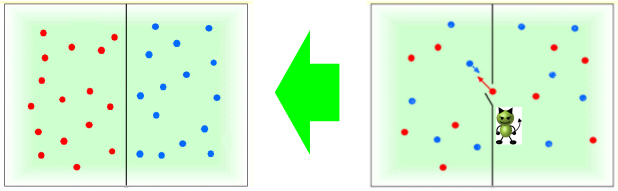
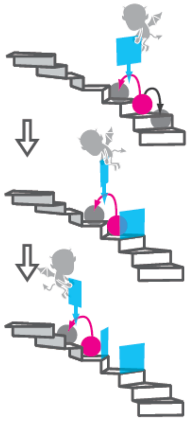


Thermodynamics of information



Takahiro Sagawa

Department of Applied Physics, University of Tokyo

ICTP-KIAS School on Statistical Physics for Life Sciences
3-4 November 2022, Seoul, Korea

Outline

- Introduction
- An introduction to information theory
- Second law of information thermodynamics
- Experimental demonstrations of Maxwell's demon
- Thermodynamics of measurement and erasure
- Entropy production
- General framework of information thermodynamics
- Autonomous Maxwell's demons
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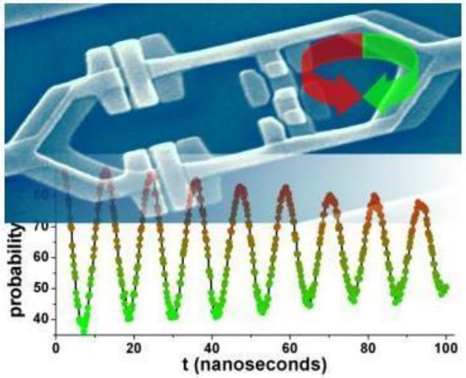
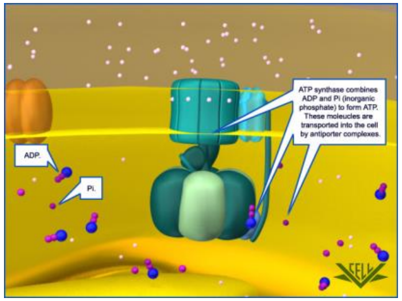
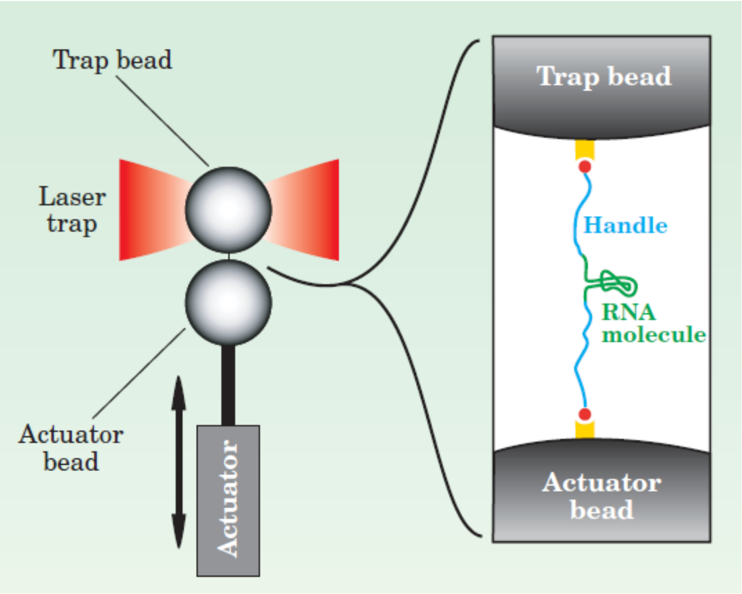
Thermodynamics in the Fluctuating World

Thermodynamics of small systems with large heat bath(s)

➡ Thermodynamic quantities are fluctuating!





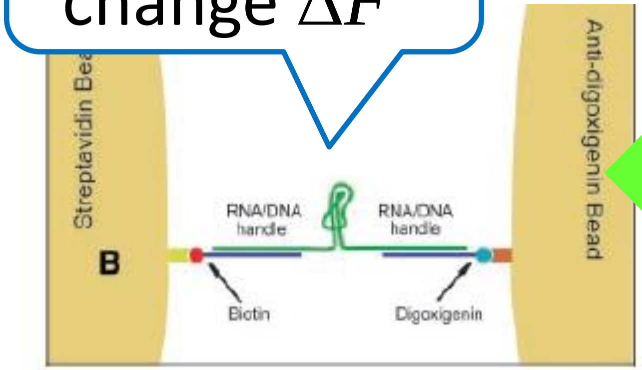


- ✓ **Second law** $\langle W \rangle \geq \Delta F$
- ✓ **Nonlinear & nonequilibrium relations**

Second Law in Small Systems

with a large heat bath

Free-energy change ΔF

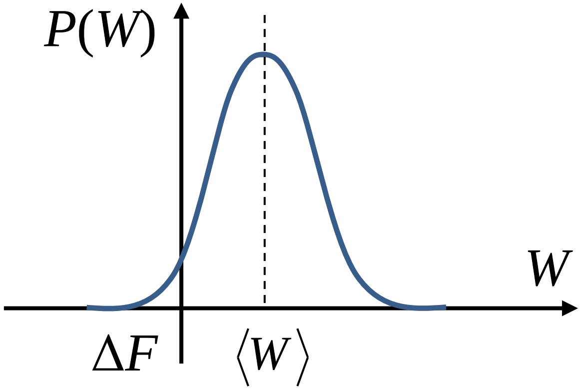


Work W

Drive the system from equilibrium

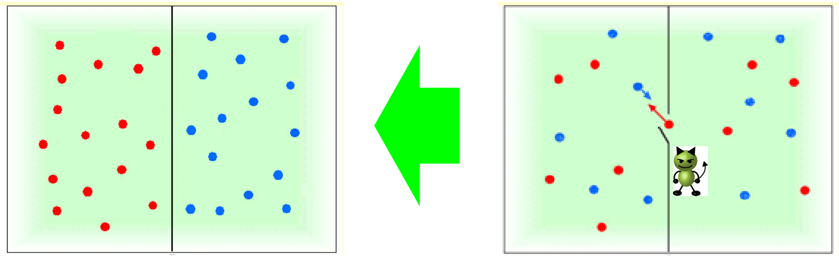
W becomes stochastic due to thermal fluctuations

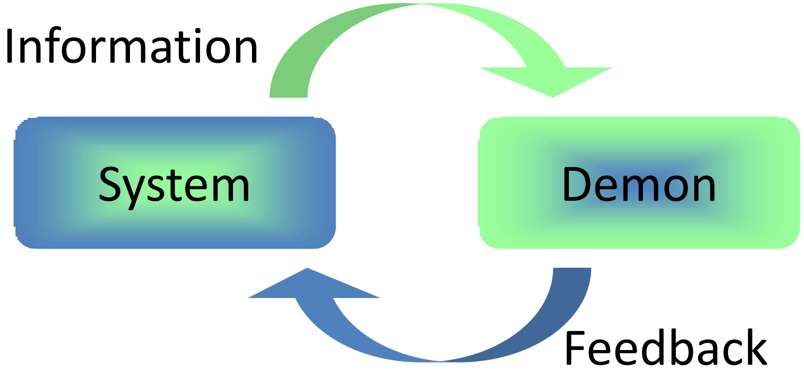
$$\langle W \rangle \geq \Delta F$$ on average



$W < \Delta F$ can occur with a small probability **(stochastic violation of the second law)**

Information Thermodynamics




Information

System

Demon

Feedback

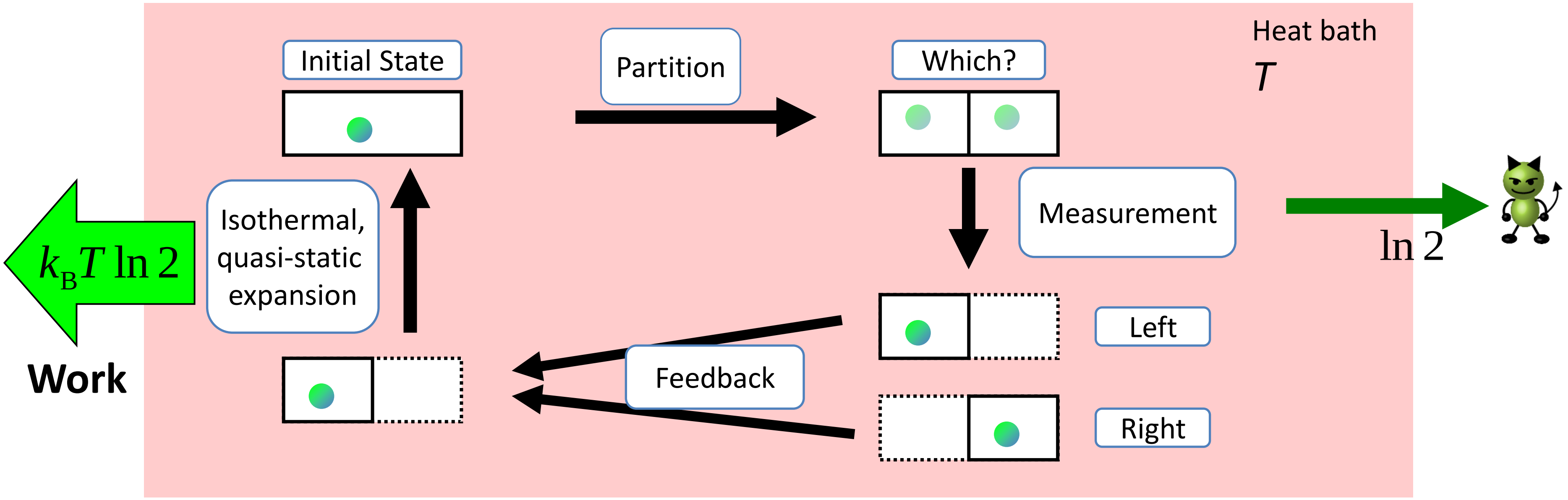
Information processing at the level of thermal fluctuations

- ✓ Foundation of the second law of thermodynamics
- ✓ Application to nanomachines and nanodevices

Review: J. M. R. Parrondo, J. M. Horowitz, & T. Sagawa, *Nature Physics* **11**, 131-139 (2015).

