양자장론과 표준모형

QUANTUM FIELD THEORY AND STANDARD MODEL

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Spinors and Gauge Bosons

LECTURE 3

In the theory with Lorentz invariance, quantum states correspond to the irreducible representation of Lorentz group SO(3,1) : spin (intrinsic angular momentum) state

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu, \quad \eta_{\mu\nu} x^\mu x^\nu = \eta_{\mu\nu} x'^\mu x'^\nu$$

6 independent generators $J_{\mu\nu}$ satisfy the algebra

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\rho} J_{\nu\sigma} - \eta_{\nu\sigma} J_{\mu\rho} + \eta_{\mu\sigma} J_{\nu\rho})$$

$$K_i = J_{i0}, \quad J_i = \frac{1}{2}\epsilon_{ijk} J_{jk}$$

$$iK_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad iK_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad iK_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

(boosts)

$$iJ_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad iJ_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad iJ_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(rotations)

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

$$[J_i, K_j] = i\epsilon_{ijk} K_k$$

$$[K_i, K_j] = -i\epsilon_{ijk} J_k$$

cf. Thomas precession

Problem : **Lorentz group is not compact**

angle $\boldsymbol{\theta} \in [\mathbf{0}, \mathbf{2}\boldsymbol{\pi})$ but rapidity $\boldsymbol{\omega} \in (-\infty, \infty)$

$\rightarrow$ no finite dimensional unitary representation

Resolution (E. Wigner) : **Since non-compactness originates from boosts, we fix the states to some specific inertial frame and find out representations of little group, the subgroup of SO(3,1) that does not change momentum.**

1. <u>Massive particle</u> : $p^\mu = (M, \mathbf{0})$ **invariant under the spatial rotation SO(3)**

Little group = SO(3) : particle states ~ SO(3) representations (spin)

Pauli-Lubanski vector : projection of $\boldsymbol{J_{\mu\nu}}$ to spin

$$W_\sigma = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}J^{\mu\nu}p^\rho \qquad \textbf{if} \quad J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu \;, \; \boldsymbol{W_\sigma = 0}$$

For $p^\mu = (M, \mathbf{0})$ $\qquad W_0 = 0, \quad W_i = \frac{1}{2}\epsilon_{ijk}MJ^{jk} = MJ_i$

($\boldsymbol{P^2}, \boldsymbol{W^2}$) : SO(3,1) quadratic Casimir (commute with all SO(3,1) generators)

2. Massless particle : $p^\mu = (p, 0, 0, p)$

Little group = ISO(2) : 2-dim. Euclidean group

Generators:

$$J_3, \quad A = K_1 + J_2, \quad B = K_2 - J_1$$

$$[A, B] = 0, \quad [J_3, A] = iB, \quad [J_3, B] = -iA$$

$$iA = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad iB = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

A, B non-compact : fix their eigenvalues to, say, $(0, 0)$

The unitary massless particle states : SO(2) (generated by J_3 representations)

since no reference to eigenvalue raising/lowering $J_{1,2}$

we only consider whether the spin is up or down in the direction of motion, $\pm j$ regardless of j value

: Helicity $\hat{\mathbf{p}} \cdot \mathbf{S}$ $\qquad W^\mu = -(\hat{\mathbf{p}} \cdot \mathbf{S}) p^\mu$

The role of A, B ? : the case of photon (two polarizations)

$$R_3 = e^{iJ_3\theta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$e^\mu(\hat{z}, \pm 1) = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0)$$

: eigenvectors with eigenvalues $e^{\pm i\theta}$ **(spin-1)**

$$S_1 = e^{i(A\alpha + B\beta)} = \begin{pmatrix} 1 + \frac{1}{2}(\alpha^2 + \beta^2) & \alpha & \beta & -\frac{1}{2}(\alpha^2 + \beta^2) \\ \alpha & 1 & 0 & -\alpha \\ \beta & 0 & 1 & -\beta \\ \frac{1}{2}(\alpha^2 + \beta^2) & \alpha & \beta & 1 - \frac{1}{2}(\alpha^2 + \beta^2) \end{pmatrix}$$

$$S_1{}^\mu{}_\lambda R_3{}^\lambda{}_\nu e^\nu(\hat{z}, \pm 1) = e^{\pm i\theta}\Big(e^\mu(\hat{z}, \pm 1) + \frac{1}{\sqrt{2}}(\alpha \pm i\beta)\frac{k^\mu}{|\mathbf{k}|}\Big)$$

$$A_\mu \to A_\mu + \partial_\mu \Lambda$$ **gauge + dual gauge transformation**

Nevertheless, operator may belong to the non-unitary representation (another reason to consider the field 'operator')

$$[J_i, J_j] = i\epsilon_{ijk}J_k$$
$$[J_i, K_j] = i\epsilon_{ijk}K_k$$
$$[K_i, K_j] = -i\epsilon_{ijk}J_k$$

$$\boxed{J_{\pm,i} = \frac{1}{2}(J_i \mp iK_i)}$$

$$[J_{+,i}, J_{+,j}] = i\epsilon_{ijk}J_{+,k}$$
$$[J_{-,i}, J_{-,j}] = i\epsilon_{ijk}J_{-,k}$$
$$[J_{+,i}, J_{-,j}] = 0$$

two independent SU(2)s cf) SO(4)=SU(2)×SU(2) (hydrogen atom)

$$J_+ \; : \; (j^+, m^+) \qquad j^+ = 0, \frac{1}{2}, 1, \cdots ,$$
$$J_- \; : \; (j^-, m^-) \qquad j^- = 0, \frac{1}{2}, 1, \cdots$$

$$J_i = J_{+,i} + J_{-,i} \quad |j^+ - j^-| \le j \le j^+ + j^-$$

$$(j^+, j^-) = (0,0), \quad \left(\frac{1}{2}, 0\right) \quad \left(0, \frac{1}{2}\right) \quad \left(\frac{1}{2}, \frac{1}{2}\right), \quad (1,0), \quad (0,1), \cdots$$

s=0	**s=1/2**	**s=1/2**	**s=1**	**s=1**	**s=1**
	(left-handed)	**(right-handed)**	**(vector)**	**(self-dual)**	**(anti-self-dual)**

Spinor (Spin-1/2)

Recommended readings:

H. K. Dreiner, H. E. Haber, S. P. Martin, Phys. Rep. 494 (2010) 1 (0812.1594)
H. J. W. Muller-Kirsten, A. Wiedemann, Introduction to supersymmetry
R. Ticciati, Quantum field theory for Mathematicians
A. Zee, Group theory in a nutshell for physicists

1. Left-handed (1/2,0)

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad \psi_\alpha \qquad \alpha = 1, 2$$

$$\boxed{J_{+i} = \frac{1}{2}(J_i - iK_i) = \frac{1}{2}\sigma_i, \quad J_{-i} = \frac{1}{2}(J_i + iK_i) = 0} \Rightarrow \boxed{J_i = \frac{1}{2}\sigma_i, \quad K_i = \frac{i}{2}\sigma_i}$$

Lorentz transformation :

$$M = e^{i(\mathbf{J}\cdot\theta + \mathbf{K}\cdot\omega)} = e^{\frac{i}{2}\sigma\cdot\theta - \frac{1}{2}\sigma\cdot\omega} \qquad \psi_\alpha \rightarrow M_\alpha{}^\beta \psi_\beta$$

M **is equivalent to** $M^{-1\,T}$

Proof: $\epsilon^{\alpha\beta} = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\epsilon_{\alpha\beta} = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta^{\alpha}_{\gamma}$

then $\epsilon^{\alpha\beta} M_{\beta}{}^{\gamma} \epsilon_{\gamma\delta} = (M^{-1T})^{\alpha}{}_{\delta}$ $M^{\alpha}{}_{\beta} = \epsilon_{\alpha\gamma} (M^{-1T})^{\gamma}{}_{\delta} \epsilon^{\delta\beta}$

Thus we have another (equivalent) way to represent (1/2, 0) :

$$\psi^{\alpha} = \epsilon^{\alpha\beta}\psi_{\beta}$$

then $\psi'^{\alpha} = (M^{-1T})^{\alpha}{}_{\beta}\psi^{\beta}$ **under Lorentz transformation**

$$\psi\chi = \psi^{\alpha}\chi_{\alpha} = -\psi_{\alpha}\chi^{\alpha} = \chi^{\alpha}\psi_{\alpha} = \chi\psi$$

if ψ, χ **fermionic (Grassmannian)**

Details:

$$\overline{\psi}^{\dot{\alpha}} \to (M^{*-1T})^{\dot{\alpha}}{}_{\dot{\beta}}\overline{\psi}^{\dot{\beta}}$$

$$\psi'^{\alpha} = \epsilon^{\alpha\beta}\psi'_{\beta} = \epsilon^{\alpha\beta}M_{\beta}{}^{\gamma}\psi_{\gamma} = (\epsilon^{\alpha\beta}M_{\beta}{}^{\gamma}\epsilon_{\gamma\delta})\epsilon^{\delta\epsilon}\psi_{\epsilon} = (M^{-1T})^{\alpha}{}_{\delta}\psi^{\delta}$$

Lorentz scalar : $\overline{\psi}\overline{\chi} = \overline{\psi}_{\dot{\alpha}}\overline{\chi}^{\dot{\alpha}} = -\overline{\psi}^{\dot{\alpha}}\overline{\chi}_{\dot{\alpha}} = \overline{\chi}_{\dot{\alpha}}\overline{\psi}^{\dot{\alpha}} = \overline{\chi}\overline{\psi}$

$$\psi'^{\alpha}\chi'_{\alpha} = (M^{-1T})^{\alpha}{}_{\beta}\psi^{\beta}(M)_{\alpha}{}^{\gamma}\psi^{\gamma} = \underbrace{(M^{-1})_{\beta}{}^{\alpha}M_{\alpha}{}^{\gamma}}_{=\delta_{\beta}^{\gamma}}\psi^{\beta}\chi_{\gamma} = \psi^{\beta}\chi_{\beta}$$

2. Right-handed (0,1/2)

$$\overline{\psi} = \begin{pmatrix} \overline{\psi}^1 \\ \overline{\psi}^2 \end{pmatrix} \qquad \overline{\psi}^{\dot{\alpha}}$$

$$\boxed{J_{+i} = \frac{1}{2}(J_i - iK_i) = 0, \quad J_{-i} = \frac{1}{2}(J_i + iK_i) = \frac{1}{2}\sigma_i} \Rightarrow \boxed{J_i = \frac{1}{2}\sigma_i, \quad K_i = -\frac{i}{2}\sigma_i}$$

