D. J. Gross in Les Houches 1975, Methods of field theory

2. Dynamical variable can take different form after integrating out invisible sector

Example : strong interaction

$$\int \mathcal{D}A^a_\mu \mathcal{D}q e^{iA^a_\mu, q} = \int \mathcal{D}\mathcal{O} e^{iS_{\rm eff}[\mathcal{O}]}$$

$\boldsymbol{O}$: color singlet operators excite hadron states from vacuum

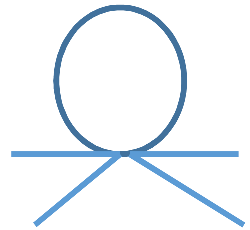
Recommended reading : J. Polchinski, Nucl. Phys. B 231 (1984) 269, hep-th/9210046

$$
\begin{aligned}
m'^2 &= (m^2 + \Delta m^2)(1+\Delta Z)^{-1} b^{-2}, \\
\lambda' &= (\lambda + \Delta\lambda)(1+\Delta Z)^{-2} b^{d-4}, \\
C' &= (C + \Delta C)(1+\Delta Z)^{-2} b^{d}, \\
D' &= (D + \Delta D)(1+\Delta Z)^{-3} b^{2d-6},
\end{aligned}
$$

Since $\boldsymbol{b < 1}$, for $\boldsymbol{d = 4}$, $m'^2 \gg m^2, \quad \lambda' \simeq \lambda. \quad C' \ll C, \quad D' \ll D$

Relevant **Marginal** **Irrelevant**

As probing scale becomes 0, irrelevant operators are suppressed : do not play the crucial role in dynamics (In fact irrelevant operators can affect relevant/marginal operators through the loop…)



We have a theory consists of relevant / marginal operators only

$$S = \int d^d x \Big(\frac{1}{2} (\partial_\mu \phi)^2 + \sum_{n \geq 0} a_n \phi^{2+n} + \sum_{n > 0} a'_n (\partial_\mu \phi)^2 \phi^n + \cdots \Big)$$

$$\equiv \int d^d x \Big(\frac{1}{2} (\partial_\mu \phi)^2 + \sum_{n \geq 0} a_n \mathcal{O}_n + \sum_{n > 0} a'_n \mathcal{O}'_n + \cdots \Big)$$

Engineering dimensions :

$$\dim(\phi) = \frac{d}{2} - 1 \quad \dim(\mathcal{O}_n) = (n+2)\big(\frac{d}{2} - 1\big), \quad \dim(\mathcal{O}'_n) = n\big(\frac{d}{2} - 1\big) + d$$

$$\dim(a_n) = 2 + n\big(1 - \frac{d}{2}\big), \qquad \dim(a'_n) = n\big(1 - \frac{d}{2}\big)$$

Dimension of 1PI graph

1. **Let E : the number of external legs**

$$d^d\delta(\sum_i k_i)\Gamma(k_1,\cdots,k_E) \sim \prod_{i=1}^{E}(k_i^2+m^2)\int d^dx_1\cdots d^dx_E\langle 0|T\phi_1\cdots\phi_n|0\rangle$$

$$-d+\dim(\Gamma) \quad = \quad 2E-dE+E(\frac{d}{2}-1)$$

$$\dim(\Gamma)=d+E(1-\frac{d}{2})$$

Dimension of 1PI graph

2. It consists of L loops, N_i vertices of O_i, N'_i vertices of O'_i

$$\Gamma \sim a_1^{N_1} a_2^{N_2} \cdots (a'_1)^{N'_1} (a'_2)^{N'_2} \cdots \int (d^d k)^L k^{D-dL}$$

D : superficial degree of divergence $\Gamma \sim \Lambda^D$

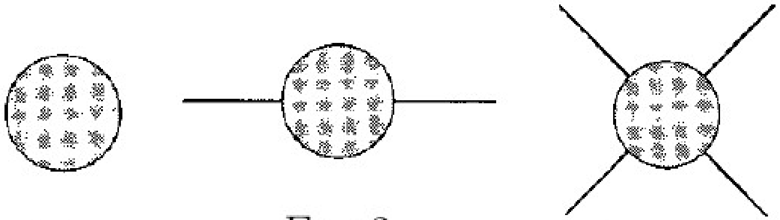
$$\dim(\Gamma) = \sum_n N_n \Big(2 + n\big(1 - \frac{d}{2}\big)\Big) + N'_n n\big(1 - \frac{d}{2}\big) + D$$

1=2 :

$$D = 4 - E + \sum_n \left((n-2)N_n + nN'_n + \cdots\right)$$

For ϕ^4 theory, $n = 0, 2$ without O'_n

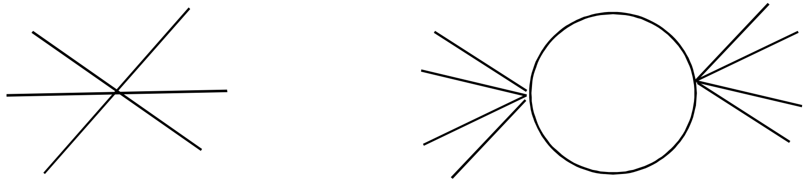
: $\boldsymbol{D = 4 - E - 2N_0}$



$E = 0$ $E = 2$ $E = 4$

For ϕ^6 theory, $n = 4$ without O'_n

: $\boldsymbol{D = 4 - E + 2N_4}$



$E = 6$ $E = 8$

incontrollable theory?

(gravity : see, e.g., T. Banks : A. Shomer, 0709.3555)