Szilard Engine (1929)

L. Szilard, Z. Phys. **53**, 840 (1929)



Heat bath

T

Initial State

Partition

Which?

Measurement

$\ln 2$

Left

Right

Feedback

Isothermal, quasi-static expansion

$k_B T \ln 2$

Work

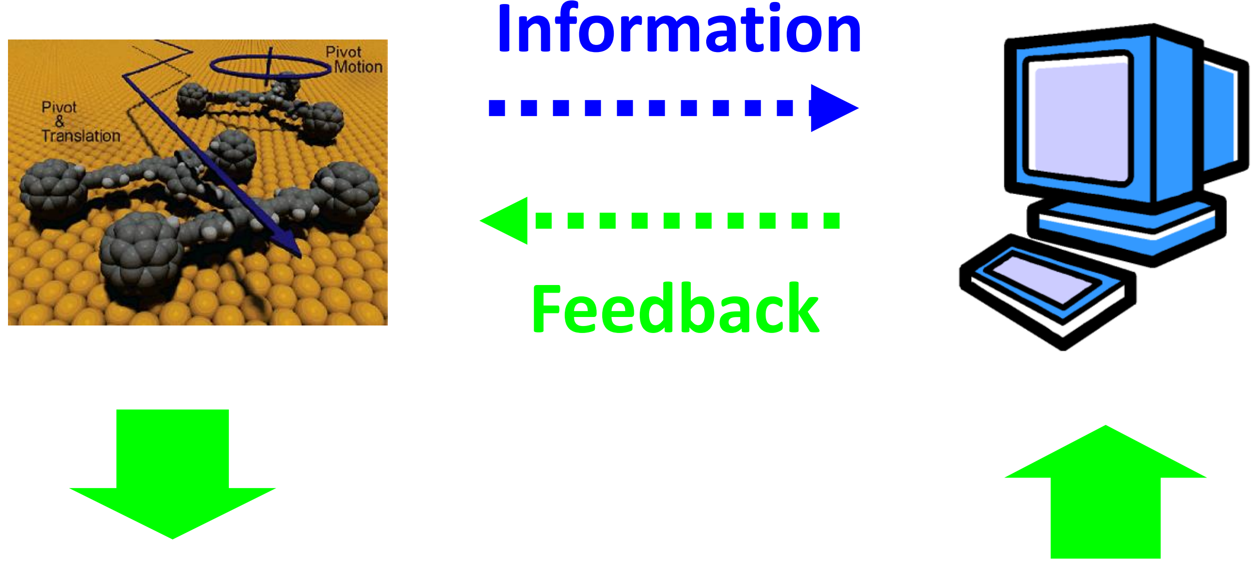
Free energy: $F = E - TS$

Increase

Decrease by feedback

Can control physical entropy by using information

Information Heat Engine



System
(Working engine)

Memory
(Controller)

Information

Feedback

Free energy / work

Entropic cost

✓ **Can increase the system's free energy even if there is no energy flow between the system and the controller**

Progress in Thermodynamics of Information

- Maxwell (1860's)
- Szilard (1920's)
- Brillouin (1950's)
- Landauer (1960's)
- Bennett (1980's)

Fundamental insights,
based on thought experiments

- Modern theory (2000's -)
- Experiments (2010 -)

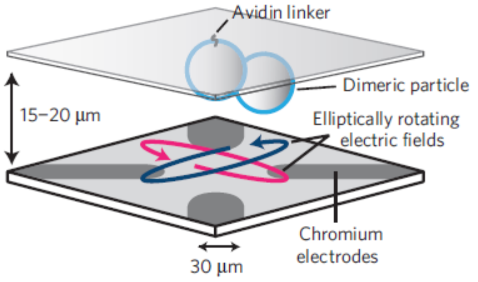
General theory based on modern
nonequilibrium statistical mechanics
(fluctuation theorem etc.)

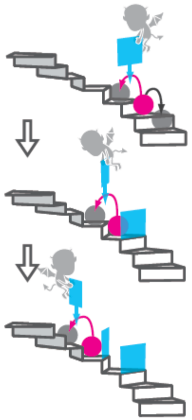
Experimental Demonstrations

- With a colloidal particle
  Toyabe, TS, Ueda, Muneyuki, & Sano, Nature Physics (2010)

  Efficiency: 30%

  Validation of $\left\langle e^{-\beta(W-\Delta F)} \right\rangle = \gamma$

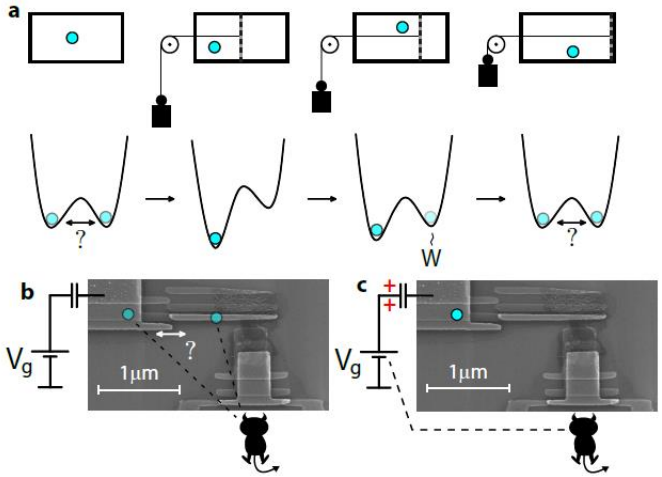




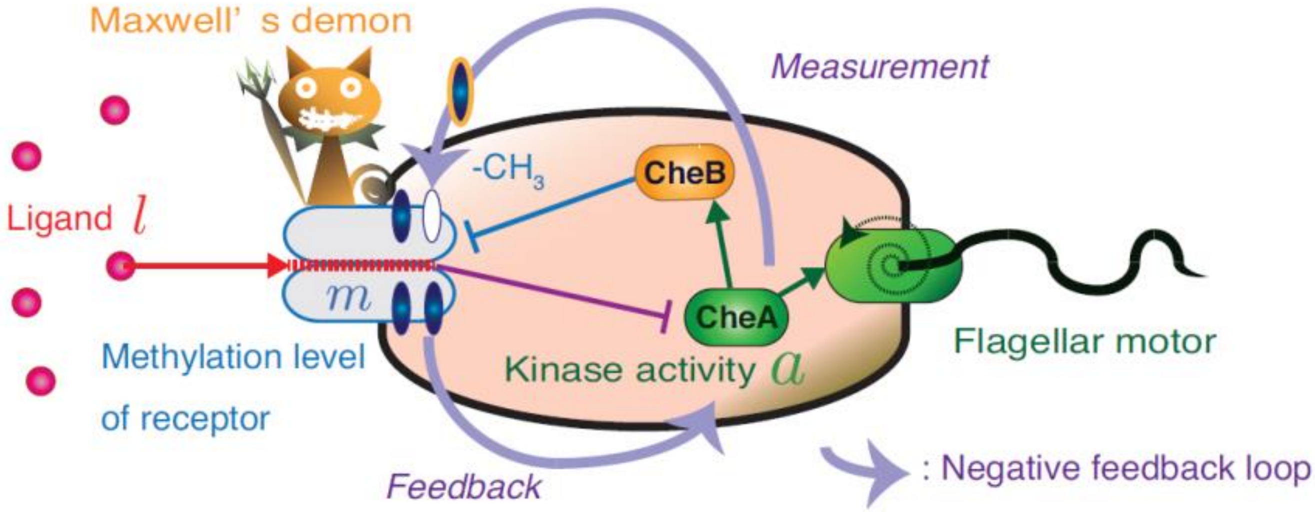
- With a single electron
  Koski, Maisi, TS, & Pekola, PRL (2014)

  Efficiency: 75%

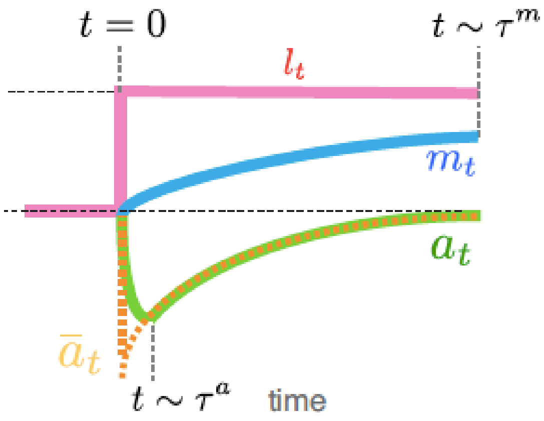
  Validation of $\left\langle e^{-\beta(W-\Delta F)-I} \right\rangle = 1$