Why upper index? In fact,

$$\boxed{\begin{matrix}\psi_\alpha \\ \textbf{(1/2,0)}\end{matrix}} \xrightarrow[\text{Complex conjugate}]{} \boxed{\overline{\psi}_{\dot{\alpha}}} \qquad \overline{\psi}_{\dot{\alpha}} \to M^{*}{}_{\dot{\alpha}}{}^{\dot{\beta}}\overline{\psi}_{\dot{\beta}}$$

Then $\overline{\psi}_{\dot{\alpha}}$ **is equivalent to** $\overline{\psi}^{\dot{\alpha}}$ **, i.e., (0,1/2) spinor**

: They are related by $\overline{\psi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\overline{\psi}_{\dot{\beta}} (= i\sigma_2\psi^*)$

This comes from the pseudo-reality of SU(2), $\sigma_2\sigma_i^*\sigma = -\sigma_i$

$$\overline{\psi} \equiv i\sigma_2\psi^* \to i\sigma_2\left(e^{\frac{i}{2}\sigma\cdot\theta-\frac{1}{2}\sigma\cdot\omega}\psi\right)^* = i\sigma_2 e^{-\frac{i}{2}\sigma^*\cdot\theta-\frac{1}{2}\sigma^*\cdot\omega}\psi^* = e^{\frac{i}{2}\sigma\cdot\theta-\frac{1}{2}\sigma\cdot\omega}(i\sigma^*\psi^*)$$
$$= e^{i\left(\frac{1}{2}\sigma\cdot\theta-\frac{i}{2}\sigma\cdot\omega\right)}(i\sigma^*\psi^*)$$

As in the case of (1/2,0), the Lorentz transformation for the (0,1/2) spinor $\overline{\psi}^{\dot\alpha}$ **is given by**

$$\overline{\psi}^{\dot\alpha} \to (M^{*-1T})^{\dot\alpha}{}_{\dot\beta}\overline{\psi}^{\dot\beta}$$

Lorentz scalar : $\overline{\psi}\overline{\chi} = \overline{\psi}_{\dot\alpha}\overline{\chi}^{\dot\alpha} = -\overline{\psi}^{\dot\alpha}\overline{\chi}_{\dot\alpha} = \overline{\chi}_{\dot\alpha}\overline{\psi}^{\dot\alpha} = \overline{\chi}\overline{\psi}$

if ψ, χ fermionic (Grassmannian)

Summary :

$$\psi_\alpha \to M_\alpha{}^\beta \psi_\beta \qquad\qquad \psi_\alpha \quad \xleftrightarrow{\text{complex conjugate}} \quad \overline{\psi}_{\dot\alpha} \qquad\qquad \overline{\psi}'_{\dot\alpha} = M^*{}_{\dot\alpha}{}^{\dot\beta}\,\overline{\psi}_{\dot\beta}$$

$$\epsilon = i\sigma_2 \qquad\qquad\qquad\qquad \epsilon = i\sigma_2$$

$$\psi'^\alpha = (M^{-1T})^\alpha{}_\beta \psi^\beta \qquad \psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta \quad \xleftrightarrow[\text{complex conjugate}]{} \quad \overline{\psi}^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\,\overline{\psi}_{\dot\beta} \qquad \overline{\psi}'^{\dot\alpha} = (M^{*-1T})^{\dot\alpha}{}_{\dot\beta}\,\overline{\psi}^{\dot\beta}$$

$$\left(\frac{1}{2}, 0\right) \qquad\qquad\qquad\qquad \left(0, \frac{1}{2}\right)$$

Construction of Lorentz vector

$$\boxed{\sigma^{\mu}_{\alpha\dot{\alpha}} = (I, \sigma_i)} \longleftrightarrow \boxed{\overline{\sigma}^{\mu\dot{\alpha}\alpha} = (I, -\sigma_i)}$$

$$\sigma^{\mu}_{\alpha\dot{\alpha}} = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\sigma^{\mu\dot{\beta}\beta}$$

$$\sigma^{\mu\dot{\alpha}\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^{\mu}_{\beta\dot{\beta}} \qquad \sigma^{\mu}_{\alpha\dot{\alpha}}\overline{\sigma}^{\dot{\beta}\beta} = 2\delta^{\beta}_{\alpha}\delta^{\dot{\alpha}}_{\dot{\beta}}$$

Any vector can be mapped to bi-spinor

$$V_{\alpha\dot{\alpha}} = V^{\mu}\sigma_{\mu\alpha\dot{\alpha}}, \qquad V^{\mu} = \frac{1}{2}\overline{\sigma}^{\dot{\alpha}\alpha}V_{\alpha\dot{\alpha}}$$

: under the Lorentz transformation,

$$V^{\mu} \to \Lambda^{\mu}{}_{\nu}V^{\nu} \qquad V_{\alpha\dot{\beta}} \to (MVM^{\dagger})_{\alpha\dot{\beta}} \qquad \Lambda^{\mu}{}_{\nu} = \frac{1}{2}\mathrm{tr}\left(M^{\dagger}\overline{\sigma^{\mu}}M\sigma_{\nu}\right)$$

Moreover, Lorentz generators are written as

$$(\sigma^{\mu\nu})_{\alpha}^{\ \beta} = \frac{i}{4}(\sigma^{\mu}_{\alpha\dot{\gamma}}\overline{\sigma}^{\nu\dot{\gamma}\beta} - \sigma^{\nu}_{\alpha\dot{\gamma}}\overline{\sigma}^{\mu\dot{\gamma}\beta})$$

$$(\overline{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}} = \frac{i}{4}(\overline{\sigma}^{\mu\dot{\alpha}\gamma}\sigma^{\nu}_{\gamma\dot{\beta}} - \overline{\sigma}^{\nu\dot{\alpha}\gamma}\sigma^{\mu}_{\gamma\dot{\beta}})$$

$$M = e^{\frac{i}{2}\theta_{\mu\nu}\sigma^{\mu\nu}} \quad \left(\frac{1}{2}, 0\right)$$

$$M^{*-1T} = e^{\frac{i}{2}\theta_{\mu\nu}\overline{\sigma}^{\mu\nu}} \quad \left(0, \frac{1}{2}\right)$$

$$\theta^{i} = \frac{1}{2}\epsilon^{ijk}\theta_{jk}, \quad \omega^{i} = \theta^{i0} = -\theta^{0i}$$

Then $\psi\sigma^{\mu}\overline{\chi} \underset{(1)}{=} -\overline{\chi}\overline{\sigma}^{\mu}\psi$ **is a Lorentz vector** **(2)**

$$\left(\frac{1}{2}, 0\right) \otimes \left(0, \frac{1}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

(1) :

$$\psi\sigma^{\mu}\overline{\chi} = \psi^{\alpha}\sigma^{\mu}_{\alpha\dot{\beta}}\overline{\chi}^{\dot{\beta}} = \epsilon^{\alpha\beta}\psi_{\beta}\sigma^{\mu}_{\alpha\dot{\beta}}\epsilon^{\dot{\beta}\dot{\gamma}}\overline{\chi}_{\dot{\gamma}} = \psi_{\beta}(\epsilon^{\alpha\beta}\epsilon^{\dot{\beta}\dot{\gamma}}\sigma^{\mu}_{\alpha\dot{\beta}})\overline{\chi}_{\dot{\gamma}} = \psi_{\beta}\overline{\sigma}^{\mu\dot{\gamma}\beta}\overline{\chi}_{\dot{\gamma}}$$

$$= -\overline{\chi}_{\dot{\gamma}}\overline{\sigma}^{\mu\dot{\gamma}\beta}\psi_{\beta} = -\overline{\chi}\overline{\sigma}^{\mu}\psi$$

(2) :

$$\psi'\sigma^\mu\overline{\chi}' = -\overline{\chi}'\overline{\sigma}^\mu\psi' = -M^{*}{}_{\dot\alpha}{}^{\dot\beta}\overline{\chi}_{\dot\beta}\overline{\sigma}^{\mu\dot\alpha\beta}M_\beta{}^\gamma\psi_\gamma = -\overline{\chi}_{\dot\beta}M^{\dagger\dot\beta}{}_{\dot\alpha}\overline{\sigma}^{\mu\dot\alpha\beta}M_\beta{}^\gamma\psi_\gamma$$
$$= -\overline{\chi}_{\dot\beta}\delta^{\dot\beta}_{\dot\delta}M^{\dagger\dot\delta}{}_{\dot\alpha}\overline{\sigma}^{\mu\dot\alpha\beta}M_\beta{}^\gamma\delta^\delta_\gamma\psi_\delta = -\frac{1}{2}\overline{\chi}_{\dot\beta}\sigma_{\nu\gamma\dot\delta}\overline{\sigma}^{\nu\dot\beta\delta}M^{\dagger\dot\delta}{}_{\dot\alpha}\overline{\sigma}^{\mu\dot\alpha\beta}M_\beta{}^\gamma\psi_\delta$$
$$= -\frac{1}{2}\left[M^{\dagger\dot\delta}{}_{\dot\alpha}\overline{\sigma}^{\mu\dot\alpha\beta}M_\beta{}^\gamma\sigma_{\nu\gamma\dot\delta}\right](\overline{\chi}_{\dot\beta}\overline{\sigma}^{\nu\dot\beta\delta}\psi_\delta) = -\Lambda^\mu{}_\nu(\overline{\chi}\sigma^\nu\psi)$$
$$= \Lambda^\mu{}_\nu(\psi\sigma^\mu\chi)$$

Then $\psi\sigma^\mu\partial_\mu\overline{\chi}, \quad \overline{\chi\sigma}^\mu\partial_\mu\psi$ **: Lorentz scalar**

Scalars made up of bi-spinor : $\psi\chi \quad \overline{\psi}\overline{\chi} \quad \psi\sigma^\mu\partial_\mu\overline{\chi}, \quad \overline{\chi\sigma}^\mu\partial_\mu\psi$