Application to *E. Coli* Chemotaxis

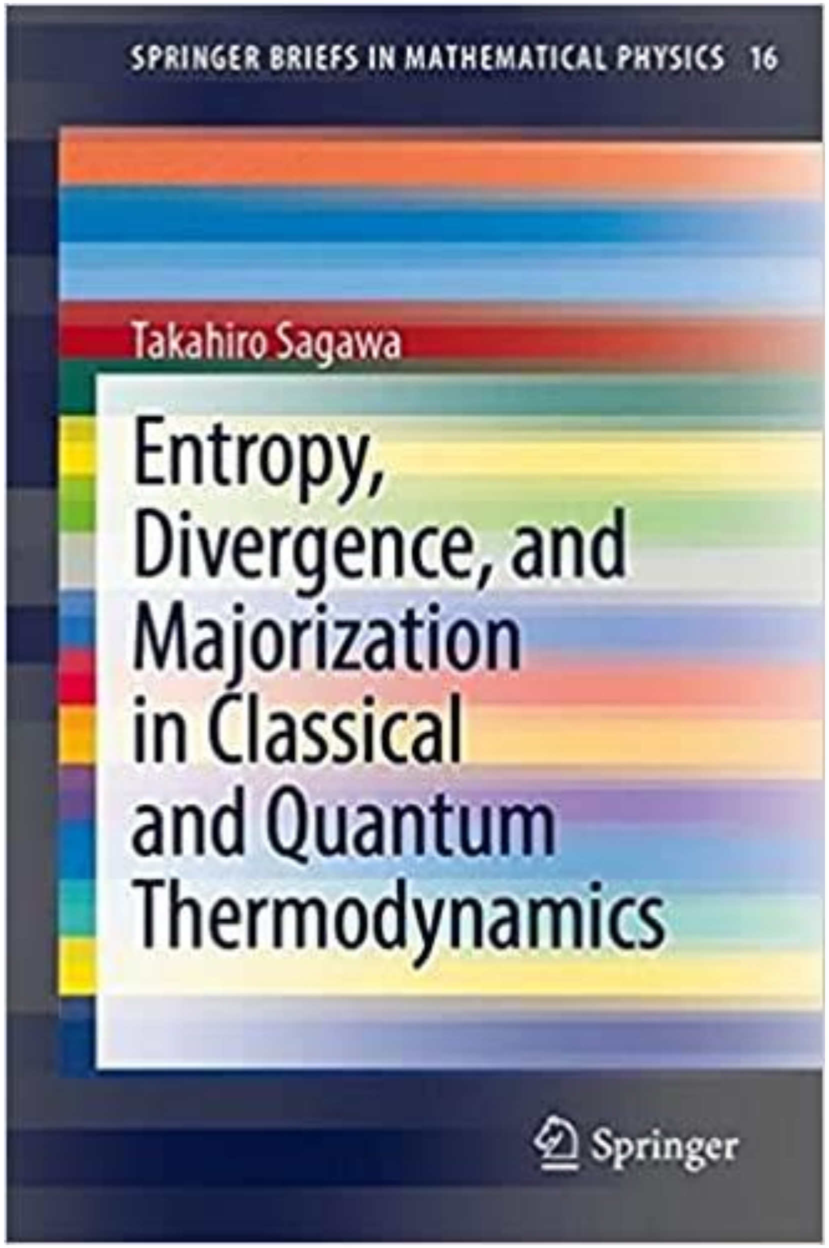


E. Coli moves toward food (ligand)



Information thermodynamics can characterize **robustness** of chemotaxis against environmental noise

S. Ito & T. Sagawa, Nat. Commu. **6**, 7498 (2015)



SPRINGER BRIEFS IN MATHEMATICAL PHYSICS 16

Takahiro Sagawa

Entropy,
Divergence, and
Majorization
in Classical
and Quantum
Thermodynamics

Springer

Now on sale!

arXiv:2007.09974

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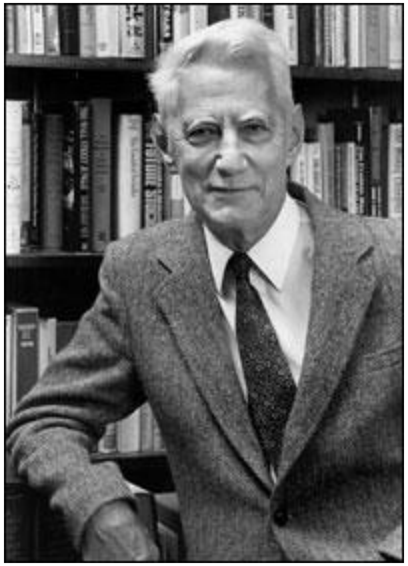
Claude Shannon（1916-2001）

Father of information science



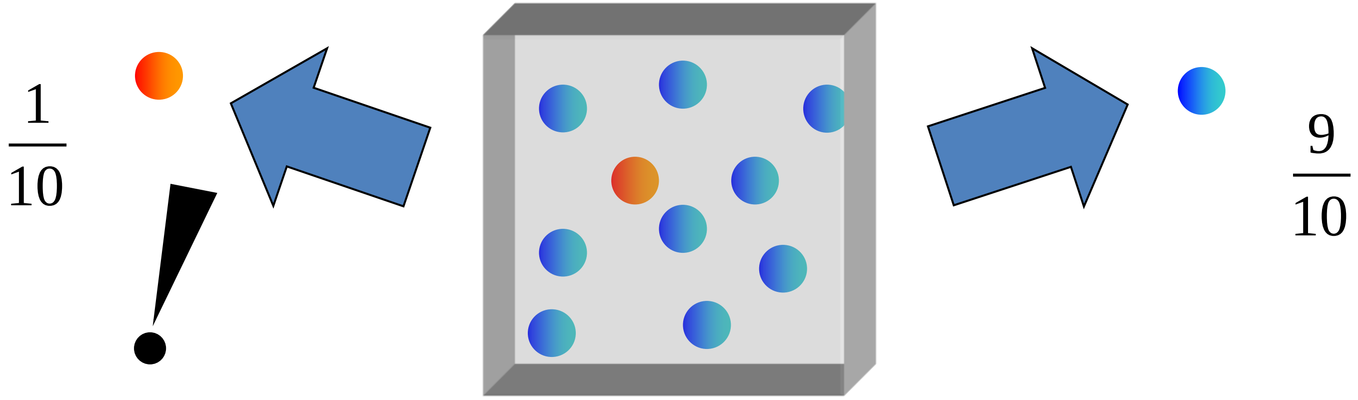
Established information theory in his 1948 papers (Introduced and analyzed the Shannon information and the mutual information)

C. Shannon, Bell System Technical Journal **27**, 379-423 and 623-656 (1948).

Standard textbooks of information theory:

- T. M. Cover and J. A. Thomas, “*Elements of Information Theory*” (John Wiley and Sons, New York, 1991).
- M. A. Nielsen and I. L. Chuang, “*Quantum Computation and Quantum Information*” (Cambridge University Press, Cambridge, 2000).

Shannon Information (1)



Information content with event k: $\ln \frac{1}{p_k}$

Average

Shannon information: $H = \sum_k p_k \ln \frac{1}{p_k}$

Shannon Information (2)

$$0 \le H \le \ln N$$

N: the number of k's

$p_i = 1$ for a single i

$p_k = 0$ for $k \neq i$

$p_k = 1/N$

for all k

$$H = -\sum_k p_k \ln p_k$$

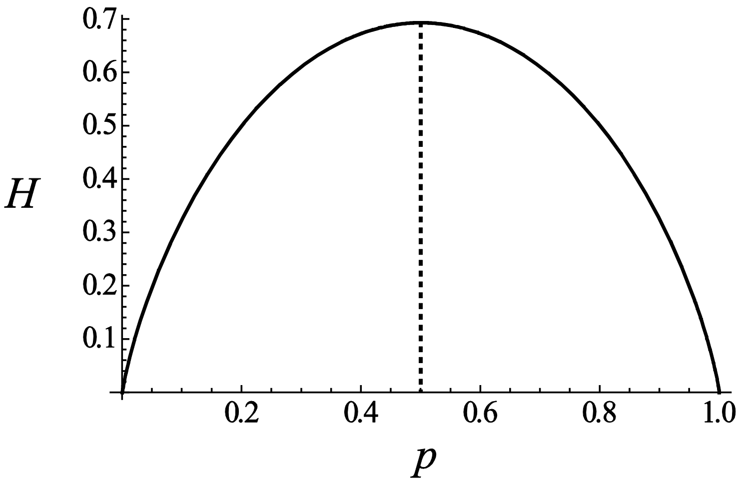
Ex. Binary system

"0": probability p

"1": probability 1-p

$$H = -p \ln p - (1-p)\ln(1-p)$$

$$0 \le H \le \ln 2$$



Shannon Entropy and Free Energy

Canonical distribution:

$$P[x] = \frac{e^{-\beta E[x]}}{Z}$$

Partition function:

$$Z = \sum_x e^{-\beta E[x]}$$

Free energy:

$$F = -k_{\mathrm{B}} T \ln Z$$

Averaged energy:

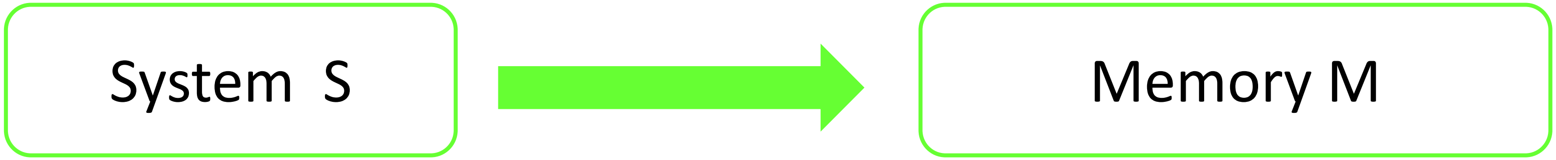
$$\langle E \rangle = \sum_x E[x] P[x]$$

⇒ $$\langle E \rangle - F = -k_{\mathrm{B}} T \sum_x P(x) \ln P(x)$$

⇔ $$\langle E \rangle - F = TS$$

The Shannon entropy is consistent with thermodynamic entropy

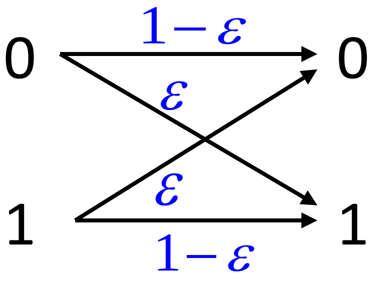
Mutual Information (1)



System S

Memory M

Measurement or communication
(with stochastic error, in general)



Ex. Binary symmetric channel

$p(s)$: distribution of the measured state of S

$p(m)$: distribution of the outcome in M

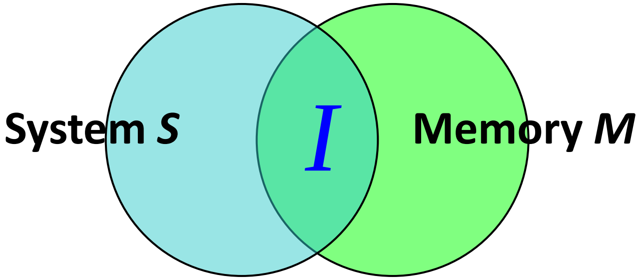
$p(m \mid s)$: conditional probability characterizing the error

$p(s,m) = p(m \mid s)\, p(s)$: joint distribution of S and M

Mutual Information (2)

$$I(S:M) \equiv H(S) + H(M) - H(SM)$$

$$= \sum_{sm} p(s,m) \ln \frac{p(s,m)}{p(s)p(m)}$$



$p(s)$: distribution of the measured state of S

$p(m)$: distribution of the outcome in M

$p(m|s)$: conditional probability characterizing the error

$p(s,m) = p(m|s)p(s)$: joint distribution of S and M

$$H(S) = -\sum_s p(s) \ln p(s) \quad H(M) = -\sum_m p(m) \ln p(m) \quad H(SM) = -\sum_{sm} p(s,m) \ln p(s,m)$$

Mutual Information (3)

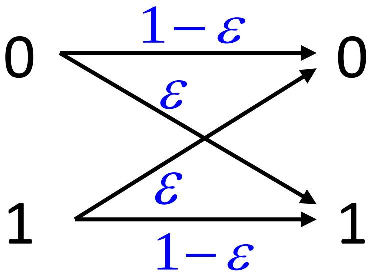
$$I(S:M) \equiv H(S) + H(M) - H(SM)$$

Correlation between S and M

$$0 \leq I \leq H(M)$$

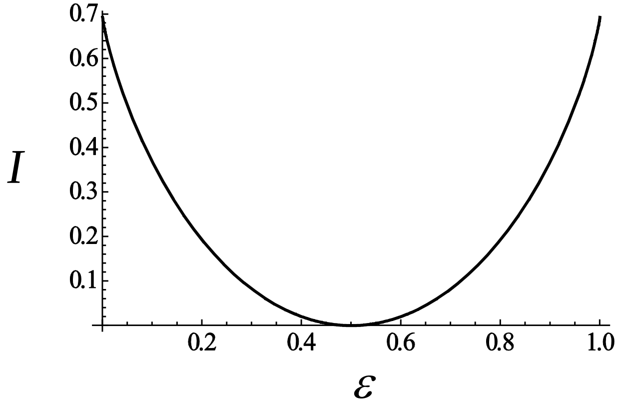
No information

No error



Ex. Binary symmetric channel

$$I = \ln 2 + \varepsilon \ln \varepsilon + (1-\varepsilon)\ln(1-\varepsilon)$$



Continuous Variable (1)

$P(x)dx$: probability $P(x)$: probability density

$$H = -\int dx P(x) \ln P(x)$$

Transformation of the variable: $P(x)dx = P(x')dx'$

$$\begin{aligned} H' &= -\int dx' P(x') \ln P(x') \\ &= -\int dx P(x) \ln P(x) + \int dx P(x) \ln\left|\frac{dx}{dx'}\right| \\ &\neq H \end{aligned}$$

Not invariant!

Continuous Variable (2)

Mutual information:

$$I = \int dxdyP(x,y)\ln\frac{P(x,y)}{P(x)P(y)}$$

$$= \int dx'dy'P(x',y')\ln\frac{P(x',y')}{P(x')P(y')}$$

Invariant!

Continuous Variable (3)

Ex. Gaussian channel

$$x \xrightarrow{\text{Gaussian noise}} y \qquad P(y \mid x) = \frac{1}{\sqrt{2\pi N}} \exp\left(-\frac{(y-x)^2}{2N} \right)$$

Input:

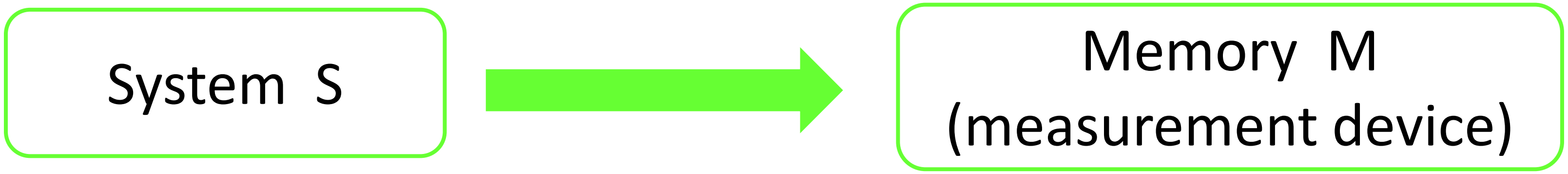
Output:

$$P(x) = \frac{1}{\sqrt{2\pi S}} \exp\left(-\frac{x^2}{2S} \right) \quad \Rightarrow \quad P(y) = \frac{1}{\sqrt{2\pi (S+N)}} \exp\left(-\frac{y^2}{2(S+N)} \right)$$

Mutual information:

$$I = \frac{1}{2} \ln\left(1 + \frac{S}{N} \right)$$

Summary



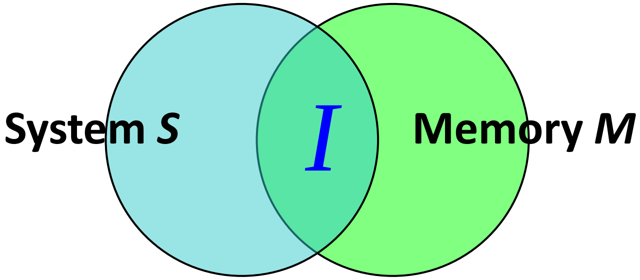
System S

Memory M

(measurement device)

Measurement with stochastic errors

$$I(S:M)\equiv H(S)+H(M)-H(SM)$$



$$0\leq I\leq H(M)$$

No information

Error-free

Shannon information:

Randomness of the system

Mutual information:

Correlation between the system and the memory

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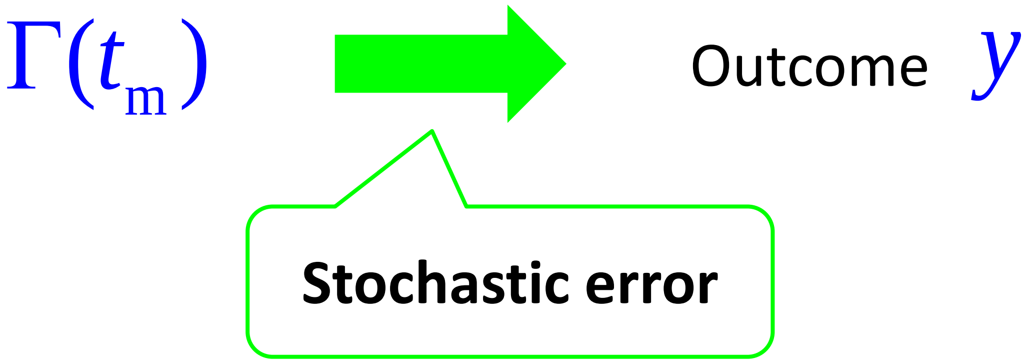
Measurement on Thermodynamic System

Classical stochastic dynamics with phase-space point Γ
(in contact with heat bath at $\beta = (k_{\mathrm{B}}T)^{-1}$)

Measurement at time t_{m}



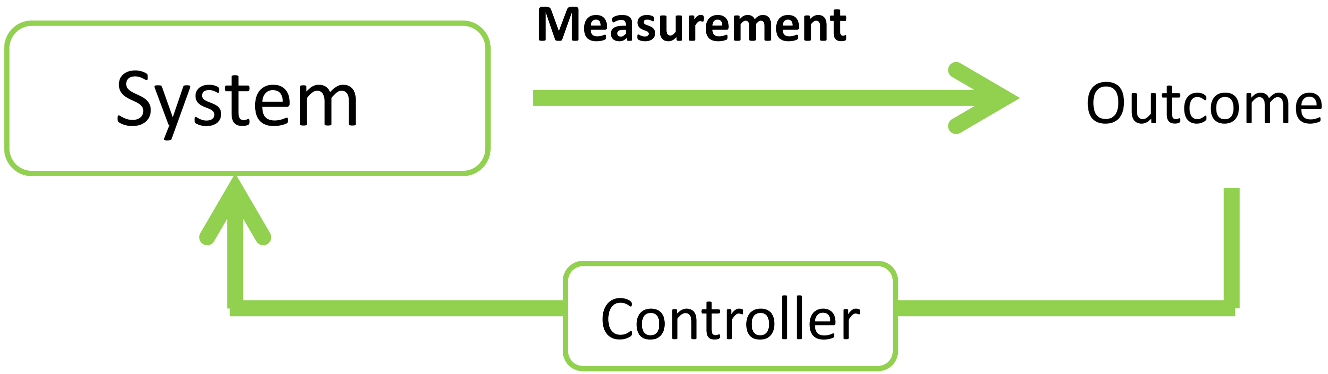
$\Gamma(t_{\mathrm{m}})$ Outcome y

Stochastic error

Conditional probability: $P[y \,|\, \Gamma(t_{\mathrm{m}})] \neq 0$

Ex. Gaussian noise $P[y \,|\, \Gamma(t_{\mathrm{m}})] \propto \exp(-(y - \Gamma(t_{\mathrm{m}}))^2 / 2N)$

General Schematics of Feedback Control



System

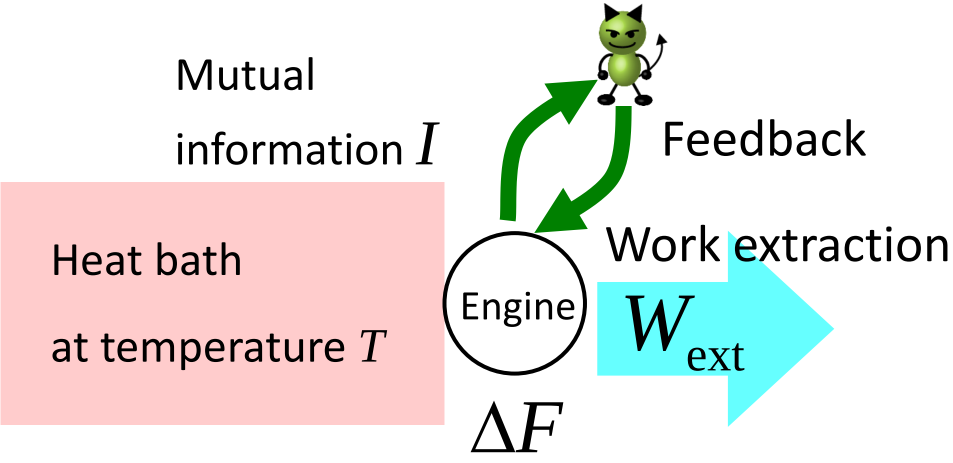
Measurement

Outcome

Controller

Control protocol can depend on outcome after the measurement

Upper Bound of Extractable Work with Feedback



Mutual information I

Feedback

Heat bath

at temperature T

Engine

ΔF

Work extraction

W_{ext}

$$W_{\mathrm{ext}} \equiv -\langle W \rangle$$

TS and M. Ueda, PRL **104**, 090602 (2010).

$$\Rightarrow \quad W_{\mathrm{ext}} \leq -\Delta F + k_{\mathrm{B}} T I$$

The upper bound of the work extracted by the demon is bounded by the mutual information.

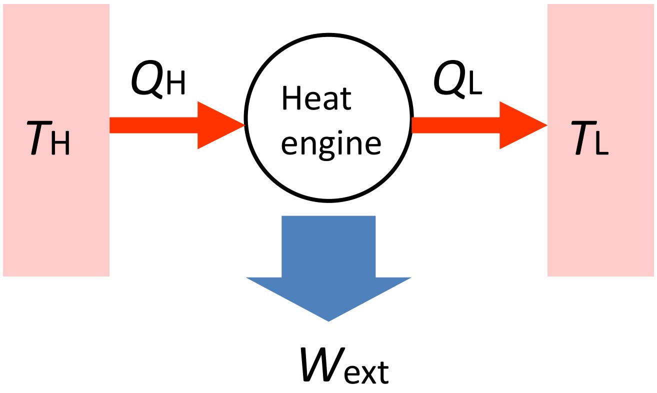
The equality can be achieved:

K. Jacobs, PRA **80**, 012322 (2009)
J. M. Horowitz & J. M. R. Parrondo, EPL **95**, 10005 (2011)
D. Abreu & U. Seifert, EPL **94**, 10001 (2011)
J. M. Horowitz & J. M. R. Parrondo, New J. Phys. **13**, 123019 (2011)
T. Sagawa & M. Ueda, PRE **85**, 021104 (2012)
M. Bauer, D. Abreu & U. Seifert, J. Phys. A: Math. Theor. **45**, 162001 (2012)

Information Heat Engine

Conventional heat engine:

Heat → Work



Heat efficiency

$$e \equiv \frac{W_{\mathrm{ext}}}{Q_{\mathrm{H}}} \le 1 - \frac{T_{\mathrm{L}}}{T_{\mathrm{H}}}$$

Carnot cycle

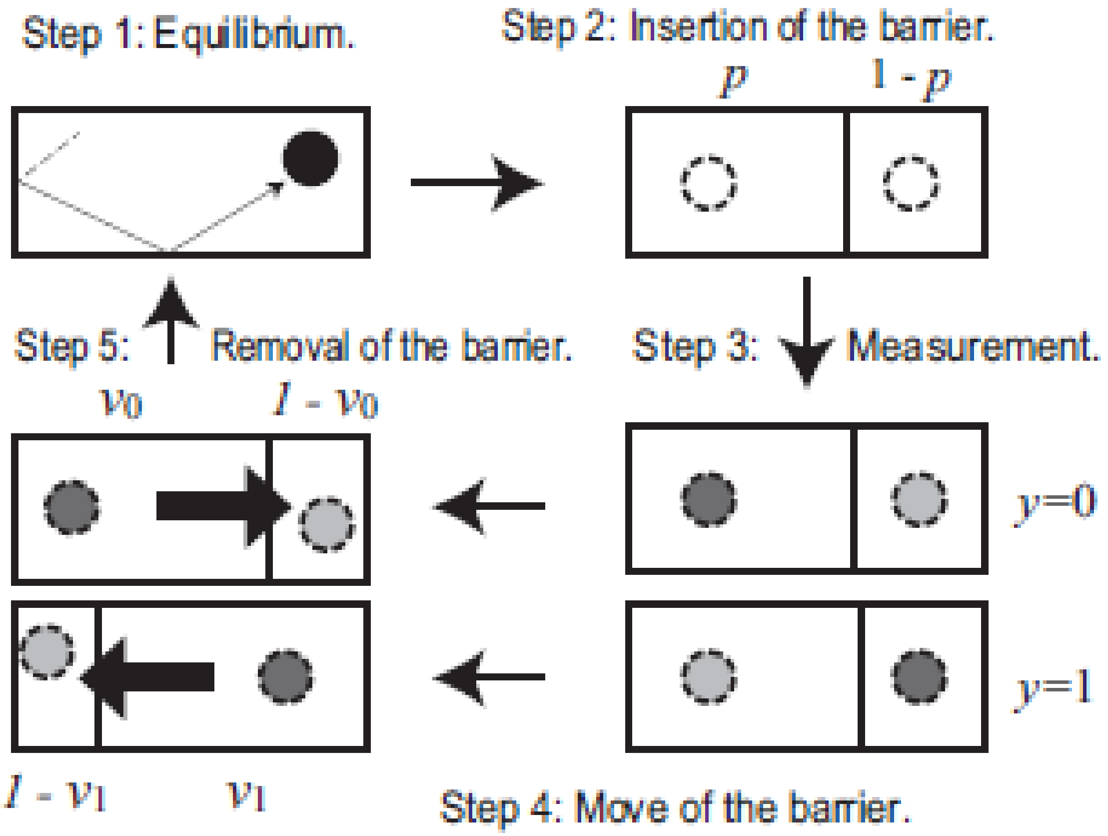
Information heat engine:

Mutual information → Work and Free energy

$$W_{\mathrm{ext}} + \Delta F \le k_{\mathrm{B}} T I$$

Szilard engine

Generalized Szilard Engine (1)