: terms in the free Lagrangian (Lorentz scalar) for free fermion

coefficients are fixed by requiring $\partial_\mu\partial^\mu + m^2 = 0$

Chiral fermion

Spinor in (1/2, 0) (or spinor in (0, 1/2)) : NOT the mixture (1/2,0)⊕(0,1/2)

$$\mathcal{L} = -\overline{\xi}\overline{\sigma^\mu}\partial_\mu\xi - \frac{m}{2}\xi\xi - \frac{m}{2}\overline{\xi\xi}$$

***m* : majorana mass**
(not allowed in U(1) symmetric theory)

Equations of motion :

$$i\overline{\sigma}^\mu\partial_\mu\xi - m\overline{\xi} = 0, \qquad i\sigma^\mu\partial_\mu\overline{\xi} - m\xi = 0$$

$$\Big\Downarrow \; (-i\sigma^\nu\partial_\nu)$$

$$\sigma^\nu\overline{\sigma}^\mu\partial_\nu\partial_\mu\xi = \frac{1}{2}(\sigma^\nu\overline{\sigma}^\mu + \sigma^\mu\overline{\sigma}^\nu)\partial_\nu\partial_\mu\xi = \frac{1}{2}\times 2\eta^{\mu\nu}\partial_\nu\partial_\mu\xi = \partial_\mu\partial^\mu\xi \Rightarrow (\partial_\mu\partial^\mu + m^2)\xi = 0$$

$$= m(-i\sigma^\nu\partial_\nu)\overline{\xi} = -m^2\xi$$

Quantization :

$$\xi_\alpha(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E(p)}}\sum_s \left(\chi_\alpha(p,s)a(p,s)e^{-ipx} + y_\alpha(p,s)a^\dagger(p,s)e^{ipx}\right)$$

(left handed particle + right handed antiparticle)

Vector-like fermion

(1/2,0)⊕(0,1/2) : (1/2,0) with U(1) charge= +1 :

(1/2,0) with U(1) charge= −1 :
= (0,1/2) particle with U(1) charge = +1 :

$$\mathcal{L} = i\overline{\chi\sigma}^{\mu}\partial_{\mu}\chi + i\overline{\eta\sigma}^{\mu}\partial_{\mu}\eta - m(\chi\eta + \overline{\chi\eta})$$

mass : left-right mixing

Equations of motion : $i\overline{\sigma}^{\mu\dot{\alpha}\alpha}\partial_{\mu}\chi_{\alpha} - m\overline{\eta}^{\dot{\alpha}} = 0 \qquad i\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu}\overline{\eta}^{\dot{\alpha}} - m\chi_{\alpha} = 0$

Define U(1) charge = +1 4-component spinor : Dirac spinor

$$\psi = \begin{pmatrix} \chi_{\alpha} \\ \overline{\eta}^{\dot{\alpha}} \end{pmatrix} \qquad i\underbrace{\begin{pmatrix} 0 & \sigma^{\mu}_{\alpha\dot{\alpha}} \\ \overline{\sigma}^{\mu\dot{\alpha}\alpha} & 0 \end{pmatrix}}_{\boxed{\gamma^{\mu}}}\partial_{\mu}\begin{pmatrix} \chi_{\alpha} \\ \overline{\eta}^{\dot{\alpha}} \end{pmatrix} - m\begin{pmatrix} \chi_{\alpha} \\ \overline{\eta}^{\dot{\alpha}} \end{pmatrix} = 0$$

(1/2,0)⊕(0,1/2)

$$i\gamma^{\mu}\partial_{\mu}\psi - m\psi = 0$$

$$i\not{\partial}\psi - m\psi = 0$$

Dirac Equation

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot{\alpha}} \\ \overline{\sigma}^{\mu\dot{\alpha}\alpha} & 0 \end{pmatrix}$$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = \begin{pmatrix} \sigma^\mu\overline{\sigma}^\nu + \sigma^\nu\overline{\sigma}^\mu & 0 \\ 0 & \overline{\sigma}^\mu\sigma^\nu + \overline{\sigma}^\nu\sigma^\mu \end{pmatrix} = 2\eta^{\mu\nu} I \quad \Rightarrow \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

: Clifford algebra

1. Lorentz generator : $J^{\mu\nu} = \dfrac{i}{4}[\gamma^\mu, \gamma^\nu]$ $\qquad [J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\nu\rho}J_{\mu\sigma} - \eta_{\mu\rho}J_{\nu\sigma} - \eta_{\nu\sigma}J_{\mu\rho} + \eta_{\mu\sigma}J_{\nu\rho})$

for SO(n) or SO(1, n-1), the spinor constructed from the Clifford algebra is the building block of representations.

$$b^0 = \frac{i}{2}(\gamma^0 + \gamma^1), \quad b^i = \frac{1}{2}(\gamma^{2i} + i\gamma^{2i+1})$$
$$b_0 = \frac{i}{2}(\gamma^0 - \gamma^1), \quad b^i = \frac{1}{2}(\gamma^{2i} - i\gamma^{2i+1})$$

$$\{b^i, b_j\} = \delta^i_j, \quad \{b_i, b_j\} = \{b^i, b^j\} = 0$$

See, e.g., App. B of J. Polchinski, String theory, Vol. II
Sec. 8.5 of R. Blumenhagen, D. Lust, S. Theisen, Basic concepts of string theory

2. Chirality projection (even dimension only)

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \qquad \{\gamma^\mu, \gamma_5\} = 0$$

$$\psi = \begin{pmatrix} \chi_\alpha \\ \overline{\eta}^{\dot\alpha} \end{pmatrix}$$

$\frac{1}{2}(I - \gamma_5)$ **: Projection to (1/2,0)** χ_α

$\frac{1}{2}(I + \gamma_5)$ **: Projection to (0,1/2)** $\overline{\eta}^{\dot\alpha}$

3. Lagrangian:

The kinetic term $i\overline{\chi}\overline{\sigma}^\mu\partial_\mu\chi + i\overline{\eta}\overline{\sigma}^\mu\partial_\mu\eta = \overline{\psi}i\gamma^\mu\partial_\mu\psi \equiv \overline{\psi}i\partial\!\!\!/\psi, \qquad \overline{\psi} = \psi^\dagger\gamma^0$

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$\boxed{\mathcal{L} = \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi}$$

4. Quantization

$$\psi(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E(p)}} \sum_s \left(u(p,s)a(p,s)e^{-ipx} + v(p,s)b^\dagger(p,s)e^{ipx} \right)$$

$$\overline{\psi}(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E(p)}} \sum_s \left(\overline{v}(p,s)b(p,s)e^{-ipx} + \overline{u}(p,s)a^\dagger(p,s)e^{ipx} \right)$$

$$\boxed{\{a(p,s), a^\dagger(p',s')\} = \{b(p,s), b^\dagger(p',s')\} = \delta^3(\mathbf{p} - \mathbf{p}')\delta_{s,s'}}$$

$$(i\partial\!\!\!/ - m)\psi = 0 \quad \Longrightarrow \quad (p\!\!\!/ - m)u(p,s) = 0 \qquad (p\!\!\!/ + m)v(p,s) = 0$$

How to solve?

1) In the rest fram $p = (m, \mathbf{0})$

$$(\not{p} - m)u = m(\gamma^0 - I) = \begin{pmatrix} -I & I \\ I & -I \end{pmatrix} u = 0$$

$$\Rightarrow \quad u = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix} \qquad \xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \cosh\omega & \sinh\omega \\ \sinh\omega & \sinh\omega \end{pmatrix}$$

boost

2) $p = (E, 0, 0, p^3)$

$$\boxed{\begin{aligned} E &= m\cosh\omega \\ p^3 &= m\sinh\omega \end{aligned}}$$

$$\boxed{u = e^{-\frac{\omega}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}} \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix} = \begin{pmatrix} \sqrt{p \cdot \sigma}\xi \\ \sqrt{p \cdot \bar{\sigma}}\xi \end{pmatrix}}$$

$$e^{\frac{\omega}{2}} = \frac{E + p^3}{m}, \quad e^{-\frac{\omega}{2}} = \frac{E - p^3}{m}$$

$$\begin{aligned} e^{-\frac{\omega}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}} &= \cosh\omega \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sinh\omega \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} \\ &= \begin{pmatrix} e^{\frac{\omega}{2}}\left(\frac{1-\sigma_3}{2}\right) + e^{-\frac{\omega}{2}}\left(\frac{1+\sigma_3}{2}\right) & 0 \\ 0 & e^{\frac{\omega}{2}}\left(\frac{1+\sigma_3}{2}\right) + e^{-\frac{\omega}{2}}\left(\frac{1-\sigma_3}{2}\right) \end{pmatrix} \end{aligned}$$

$$u(p,s)=\begin{pmatrix}\sqrt{p\cdot\sigma}\xi_s\\ \sqrt{p\cdot\overline{\sigma}}\xi_s\end{pmatrix}$$

Spin sum (the sum of intermediate states in propagator) :

$$\begin{aligned}\sum_{s=1,2}u(p,s)\overline{u}(p,s)&=\sum_s\begin{pmatrix}\sqrt{p\cdot\sigma}\xi_s\\ \sqrt{p\cdot\overline{\sigma}}\xi_s\end{pmatrix}\begin{pmatrix}\xi_s^\dagger\sqrt{p\cdot\overline{\sigma}} & \xi_s^\dagger\sqrt{p\cdot\sigma}\end{pmatrix}\\ &=\begin{pmatrix}\sqrt{p\cdot\sigma}\sqrt{p\cdot\overline{\sigma}} & \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma}\\ \sqrt{p\cdot\overline{\sigma}}\sqrt{p\cdot\overline{\sigma}} & \sqrt{p\cdot\overline{\sigma}}\sqrt{p\cdot\sigma}\end{pmatrix}=\begin{pmatrix}m & p\cdot\sigma\\ p\cdot\overline{\sigma} & m\end{pmatrix}\\ &=\not{p}+m\end{aligned}$$

In the same way,

$$v(p,s)=\begin{pmatrix}\sqrt{p\cdot\sigma}\xi_s\\ -\sqrt{p\cdot\overline{\sigma}}\xi_s\end{pmatrix}\qquad\qquad \sum_{s=1,2}v(p,s)\overline{v}(p,s)=\not{p}-m$$

Propagator

$$\langle 0|\psi_a(x)\overline{\psi}_b(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3 2E(p)} \sum_s u(p,s)\overline{u}(p,s)e^{-ipx} = (i\partial\!\!\!/_x + m)_{ab} \int \frac{d^3p}{(2\pi)^3 2E(p)} e^{-ip(x-y)}$$ **particle**

$$\langle 0|\overline{\psi}_b(y)\psi_a(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3 2E(p)} \sum_s v(p,s)\overline{v}(p,s)e^{-ipx} = -(i\partial\!\!\!/_x + m)_{ab} \int \frac{d^3p}{(2\pi)^3 2E(p)} e^{-ip(y-x)}$$ **anti-particle**

Spin sum (the sum of intermediate states in propagator) :

$$\langle 0|T(\psi(x)\overline{\psi}(y))|0\rangle = \begin{cases} \langle 0|\psi(x)\overline{\psi}(y)|0\rangle & x^0 > y^0 \\ -\langle 0|\overline{\psi}(y)\psi(x)|0\rangle & x^0 < y^0 \end{cases} = \int \frac{d^4p}{(2\pi)^4} \frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\epsilon} = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p\!\!\!/ - m}$$

inverse

$$\mathcal{L} = \overline{\psi}\boxed{(i\gamma^\mu\partial_\mu - m)}\psi$$

Generic structure of Lorentz covariant field : spin-statistics theorem

$$\varphi(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E(p)}} \sum_s \left(f(p,s)a(p,s)e^{-ipx} + h(p,s)b^\dagger(p,s)e^{ipx} \right)$$

$$\begin{cases} \langle 0|\varphi(x)\varphi^\dagger(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3 2E(p)} \mathcal{F}(p) e^{-ip(x-y)} = \mathcal{F}(i\partial_x)\Delta_+(x-y) & \mathcal{F} = \sum_s f(p,s)f^*(p,s) \quad \textbf{particle} \\ \langle 0|\varphi^\dagger(y)\varphi(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3 2E(p)} \mathcal{H}(p) e^{+ip(x-y)} = \mathcal{H}(-i\partial_x)\Delta_+(y-x) & \mathcal{H} = \sum_s h(p,s)h^*(p,s) \quad \textbf{anti-particle} \end{cases}$$

For $(x-y)^2 < 0$, $\Delta_+(x-y) = \Delta_+(y-x)$

$$\langle 0|\varphi(x)\varphi^\dagger(y)|0\rangle = \mathcal{F}(i\partial_x)\Delta_+(x-y)$$

$$\langle 0|\varphi^\dagger(y)\varphi(x)|0\rangle = \mathcal{H}(-i\partial_x)\Delta_+(x-y)$$

$x \to -x\ (\partial_x \to -\partial_x)$ **is equivalent to** $B(\hat{z}\,(i\pi))R(\hat{z}\pi) = e^{i(J_3\pi + K_3(i\pi))} = e^{2iJ_3^-\pi}$

$x^{0,3} \to -x^{0,3}, x^{1,2} \to x^{1,2}$ $x^{0,3} \to x^{0,3}, x^{1,2} \to -x^{1,2}$ $J_{-i} = \frac{1}{2}(J_i + iK_i)$

CPT invariance : (local) Lorentz invariant theory is invariant under CPT

Indeed, **PT:** $x \to -x$

C: particle ↔ antiparticle

$$\mathcal{H}(-i\partial_x) = \mathcal{F}(i\partial_x)e^{2i(J_3^- + J_3^+)\pi} = \mathcal{F}(i\partial_x)e^{2\pi i J_3}$$

$f(p,s)$ $f^*(p,s)$

$$j = 0, 1, \cdots \quad : \quad \mathcal{H}(-i\partial_x) = \mathcal{F}(i\partial_x) \quad \longrightarrow \quad \langle 0|\varphi(x)\varphi(y)^\dagger|0\rangle = \langle 0|\varphi(y)^\dagger\varphi(x)|0\rangle$$

$$j = \frac{1}{2}, \frac{3}{2}, \cdots \quad : \quad \mathcal{H}(-i\partial_x) = -\mathcal{F}(i\partial_x) \quad \longrightarrow \quad \langle 0|\varphi(x)\varphi(y)^\dagger|0\rangle = -\langle 0|\varphi(y)^\dagger\varphi(x)|0\rangle$$

For $(x-y)^2 < 0$

Thus, the causality conditions :

$$j = 0, 1, \cdots \quad : \quad \langle 0|[\varphi(x), \varphi(y)^\dagger]|0\rangle = 0$$

$$j = \frac{1}{2}, \frac{3}{2}, \cdots \quad : \quad \langle 0|\{\varphi(x), \varphi(y)^\dagger\}|0\rangle = 0$$

For $(x-y)^2 < 0$

This (*$VERY^N$* ROUGH) argument shows that

$j = 0, 1, \cdots$ **: boson**

$j = \frac{1}{2}, \frac{3}{2}, \cdots$ **: fermion**

spin-statistics theorem

Regardless of spin, physical observables (energy, momentum, angular momentum…) are bosonic composite of elementary particle fields : causality is defined by their commutation relation (outside the light-cone, observable operators commute)

Integration of anti-commuting number (Grassmann variable)

For the integration $I[f] = \int dw f(w)$ **(** $f(w) = f^{(0)} + f^{(1)} w$ **) to satisfy**

1) $\int dw f(w+a) = \int dw f(w)$

2) $\int dw(\alpha f_1(w) + \beta f_2(w)) = \alpha \int dw f_1(w) + \int dw \beta f_2(w)$

$$\int dw 1 = 0, \qquad \int dw w = 1$$

That is, integration = derivative

Example : $\int dw_2 dw_1 e^{-w_1 a w_2} = \int dw_2 dw_1 (1 - w_1 a w_2) = -a$

Abelian (U(1)) gauge field (Spin-1)

Lagrangian $\mathcal{L} = \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi$

invariant under the global phase (U(1)) transformation $\psi(x) \to e^{i\alpha}\psi(x)$

It is not invariant under the local phase transformation,

$$\partial_\mu \psi \to e^{i\alpha}(\boxed{i\partial_\mu \alpha} + \partial_\mu)\psi$$

but can be invariant by modifying the derivative into the covariant derivative and introduce the vector field which also transforms:

$$D_\mu \psi(x) = \partial_\mu \psi(x) + ieA_\mu \psi(x)$$

$$\boxed{\begin{aligned} &\psi(x) \to e^{i\alpha(x)}\psi(x) \\ &A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x) \end{aligned}}$$

Gauge transformation

$$D_\mu \psi \to \left[\partial_\mu + ie\left(A_\mu - \frac{1}{e}\partial\alpha\right)\right] e^{i\alpha}\psi = e^{i\alpha}(\partial_\mu + ieA_\mu)\psi = e^{i\alpha(x)} D_\mu \psi(x)$$

Dynamics of A_μ

Requirement : Lorentz + Gauge invariant kinetic term

Gauge invariant derivative : $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ **(field strength)**

This corresponds to the 'curvature'

$$[D_\mu, D_\nu]\psi = \cancel{[\partial_\mu, \partial_\nu]}\psi + ie\big([\partial_\mu, A_\nu] - [\partial_\nu, A_\mu]\big)\psi - e^2\cancel{[A_\mu, A_\nu]}\psi$$
$$= ie(\partial_\mu A_\nu - \partial_\nu A_\mu)\psi$$

$$\Rightarrow \quad [D_\mu, D_\nu] = ieF_{\mu\nu}$$

In fact, given 4-vector potential, $A^\mu = (\phi, \mathbf{A})$ **, field strength : electric/magnetic fields**

$$\mathbf{E} = -\nabla A^0 - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \qquad F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

one can also define 'dual field strength'

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{pmatrix}$$

$$E \to B,\ B \to -E, \qquad F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

Maxwell's equation :

$$\boxed{\begin{aligned} \nabla \cdot \mathbf{E} &= \rho \\ \nabla \times \mathbf{B} - \frac{\mathbf{E}}{\partial t} &= \mathbf{j} \end{aligned}} \Rightarrow \partial_\mu F^{\mu\nu} = j^\nu$$

$$\boxed{\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} - \frac{\mathbf{B}}{\partial t} &= 0 \end{aligned}} \Rightarrow \partial_\mu \tilde{F}^{\mu\nu} = 0$$

Lorentz scalars:

$$F_{\mu\nu}F^{\mu\nu} = -\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} = -2(\mathbf{E}^2 - \mathbf{B}^2)$$

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = -4\mathbf{E}\cdot\mathbf{B} = 2\partial_\mu(\epsilon^{\mu\nu\rho\sigma}A_\nu\partial_\rho A_\sigma)$$

Surface term (total derivative) : no contribution to the Feynman rule but provides a topological term

For details, see, e.g., J. D. Jackson, Classical electrodynamics

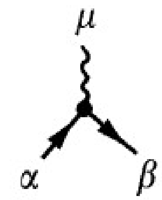
C. Itzykson, J.-B. Zuber, Quantum field theory

Lagrangian for Quantum Electrodynamics (QED)

(Dirac (vector-like) fermion charged under U(1) Abelian gauge interaction)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\not{D} - m)\psi = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\not{\partial} - m)\psi - \boxed{eA_\mu\overline{\psi}\gamma^\mu\psi}$$

Vertex

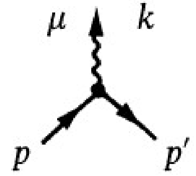


$$-ie(\gamma_\mu)_{\beta\alpha}(2\pi)^4\delta^4(\Sigma p)$$

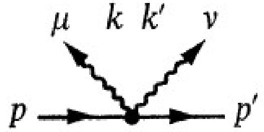
Adding U(1) charged scalar field,

$$\underline{D_\mu\phi^\dagger D_\mu\phi} - m^2\phi^\dagger\phi - V(\phi)$$

$$\boxed{\left[(\partial_\mu\phi^\dagger - ieA_\mu\phi^\dagger\right]\left[(\partial_\mu\phi + ieA_\mu\phi\right]}$$



$$-ie(p_\mu + p'_\mu)(2\pi)^4\delta^4(p - p' - k)$$



$$2ie^2 g_{\mu\nu}(2\pi)^4\delta^4(p - p' - k - k')$$

current

Issues :

1. Gauge field kinetic term : while it gives the correct Maxwell's equations, the inverse of quadratic derivative term does not exist : how to define propagator?