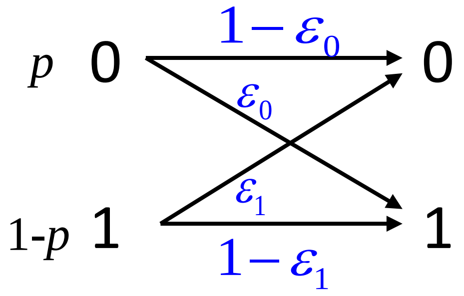
Szilard engine

$(v_0, v_1)=(1,1)$

$\varepsilon_0 = \varepsilon_1 = 0$

$p=1/2$

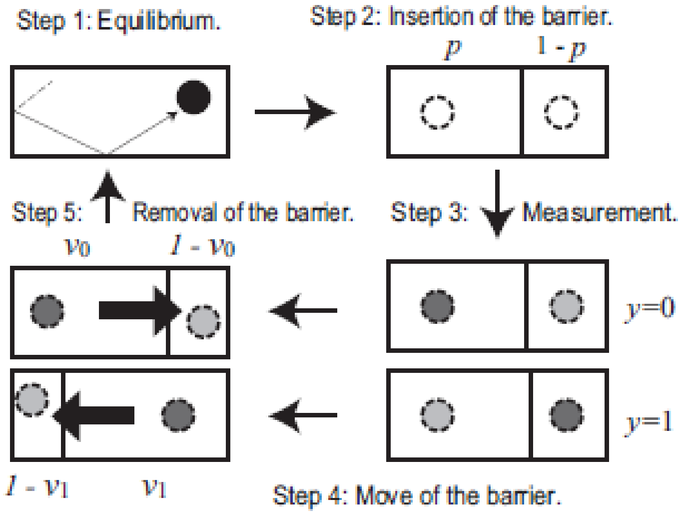
Error rates $\varepsilon_0, \varepsilon_1$



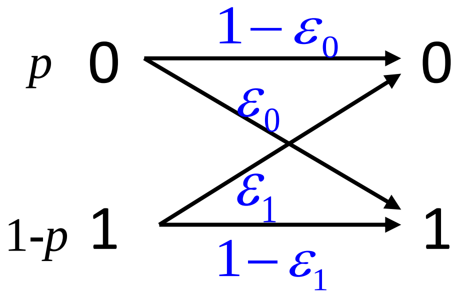
Feedback protocol is determined by (v_0, v_1)

T. Sagawa & M. Ueda, PRE **85**, 021104 (2012)

Generalized Szilard Engine (2)



Error rates ε_0, ε_1



$$I = H(Y) - pH(\varepsilon_0) - (1-p)H(\varepsilon_1)$$

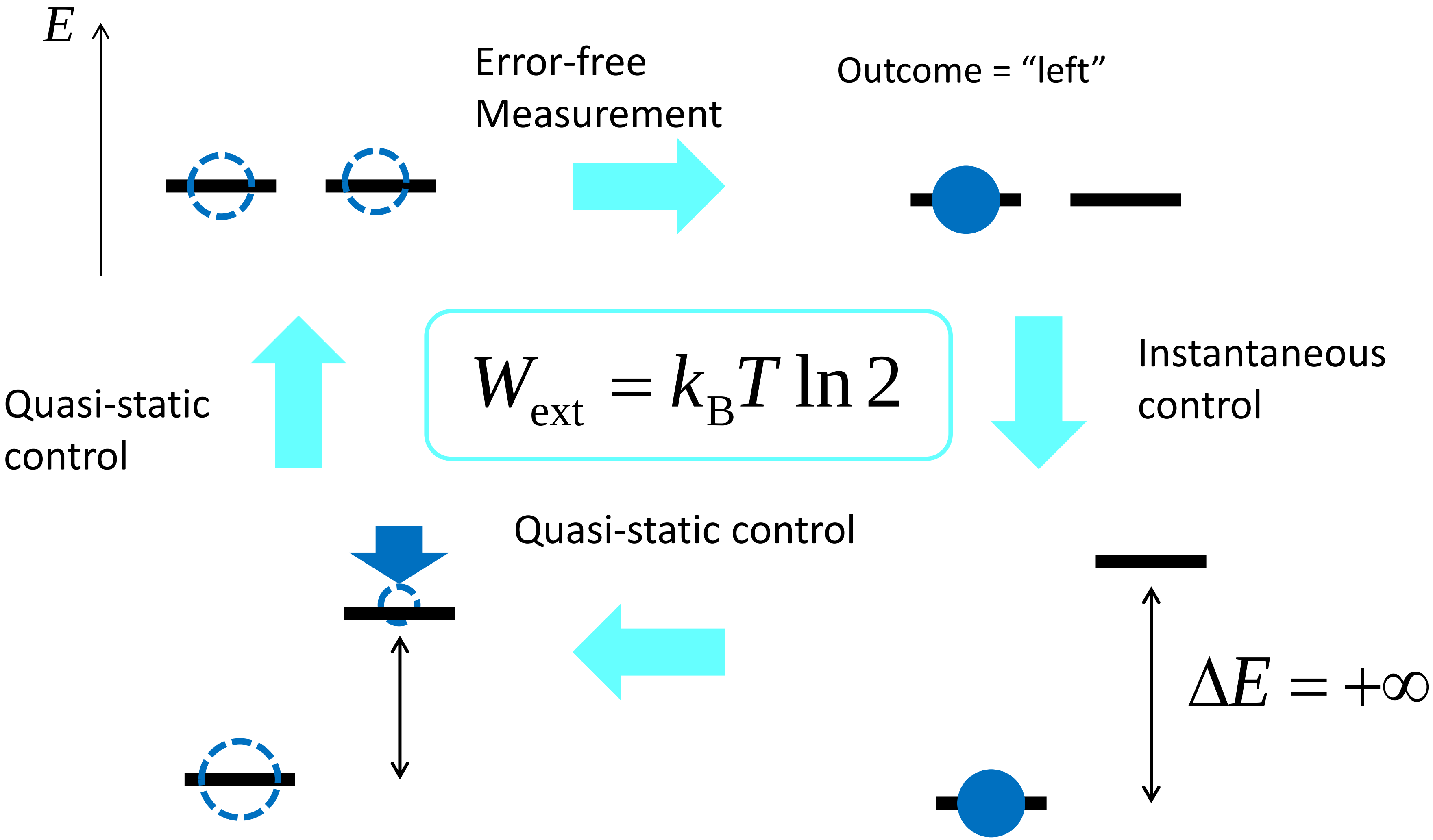
Work extraction:

$$\frac{W_{\mathrm{ext}}}{k_{\mathrm{B}}T} = p(1-\varepsilon_0)\ln\frac{v_0}{p} + p\varepsilon_0\ln\frac{1-v_1}{p} + (1-p)\varepsilon_1\ln\frac{1-v_0}{1-p} + (1-p)(1-\varepsilon_1)\ln\frac{v_1}{1-p}.$$

Maximization ($\partial W_{\mathrm{ext}}/\partial v_0 = 0$ and $\partial W_{\mathrm{ext}}/\partial v_1 = 0$):

$$\boxed{W_{\mathrm{ext}} = k_{\mathrm{B}}TI}$$

Szilard-like Engine with Two Sites (1)



Error-free Measurement

Outcome = “left”