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \sim \frac{1}{2}A_\mu(\partial^2\eta^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu$$

$$\delta^\mu_\rho = (\eta^{\mu\nu}k^2 - k^\mu k^\nu)\left(a\eta_{\nu\rho} + b\frac{k_\nu k_\rho}{k^2}\right) = ak^2\delta^\mu_\rho - ak^\mu k_\rho$$

No solutions $(\boldsymbol{a}, \boldsymbol{b})$

This has to do with the nature of gauge invariance :

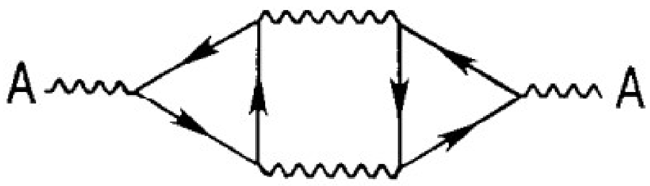
as the gauge invariance forbids the photon mass $\frac{1}{2}m^2A_\mu A^\mu$

the photon has two degrees of freedom (two polarizations), less than 4 as can be found from $\boldsymbol{A_\mu}$

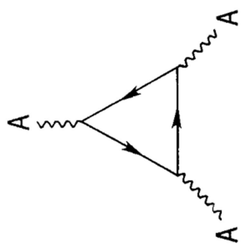
: In $\boldsymbol{A_\mu}$, some unphysical degrees of freedom are mixed, which makes a problem.

2. Does quantum mechanics preserve gauge invariance?

Consider the photon mass generated by the loop (= QM effect)



For Dirac (=vector-like) fermion, this term vanishes by the cancellation between contributions of (1/2,0) and (0, 1/2) to



For chiral fermion, cancellation cannot take place unless a number of chiral fermions have ‘carefully assigned’ U(1) charges. (in the Standard Model, the cancellation arises! But why? Grand unification?)

Recommended readings : A. Bilal, 0802.0634

R. A. Bertlmann, Anomalies in quantum field theory

J. Preskill, Ann. Phys. 210 (1991) 323

3. Photon mass compatible with gauge invariance?

Stuckelberg mechanism : introduce a real scalar (called axion) a

under gauge transformation,

$$A_\mu \to A_\mu - \frac{1}{e}\partial\alpha, \quad a \to a + \frac{1}{e}\alpha$$

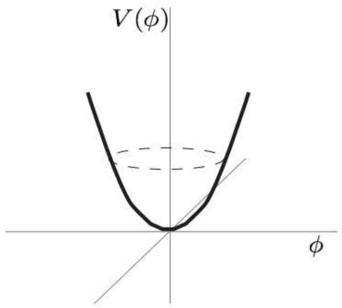
$$\Rightarrow \quad \boxed{\frac{1}{2}m^2(A_\mu + \partial_\mu a)^2}$$

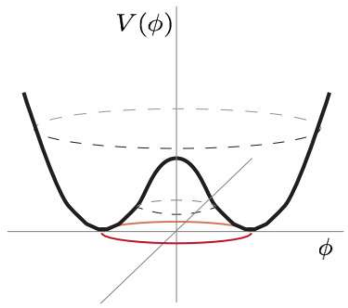
Gauge invariant mass term

Realization : Higgs mechanism (Anderson-Higgs-Brout-Englert-Guralnik-Hagen-Kibble)

complex scalar having vacuum expectation value (VEV) v

$$V(\phi) = \frac{1}{2}(-m^2)|\phi|^2 + \frac{\lambda}{4}(|\phi|^2)^2 \quad \Rightarrow \quad \phi(x) = \frac{1}{\sqrt{2}}(v+\rho)e^{iea(x)}$$





$$|D_\mu\phi|^2 = \frac{1}{2}(ev)^2(A_\mu + \partial_\mu a)^2 + \frac{1}{2}(v+\rho)^2 + \cdots$$

Axion absorbed by gauge boson (3 degrees of freedom)

Higgs particle

Non-Abelian gauge field (Spin-1)

Example :

$$\begin{pmatrix} p(x) \\ n(x) \end{pmatrix}$$ **SU(2) isospin between p and n**

$$\begin{pmatrix} u^r(x) \\ u^g(x) \\ u^b(x) \end{pmatrix}$$ **SU(3) color of quarks**

$$\boxed{\begin{aligned} &\psi(x) \to U(x)\psi(x), \\ &A_\mu(x) \to U(x)A_\mu(x)U^\dagger(x) + \frac{i}{g}U(x)\partial_\mu U^\dagger(x), \end{aligned}}$$

$$U(x) = e^{i\alpha^a(x)t^a}$$

$$[t^a, t^b] = i f_{abc} t^c.$$

t^a: generator of non-Abelian gauge group

f_{abc} : structure constant
For SU(2), $f_{abc} = \epsilon_{abc}$

$$A_\mu = A_\mu^a(x) t^a$$

matrix

$$D_\mu = \partial_\mu - igA_\mu$$

$$D_\mu\psi \to (\partial_\mu - ig(-\partial_\mu U)U^\dagger)U\psi = UD_\mu\psi$$

$$\boxed{D_\mu \to UD_\mu U^\dagger}$$

“Curvature”

$$[D_\mu, D_\nu] = [\partial_\mu - igA_\mu, \partial_\nu - igA_\nu] = -ig(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]) \equiv -igF_{\mu\nu}$$

$$\boxed{[A_\mu, A_\nu] = A^a_\mu A^a_\nu [t^a, t^b] = i f_{abc} A^a_\mu A^b_\nu t^c}$$

$$F_{\mu\nu} = F^a_{\mu\nu} t^a. \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu$$

Under the gauge transformation,

$$D_\mu \to U D_\mu U^\dagger \quad \Rightarrow \quad [D_\mu, D_\nu] \to U[D_\mu, D_\nu]U^\dagger \quad \textbf{thus} \quad F_{\mu\nu} \to U F_{\mu\nu} U^\dagger$$

Unlike the Abelian case, gauge boson is charged under the gauge group
(field strength is NOT gauge invariant)

Lorentz, gauge invariant kinetic term : $-\frac{1}{2}\mathrm{tr}[F_{\mu\nu}F^{\mu\nu}] = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}$ $\qquad \boxed{\mathrm{tr}[t^a t^b] = \frac{1}{2}\delta_{ab}}$

Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \overline{\psi}(i\not{D} - m)\psi$$

Equations of motion :

$$\underline{\partial^\mu F^a_{\mu\nu} + gf_{abc}A^{b\mu}F^c_{\mu\nu}} = -gj^a_\nu, \qquad j^a_\nu = \overline{\psi}\gamma_\nu t^a \psi$$

$$(\partial^\mu \delta^{ac} - ig(if_{abc})A^{b\mu})F^c_{\mu\nu}$$

This shows that gauge boson belongs to the adjoint representation ("charged")

$$(t^b_G)_{ac} = if_{abc}$$

Jacobi identity

$$0 = [t^a, [t^b, t^c]] + [t^b, [t^c, t^a]] + [t^c, [t^a, t^b]]$$
$$= f^{ade}f^{bcd} + f^{bde}f^{cad} + f^{cde}f^{abd}$$

$$\Rightarrow \quad ([t^b_G, t^c_G])_{ae} = if_{bcd}(t^d_G)_{ae}$$

Useful representations:

Recommended readings : A. Zee, Group theory in a nutshell for physicists

H. Georgi. Lie algebras in particle physics

1. Fundamental representation (or defining representation) V
2. Anti-fundamental representation V^*

   : equivalent to V for SU(2)

3. Adjoint representation : $V \otimes V^*$ for SU(N), $N^2 - 1$ –dimensional

Quantization of gauge boson field

$$-\frac{1}{2}\mathrm{tr}[F_{\mu\nu}F^{\mu\nu}] = -\mathrm{tr}\Big[(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu])F^{\mu\nu} - \frac{1}{2}F_{\mu\nu}F^{\mu\nu}\Big]$$

$$= -\mathrm{tr}\Big[2\underbrace{\partial_0 A_k}_{E^k}F^{0k} + 2\partial_k A_0 F^{k0} - 2ig(A_0A_k - A_kA_0)F^{0k} + \underbrace{F_{ij}F^{ij}}_{2\mathbf{B}^2} - \frac{1}{2}(\underbrace{F_{0i}F^{0i} + F_{i0}F^{i0}}_{-\,2\mathbf{E}^2} + \underbrace{F_{ij}F^{ij}}_{2\mathbf{B}^2})\Big]$$

$$\sim A_0\partial_k F^{k0} = A_0\partial_k E^k$$

$$\boxed{-2ig\mathrm{tr}[A_0A_kF^{0k} - A_0F^{0k}A_k] = -2ig\mathrm{tr}[A_0[A_k, F^{0k}]] = -2ig\mathrm{tr}[A_0[A_k, E^k]]}$$

$$= -\mathrm{tr}\big[-2\partial_0 A^k E^k + (\mathbf{E}^2 + \mathbf{B}^2) + 2A_0(\partial_k E^k - ig[A_k, E^k])\big]$$

$$= \partial_0 A^{ak}E^{ak} - \frac{1}{2}(\mathbf{E}^{a2} + \mathbf{B}^{a2}) - A_0^a(\partial_k E^{ak} - ig[A_k, E^k]^a)$$

$$L = \dot{q}p - H - \lambda(\text{constraint})$$

While A_0 does not have dynamics, it plays a role of Lagrange multiplier accompanying the constraint

$$Q^a \equiv \partial_k E^{ak} - ig[A_k, E^k]^a - j^{a0} = 0$$

: Coulomb's law

1) Constraint is the generator of gauge transformation :

Under the quantization $[A_i^a(x), E_j^b(y)]_{x^0=y^0} = i\delta_{ab}\delta_{ij}\delta^3(x-y)$

$$
\begin{aligned}
i[\epsilon^a Q^a, A_i^b(x)] &= i\int d^3y \epsilon^a(y)[\partial_k E^{ak} + g f_{acd} A_k^c E^{dk}, A_i^b(x)] \\
&= i\int d^3y \epsilon^b(y)\partial_{y,i}(-i\delta^3(x-y)) + g\epsilon^a f_{acb} A_i^c(-i\delta_{ab}\delta^3(x-y)) \\
&= -(\partial_i \delta^{ab} + g f_{bca} A_i^c)\epsilon^a = -D_i\epsilon^a
\end{aligned}
$$

This coincides with the infinitesimal gauge transformation , $U \simeq 1 + i\epsilon^a t^a$ **:**

$$
\begin{aligned}
\frac{i}{g} U\partial_\mu U^\dagger + U A_\mu U^\dagger &\simeq \frac{i}{g}(-i\partial_\mu \epsilon^a t^a) + i\epsilon^a [t^a, t^b] A_\mu^b \\
&= \frac{1}{g} t^a(\partial\mu\epsilon^a + f_{abc} A_\mu^b \epsilon^c) = \frac{1}{g} t^a D_\mu \epsilon^a
\end{aligned}
$$

2) Coulomb’s law as a constraint is ‘conserved’ over the whole time evolution

3) Among A^0, A^1, A^2, A^3,

A^0 is not dynamical : Lagrange multiplier accompanying constraint

By imposing the constraint, one more degrees of freedom can be eliminated (?)

to be consistent with 2 polarization (as predicted by little group)

: Not always. (Proca equation for massive spin-1 field : not commute with $\pi_{A0} = 0$)

Moreover, even if we eliminate one degree of freedom,

we need to eliminate one of conjugate momenta E^1, E^2, E^3 : gauge fixing condition

4) Gauge redundancy? Symmetry?

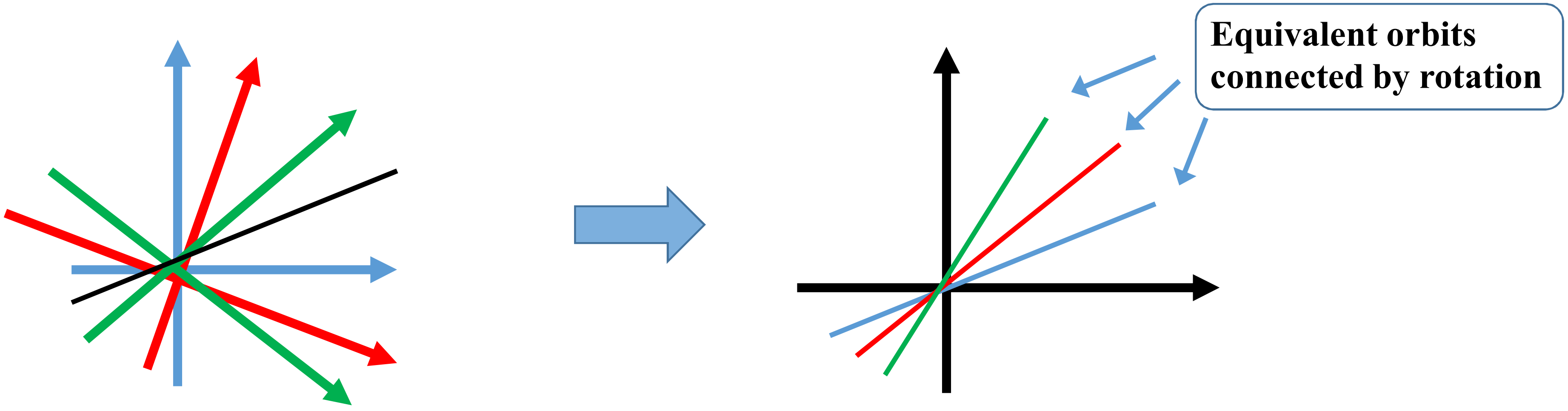
Symmetry : invariance of dynamics under the transformation between different states

But…. States related by gauge transformation are not distinct, but equivalent

(polarizations in Lorenz gauge and Coulomb gauge are NOT different states)

e.g., different coordinate systems for the same spacetime manifold

Illustration :



passive picture

active picture

If we try to describe 1-dimensional motion in terms of 2-dimensional coordinates which contain unphysical degrees of freedom, the rotation (gauge transformation) connecting equivalent orbits looks like a symmetry

See, e.g., Ch. 15 of A. Duncan, The conceptual framework of quantum field theory

M. Henneaux, C. Teitelboim, Quantization of gauge systems

Hamiltonian with r constraints χ_m $(m = 1, \dots, r)$

$$H = h(q_1, \cdots q_f; p_1, \cdots q_f) + \sum_{m=1}^{r} \lambda_m \chi_m$$

Total (including unphysical) degrees of freedom = 2 f

While constraints eliminate r degrees of freedom, the requirement that physical degrees of freedom must be even (configurations + conjugate momenta) suggests that we must eliminate r more degrees of freedom (physical degrees of freedom = $2(f - r)$, not $2f - r$.

So, we need to impose r more 'gauge fixing conditions' ψ_m $(m = 1, \dots, r)$ by hand.

(Choosing one orbit among equivalent orbits)

Cf. Noether theorem for global symmetry: promoting the transformation parameter α to local one $\alpha(x)$

Global symmetry

For a classical solution

$$0 = \delta S = \cancel{\int d^4x \alpha(x) \frac{\delta S}{\delta \alpha(x)}} + \int d^4x \partial_\mu \alpha(x) J^\mu(x)$$

$$= - \int d^4x \alpha(x) \partial_\mu J^\mu$$

$$\boxed{\partial_\mu J^\mu = 0}$$

$$Q = \int d^3x J^0 \quad \Longrightarrow \quad \frac{dQ}{dt} = \int d^3x \partial_t J^0 = - \int d^3x \nabla \cdot \mathbf{J} = 0$$

$$(q_1, \cdots, q_f; p_1, \cdots p_f)$$

$$\Big\Downarrow \; \chi_m, \;\; \psi_m$$

$$(Q_i^*, P_i^* | Q_m, P_m)$$

Physical $i = 1, \ldots, f - r$	Unphysical $m = 1, \ldots, r$

One can always choose the gauge fixing conditions to be unphysical momenta (by canonical transformation) such that

$$P_m = \psi_m(q, p) \qquad [\psi_m, \psi_n] = 0$$

If $$\det[\chi_m, \psi_n] = \det[\chi_m, P_n] = \det\left(i\frac{\partial \chi_m}{\partial Q_n}\right) \neq 0$$

Faddeev-Popov determinant

$\chi_m(Q_i^*, P_i^*; Q_m, P_m) = 0$ **can be solved to give** $Q_m = Q_m(Q_i^*, P_i^*; P_m = 0) \equiv f_m(Q_i^*, P_i^*)$

: all unphysical degrees of freedom are written in terms of physical degrees of freedom

Then from

$$\prod_{i=1}^{f-r} \mathcal{D}Q_i^*\mathcal{D}P_i^*\Big|_{\chi_m=\psi_m=0} = \prod_{i=1}^{f-r} \mathcal{D}Q_i^*\mathcal{D}P_i^* \prod_{m=1}^{r} \mathcal{D}Q_m\mathcal{D}P_m \delta(P_m)\delta(Q_m - f_m(Q_i^*, P_i^*))$$

$$= \prod_{i=1}^{f-r} \mathcal{D}Q_i^*\mathcal{D}P_i^* \prod_{m=1}^{r} \mathcal{D}Q_m\mathcal{D}P_m \delta(\psi_m) \prod_{m=1}^{r} \delta(\chi_m)\det\frac{\partial\chi_m}{\partial Q_n}$$

and replacing $\delta(\chi_m)$ by equations of motion for Lagrange multiplier,

Two partition functions are equivalent :

$$Z = \int \prod_{i=1}^{f} \mathcal{D}q_i\mathcal{D}p_i \prod_{m=1}^{r} \delta(\psi_m)\det[\chi_m, \psi_n] e^{i\int dt(p_i\dot{q}_i - H - \sum_m \lambda_m\chi_m)}$$

$$Z = \int \prod_{i=1}^{f-r} \mathcal{D}Q_i^*\mathcal{D}P_i^* e^{i\int dt\left[P_i^*\dot{Q}_i^* - H^*(Q_i^*, P_i^*)\right]}$$

In gauge theory,

$$\chi_m \to Q^a \equiv \partial_k E^{ak} - ig[A_k, E^k]^a - j^{a0}$$

Gauge transformation generator

$$[\chi_m, \psi_n] \to [Q^a, \psi_b]$$

Gauge transformation of gauge fixing condition

Example : Lorenz gauge $\psi^a = \partial_\mu A^{a\mu} - f^a(x)$

$$[Q^a, \psi^b] = \partial_\mu(\partial^\mu \delta_{ab} - g f_{abc} \partial_\mu A^{c\mu}) \Rightarrow \Delta_{\mathrm{FP}} \equiv \det[Q^a, \psi^b] = \det[\partial^2 - g f_{abc} \partial_\mu A^{c\mu}]$$

$$Z = \int \mathcal{D}A^a_\mu \mathcal{D}\psi \mathcal{D}\overline{\psi} \delta(\partial_\mu A^{a\mu} - f^a) \Delta_{\mathrm{FP}} e^{iS}$$

1. Z can be multiplied by $1 = \int \mathcal{D}f^a e^{-\frac{i}{2\xi}\int d^4x (f^a)^2}$

(f^a is irrelevant to dynamics of physical degrees of freedom :treated as a constant)

2. Introducing 'ghosts' $\omega_a, \overline{\omega_a}$: fermionic (Grassmannian) field

$$\Delta_{\mathrm{FP}} = \int \mathcal{D}\omega_a \mathcal{D}\overline{\omega}_a e^{i\int d^4x \overline{\omega}_a (\partial^2 \delta_{ab} - g f_{abc} \partial^\mu A^c_\mu)\omega_b}$$

$$\boxed{\int d\omega d\overline{\omega} e^{\overline{\omega} X \omega} = \mathrm{det}X \quad \mathrm{NOT}\ \frac{1}{\mathrm{det}X}}$$

From 1 and 2, the partition function for the gauge interaction is given by

$$\begin{aligned} Z &= \int \mathcal{D}A^a_\mu \mathcal{D}\psi \mathcal{D}\overline{\psi} \mathcal{D}\omega_a \mathcal{D}\overline{\omega}_a \mathcal{D}f^a \delta(\partial_\mu A^{a\mu} - f^a) e^{iS - \frac{i}{2\xi}\int d^4x (f^a)^2 + i\int d^4x \overline{\omega}_a (\partial^2 \delta_{ab} - g f_{abc} \partial^\mu A^c_\mu)\omega_b} \\ &= \int \mathcal{D}A^a_\mu \mathcal{D}\psi \mathcal{D}\overline{\psi} \mathcal{D}\omega_a \mathcal{D}\overline{\omega}_a e^{iS - \frac{i}{2\xi}\int d^4x (\partial^\mu A^a_\mu)^2 + i\int d^4x \overline{\omega}_a (\partial^2 \delta_{ab} - g f_{abc} \partial^\mu A^c_\mu)\omega_b} \end{aligned}$$