Instantaneous control

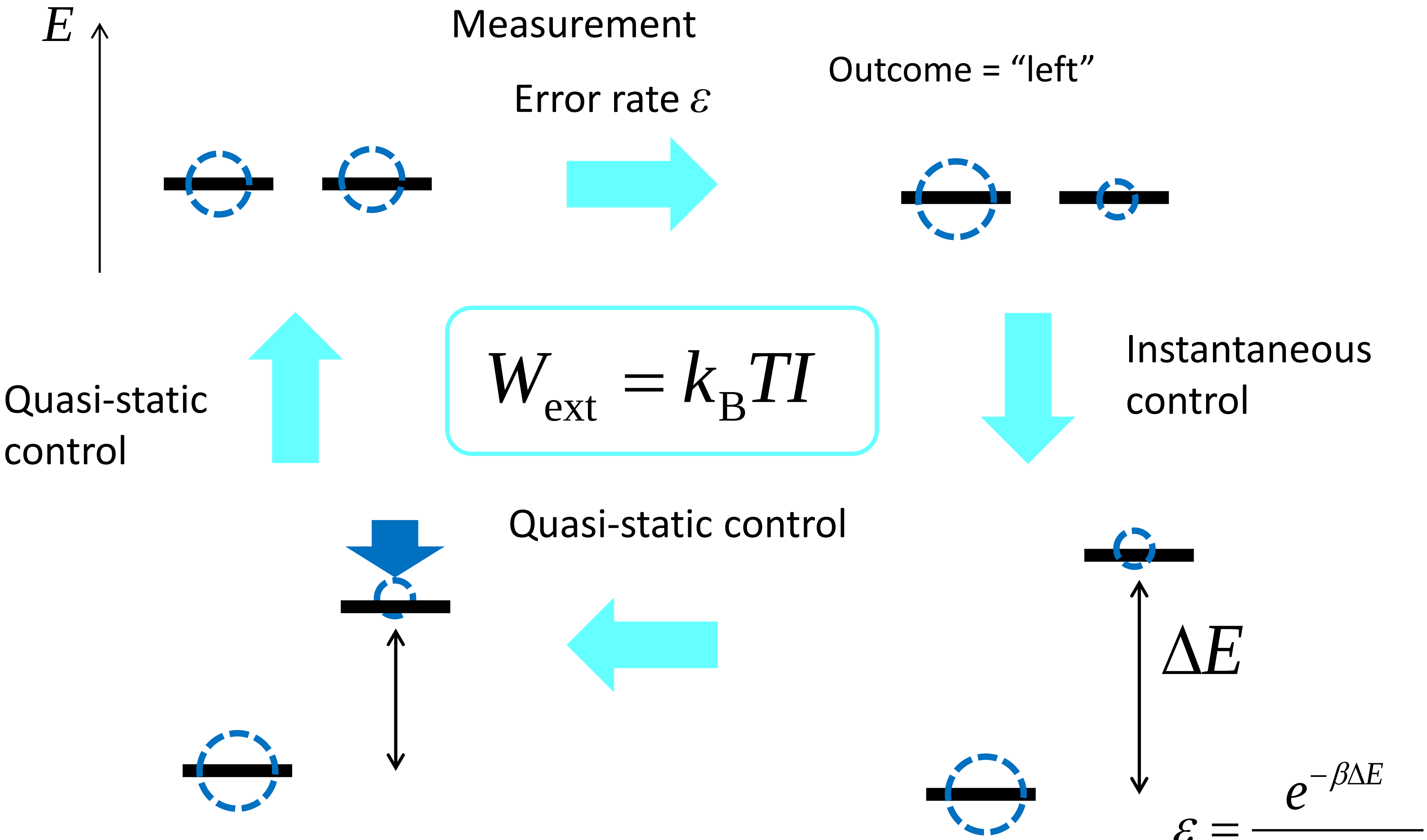
$$\Delta E = +\infty$$

Quasi-static control

Quasi-static control

$$\boxed{W_{\text{ext}} = k_{\text{B}} T \ln 2}$$

Szilard-like Engine with Two Sites (2)



E

Measurement

Error rate ε

Outcome = “left”

$$W_{\text{ext}} = k_{\text{B}} T I$$

Instantaneous control

ΔE

$$\varepsilon = \frac{e^{-\beta \Delta E}}{1 + e^{-\beta \Delta E}}$$

Quasi-static control

Quasi-static control

Horowitz & Parrondo, *EPL* **95** 10005 (2011)

Langevin System (1)

Overdamped Langevin equation with a harmonic potential

$$\eta \frac{dx}{dt} = -\lambda_1 (x - \lambda_2) + \xi$$

Gaussian white noise $\langle \xi(t)\xi(t') \rangle = 2\eta k_{\mathrm{B}} T \delta(t - t')$

$$V(x, \lambda_1, \lambda_2) = \frac{\lambda_1}{2}(x - \lambda_2)^2$$

D. Abreu and U. Seifert, Europhys Lett. **94**, 10001 (2011).

Langevin System (2)

Step 1: Equilibrium with

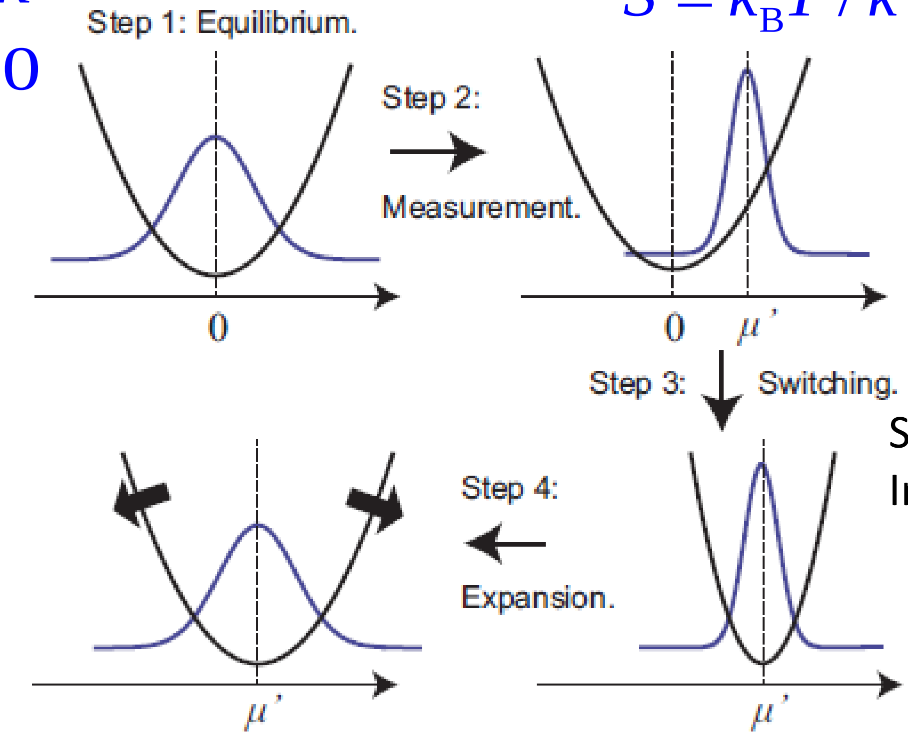
$$\lambda_1 = k$$

$$\lambda_2 = 0$$

Step 2: Measurement with Gaussian noise:

$S = k_{\mathrm{B}}T / k$: variance of the initial equilibrium

N : noise intensity



Step 3:
Instantaneous switching to

$$\lambda_1 = k' = k(1 + S / N)$$

$$\lambda_2 = \mu' = Sy / (S + N)$$

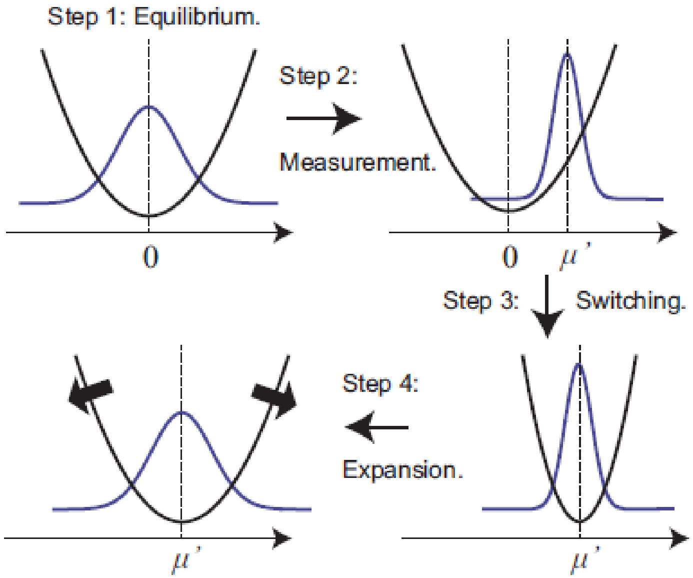
Step 4: Quasi-static expansion to

$\lambda_1 = k \qquad \lambda_2 = \mu'$

(unchanged)

$$V(x, \lambda_1, \lambda_2) = \frac{\lambda_1}{2}(x - \lambda_2)^2$$

Langevin System (3)



Work extraction:

$$W_{\text{ext}}^{(3)} = \int [V(x,k,0) - V(x,k',\mu_y)]\, P[x,0|y]P[y]dxdy = 0$$

$$\begin{aligned} W_{\text{ext}}^{(4)} &= -\int_0^\tau dt \frac{d\lambda_1(t)}{dt} \int dxdy P[x,t|y]P[y] \frac{\partial V}{\partial \lambda_1}(x,\lambda_1(t),\lambda_2(t)) \\ &= -\frac{1}{2}\int_{k'}^{k} d\lambda_1 \int dxdy P[x,t|y]P[y](x-\mu_y)^2 = -\frac{1}{2}\int_{k'}^{k} d\lambda_1 \frac{k_{\text{B}}T}{\lambda_1} \\ &= \frac{k_{\text{B}}T}{2} \ln\frac{k'}{k} = \frac{k_{\text{B}}T}{2} \ln\left(1+\frac{S}{N}\right), \end{aligned}$$

$$W_{\text{ext}} := W_{\text{ext}}^{(3)} + W_{\text{ext}}^{(4)}$$

Mutual information

$$I = \frac{1}{2}\ln\left(1+\frac{S}{N}\right)$$

$$\boxed{W_{\text{ext}} = k_{\text{B}}TI}$$

Jarzynski Equality (1997)

$$\left\langle e^{-\beta(W-\Delta F)} \right\rangle = 1$$

C. Jarzynski, PRL **78**, 2690 (1997)

Second law can be expressed by an **equality** with full cumulants

1st cumulant: the second law $\langle W \rangle \geq \Delta F$

2nd cumulant: a fluctuation-dissipation theorem

$$\langle W \rangle - \Delta F = \frac{\beta}{2}\left(\langle W^2 \rangle - \langle W \rangle^2\right)$$

if the work distribution is Gaussian.

Jarzynski Equality with Feedback?