Now the photon quadratic term has an inverse : photon propagator in this case is,

$$\langle 0|T(A^a_\mu(x) A^b_\nu(y))|0\rangle = -i\delta_{ab} \int \frac{d^4k}{(2\pi)^4} \left[\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right] \frac{e^{-ik(x-y)}}{k^2 + i\epsilon}$$

Physical states : Becchi-Rouet-Stora-Tyutlin (BRST) cohomology

1. **Simple +generic version of what we have done:**

$$Z = \int \mathcal{D}\Phi e^{iS}\delta(\psi^A)\mathrm{det}\frac{\delta\psi^A}{\delta\alpha^a} = \int \mathcal{D}\Phi\mathcal{D}b_A\mathcal{D}c^a\delta(\psi^A)e^{i(S+b_A\frac{\delta\psi^A}{\delta\alpha^a}c^a)}$$

$$\boxed{\mathcal{D}\Phi \equiv \mathcal{D}A^a_\mu\mathcal{D}\psi\mathcal{D}\overline{\psi}}$$

$$= \int \mathcal{D}\Phi\mathcal{D}b_A\mathcal{D}c^a\mathcal{D}B_A e^{i(S_0+\underbrace{i(\frac{\xi}{2}B_A^2+B_A\psi^A)+b_A\frac{\delta\psi^A}{\delta\alpha^a}c^a)}_{=S'}}$$

$$\boxed{\int \mathcal{D}f^A e^{\frac{f^{A2}}{2\xi}}\delta(\psi^A - f^A) = e^{\frac{\psi^{A2}}{2\xi}} = \int \mathcal{D}B_A e^{-(\frac{\xi}{2}B_A^2+\psi^A B_A)}}$$

(B_A : bosonic, b_A, c^a : fermionic) **Term containing ψ^A**

2. **Even though gauge invariance is 'broken' by gauge fixing, it appears in another form**

: BRST invariance (ϵ : fermionic)

$$\delta\Phi = -i\epsilon c^a\frac{\delta\Phi}{\delta\alpha^a} \Rightarrow$$

Gauge transformation with ghost c^a parameter

$$\delta b_A = -\epsilon B_A \quad \delta c^a = -\frac{i}{2}\epsilon f_{abc}c^b c^c, \quad \delta B_A = 0 \qquad [\delta_a, \delta_b] = f_{abc}\delta_c$$

3. **BRST is nilpotent : $\delta^2 = 0$ or $Q^2 = 0$ (Q : BRST generator)**

$$Q(Q|\text{arbitrary state}\rangle) = 0$$

4.

$$\delta(b_A\psi^A) = i\epsilon\left(iB_A\psi^A + b_A\frac{\delta\psi^A}{\delta\alpha^a}c^a\right) = i\epsilon S'$$

Physics is not changed under different choices of ψ^A (gauge fixing)

: under the change of gauge fixing condition $\psi^A \to \psi^A + \Delta\psi^A$ probability amplitude between physical state $\langle\phi|\phi'\rangle$ is invariant

ψ^A is contained in S' only

$$0 = i\epsilon\Delta\langle\phi|\phi'\rangle = \langle\phi|(i\epsilon\Delta S')|\phi'\rangle = \langle\phi|\Delta\delta(b_A\psi^A)|\phi'\rangle = \langle\phi|\delta(b_A\Delta\psi^A)|\phi'\rangle$$

$$= \langle\phi|\{Q, b_A\Delta\psi^A\}|\phi'\rangle$$

$$Q|\phi\rangle = Q|\phi'\rangle = 0 \quad \Rightarrow \quad Q|\text{physical state}\rangle = 0$$

Physical state is BRST (~gauge) invariant

Moreover, $\delta^2 = 0$ or $Q^2 = 0$: $|\text{physical state}\rangle$ **is equivalent to** $|\text{physical state}\rangle + Q|\text{arbitrary state}\rangle$ **(cohomology class)**

S-matrix is also required to be BRST invariant :

$$[Q, S] = 0 \quad \Longrightarrow \quad Q(S|\text{physical state}\rangle) = S(Q|\text{physical state}\rangle) = 0$$

: $S|\text{physical state}\rangle$ **is BRST invariant , thus** $\langle\text{unphysical state}|S|\text{physical state}\rangle = 0$

$$\begin{aligned}\langle A : \text{phys}|B : \text{phys}\rangle &= \langle A : \text{phys}|S^\dagger S|B : \text{phys}\rangle \\ &= \sum_{\text{phyical}} \langle A : \text{phys}|S^\dagger|C : \text{phys}\rangle\langle C : \text{phys}|S|B : \text{phys}\rangle\end{aligned}$$

Unitarity of S-matrix $S^\dagger S = I$ **is implemented among physical states only!**

Relation between 1PIs imposed by gauge invariance : Ward-Takahashi identity / Slavnov-Taylor identity

For simplicity, we consider the Abelian case, QED.

$$\delta A_\mu = -\partial_\mu \chi, \quad \delta\psi = ie\chi\psi$$

Quantum effective action (1PI generator):

$$\Gamma(A_\mu, \psi, \overline{\psi}) = \int d^4x d^4y \big(\frac{1}{2}\Gamma_{\mu\nu}(x,y)A^\mu(x)A^\nu(y) + \overline{\psi}(x)\Gamma^{1,1}(x,y)\psi(y)\big)$$
$$+ \int d^4x d^4y d^4z \overline{\psi}(x)\Gamma^{1,1}_\mu(x,y,z)\psi(y)A^\mu(z) + \cdots$$

Under the gauge transformation,

$$\delta\Gamma = -\int d^4xd^4y\Gamma_{\mu\nu}(x,y)A^\mu(x)\partial_y^\nu\chi(y) - \int d^4xd^4y\overline{\psi}(x)\Gamma^{1,1}(x,y)\psi(y)ie(\chi(x)-\chi(y))$$
$$- \int d^4xd^4yd^4z\overline{\psi}(x)\Gamma_\mu^{1,1}(x,y,z)\psi(y)\big[ie(\chi(x)-\chi(y))A^\mu(z)+\partial_z^\mu\chi(z)\big]+\cdots$$
$$=0$$

Choosing $\chi(x) = \delta^4(x-a)$,

$$\delta\Gamma[a] = \int d^4x\partial_y^\nu\Gamma_{\mu\nu}(x,a)A^\mu(x)$$
$$- \int d^4yie\overline{\psi}(a)\Gamma^{a,1}(a,y)\psi(y) + \int d^4xie\overline{\psi}(x)\Gamma^{a,1}(x,a)\psi(a)$$
$$- \int d^4yd^4zie\overline{\psi}(a)\Gamma_\mu^{1,1}(a,y,z)\psi(y)A^\mu(z) + \int d^4xd^4zie\overline{\psi}(x)\Gamma_\mu^{1,1}(x,a,z)\psi(a)A^\mu(z)$$
$$+ \int d^4xd^4yie\overline{\psi}(x)\partial_z^\mu\Gamma_\mu^{1,1}(x,y,a)\psi(y)+\cdots$$
$$=0$$

1.

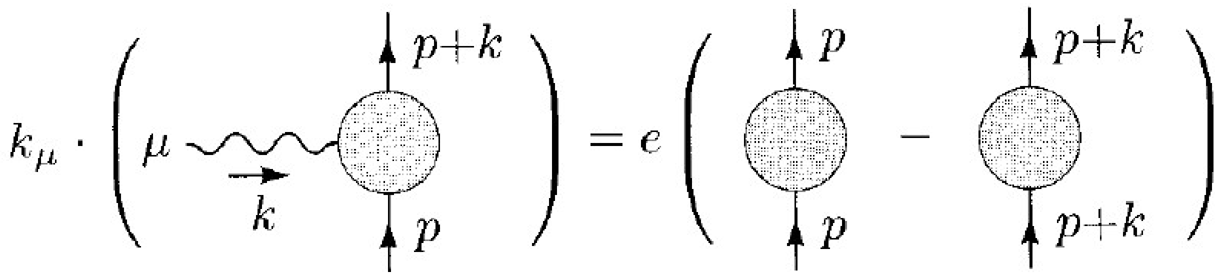
$$\frac{\delta}{\delta A^\mu(b)}\delta\Gamma[a]\Big|_{\Phi=0} = \partial_y^\nu \Gamma_{\mu\nu}(b,a) = 0 \quad \Longrightarrow \quad k^\mu \Gamma_{\mu\nu} = 0$$

transversality : only states orthogonal to k^μ contribute (two polarizations)

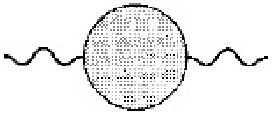
2.

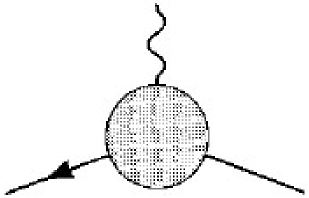
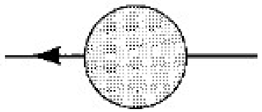
$$\frac{\delta^2}{\delta\overline{\psi}(b)\delta\psi(c)}\delta\Gamma[a]\Big|_{\Phi=0} = -ie\delta(a-b)\Gamma^{1,1}(a,c) + ie\Gamma^{1,1}\delta(a-c) + \partial_z^\mu \Gamma_\mu^{1,1}(b,c,a) = 0$$

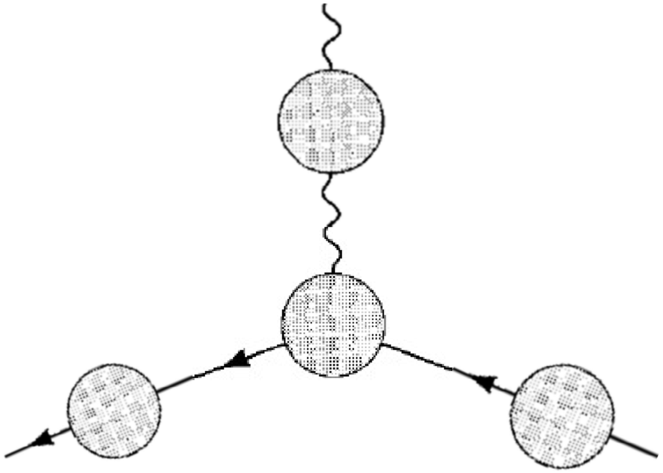
$$\Longrightarrow \quad (p+q)^\mu \Gamma_\mu^{1,1}(p,q) = -e(\not p + \not q) - e\Sigma(-p) + e\Sigma(q)$$



Interpretation:

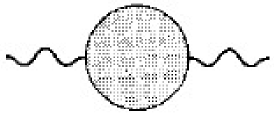
Unlike  **,**

 **and**  **reflect the nature of U(1) charged particle**



In the charge renormalization, without the Ward identity 2, the renormalized charge depends on the nature of the particle carrying the charge

e.g., positron and proton have different charges ?

:In the charge renormalization, 
is crucial.

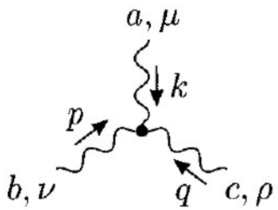
$$eA_\mu \to A_\mu$$

$$-\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu}$$

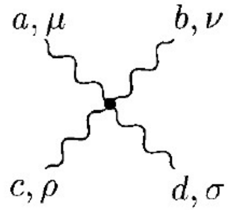
Charge renormalizaion : how Abelian and non-Abelian different

Unlike the Abelian case, non-Abelian allows the gauge boson self-interaction:

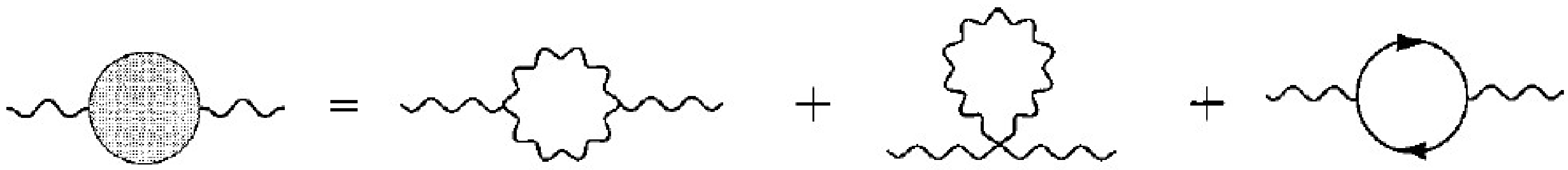
$$-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu)(\partial_\mu A^{a\nu} - \partial_\nu A^{a\mu} + g f_{abc} A^{b\mu} A^{c\nu})$$



$$= \quad g f^{abc}\big[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\big]$$



$$= \quad -ig^2\big[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho}) + f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\sigma}g^{\nu\rho}) + f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma})\big]$$



$$\alpha = \frac{g^2}{4\pi} \qquad\qquad \mu\frac{d}{d\mu}\alpha = -\beta_0\frac{\alpha^2}{4\pi} + \cdots$$

For SU(N) $\beta_0 = \frac{11}{3}N - \frac{2}{3}N_f - \frac{1}{3}N_b$ **N_f : the number of fermionic degrees of freedom**

N_b : the number of bosonic degrees of freedom

Note the contributions from gauge self-interaction and matter have different sign!

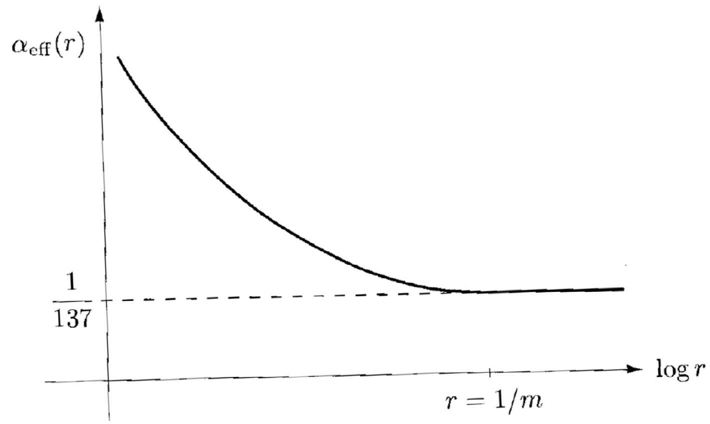
: Asymptotic freedom

$$\alpha \simeq \frac{\alpha_0}{1 + \frac{\beta_0}{2\pi}\alpha_0 \log\frac{\mu}{\Lambda}}$$

$$\alpha \simeq \frac{\alpha_0}{1 + \frac{\beta_0}{2\pi}\alpha_0 \log\frac{\mu}{\Lambda}}$$

For Abelian case,

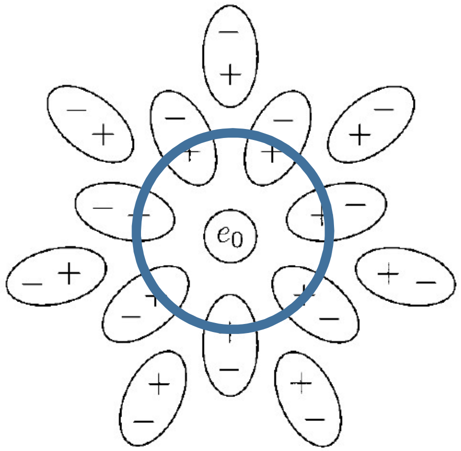
$$\alpha \simeq \frac{\alpha_0}{1 - \frac{1}{3\pi}\alpha_0 \log\frac{\mu}{\Lambda}}$$



1) The force becomes weaker as length scale increases

2) For $1/\Lambda > 1/m$ ($m < \Lambda$) 'electron' is not dynamical : integrated out effective theory for photon only in which coupling is frozen (photon is not charged)

3) Screening by vacuum polarization



dipoles (=pair created particle-antiparticle pair) array to screen the 'bare charge'

cf. In dielectric, effective description of the same effect is (polarization) :

$$\frac{q^2}{4\pi\epsilon_0} \longrightarrow \frac{q^2}{4\pi\epsilon}$$

$$\boxed{q^2 \rightarrow \frac{\epsilon_0}{\epsilon} q^2}$$

Asymptotic freedom in non-Abelian gauge interaction means that anti-screening arises by the gauge boson self-interaction

see, e.g., V. A. Novikov, L. B. Okun, M. A. Shifman, A. I. Vainshtein, M. B. Voloshin, Z. I. Zakharov, Phys. Rep. 41 (1978) 1

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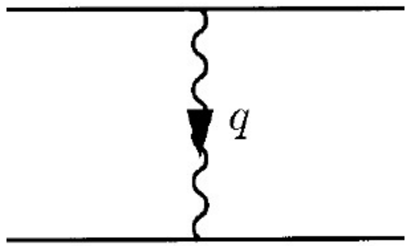


$$\mathcal{M}_0 = \frac{e_0^2}{q^2}\Gamma_\mu^{(1)}\Gamma_\nu^{(2)}\eta^{\mu\nu} = \frac{e_0^2}{q^2}\left(\Gamma_0^{(1)}\Gamma_0^{(2)} - \Gamma_1^{(1)}\Gamma_1^{(2)} - \Gamma_2^{(1)}\Gamma_2^{(2)} - \Gamma_3^{(1)}\Gamma_3^{(2)}\right)$$

$$= \frac{e_0^2}{q^2}\left[\Gamma_0^{(1)}\Gamma_0^{(2)}\left(1 - \frac{q_0^2}{q_3^2}\right) - \sum_{i=1,2}\Gamma_i^{(1)}\Gamma_i^{(2)}\right]$$

$$\boxed{\begin{gathered} q^\mu\Gamma_\mu^{(1)} = q^\mu\Gamma_\mu^{(2)} = 0 \\ q^\mu = (q^0, 0, 0, q^3) \end{gathered}} \Rightarrow \boxed{q_0\Gamma_0^{(1,2)} = q_3\Gamma_3^{(1,2)}}$$

$$q^2 = q_0^2 - q_3^2$$

$$\mathcal{M}_0 = -e^2\left(\underbrace{\frac{1}{q_3^2}\Gamma_0^{(1)}\Gamma_0^{(2)}}_{\text{Coulomb (instantaneous)}} + \underbrace{\frac{1}{q^2}\sum_{i=1,2}\Gamma_i^{(1)}\Gamma_i^{(2)}}_{\text{EM wave propagation}}\right)$$

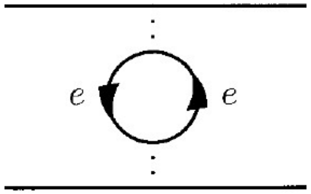
Let • • • • • • • • : Coulomb

– – – – – – – – : transverse propagation

Caution : the instantaneous Coulomb does not violate causality as electric field (gauge invariant quantity) respects causality

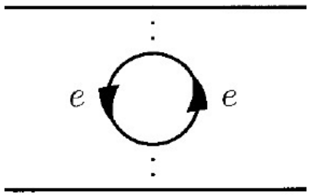
Focusing on Coulomb case only,

Abelian:

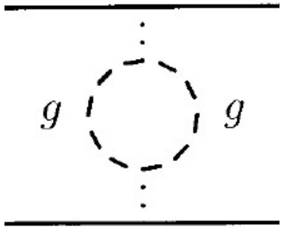


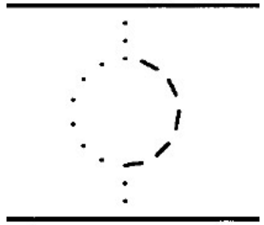
only

Non-Abelian :

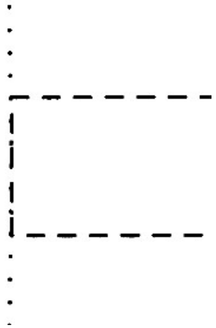


+ self-interaction

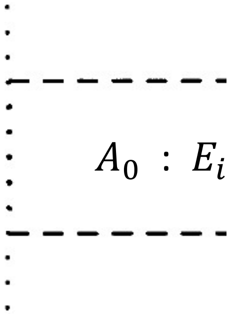




Same type



$A_i \ : \ E_i, B_i$



$A_0 \ : \ E_i$

Note the opposite behaviors between E and B : While the same charges repel, the same currents attract