Without feedback:

$$\langle W \rangle \geq \Delta F \quad \Longleftarrow \quad \left\langle e^{-\beta(W-\Delta F)} \right\rangle = 1$$

With feedback:

$$\langle W \rangle \geq \Delta F - k_{\mathrm{B}} T I \quad \Longleftarrow \quad \left\langle e^{-\beta(W-\Delta F)-I} \right\rangle = 1$$

T. Sagawa & M. Ueda, PRL **104**, 090602 (2010)

Generalized Jarzynski Equality

With feedback control $$\left\langle e^{-\beta(W-\Delta F)-I} \right\rangle = 1$$

Stochastic mutual information:

$$I[y,\Gamma(t_{\mathrm{m}})] \equiv \ln\left(P[y \mid \Gamma(t_{\mathrm{m}})] / P[y] \right)$$

Mutual information:

$$\left\langle I \right\rangle = \int I[y,\Gamma(t_{\mathrm{m}})] P[y,\Gamma(t_{\mathrm{m}})] dy d\Gamma(t_{\mathrm{m}})$$

$P[y]$: probability of obtaining outcome y $P[y,\Gamma(t_{\mathrm{m}})]$: joint probability

➡ Reproduce the generalized second law:

$$\left\langle W \right\rangle \geq \Delta F - k_{\mathrm{B}} T \left\langle I \right\rangle$$

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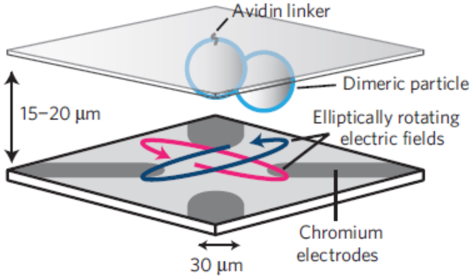
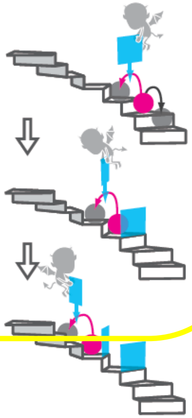
Experimental Demonstrations

- With a colloidal particle
  Toyabe, TS, Ueda, Muneyuki, & Sano, Nature Physics (2010)

  Efficiency: 30%

  Validation of $\left\langle e^{-\beta(W-\Delta F)} \right\rangle = \gamma$

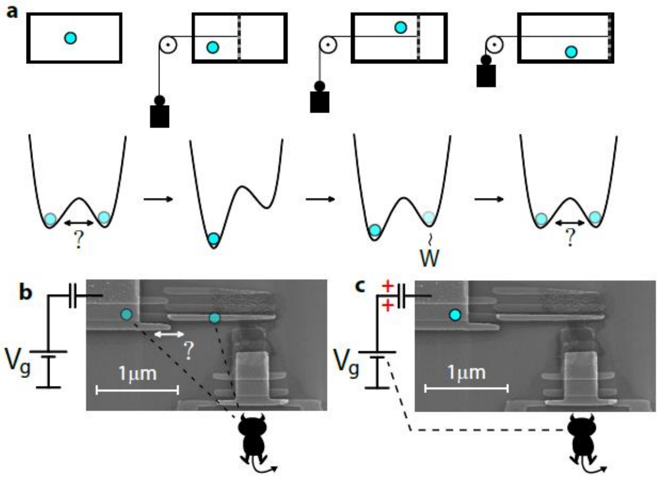




- With a single electron
  Koski, Maisi, TS, & Pekola, PRL (2014)

  Efficiency: 75%

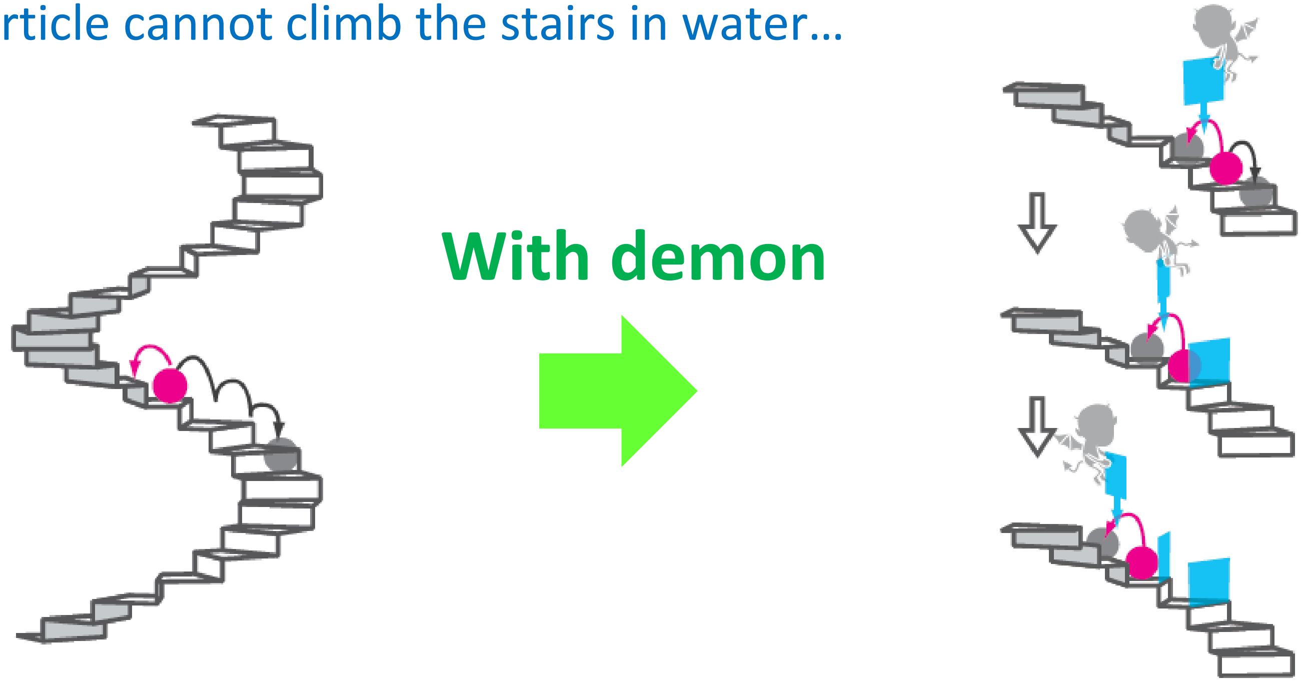
  Validation of $\left\langle e^{-\beta(W-\Delta F)-I} \right\rangle = 1$



Idea of Experiment

How to realize information heat engines?

A Brownian particle cannot climb the stairs in water…

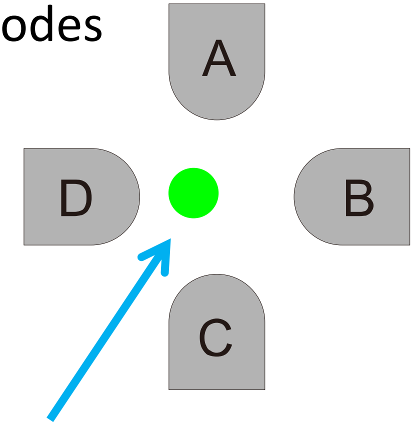


With demon

The demon gets information about the position of the particle, and gets it to climb stairs.

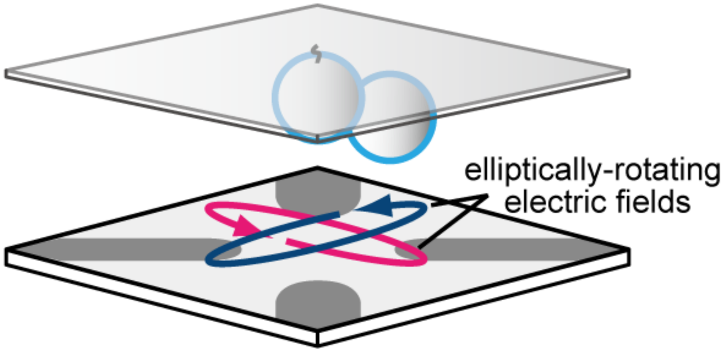
Setup

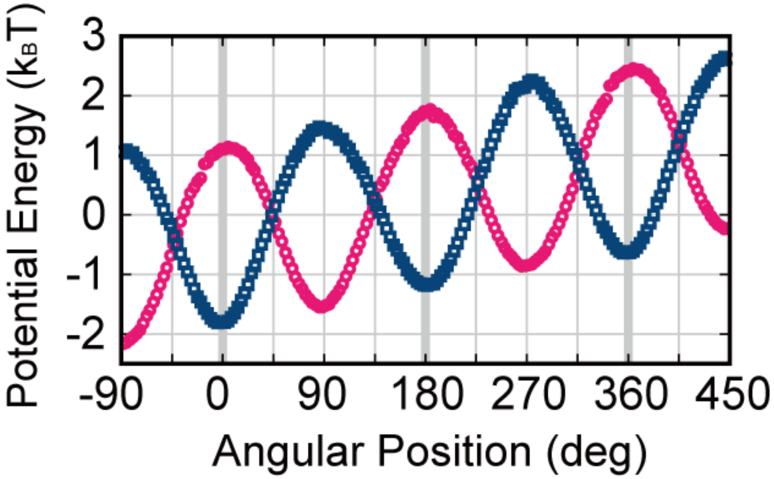
A-D: electrodes



Rotating Brownian particle
(a 287nm polystyrene bead)

S. Toyabe, TS, M. Ueda, E. Muneyuki, and
M. Sano, Nature Physics **6**, 988 (2010).

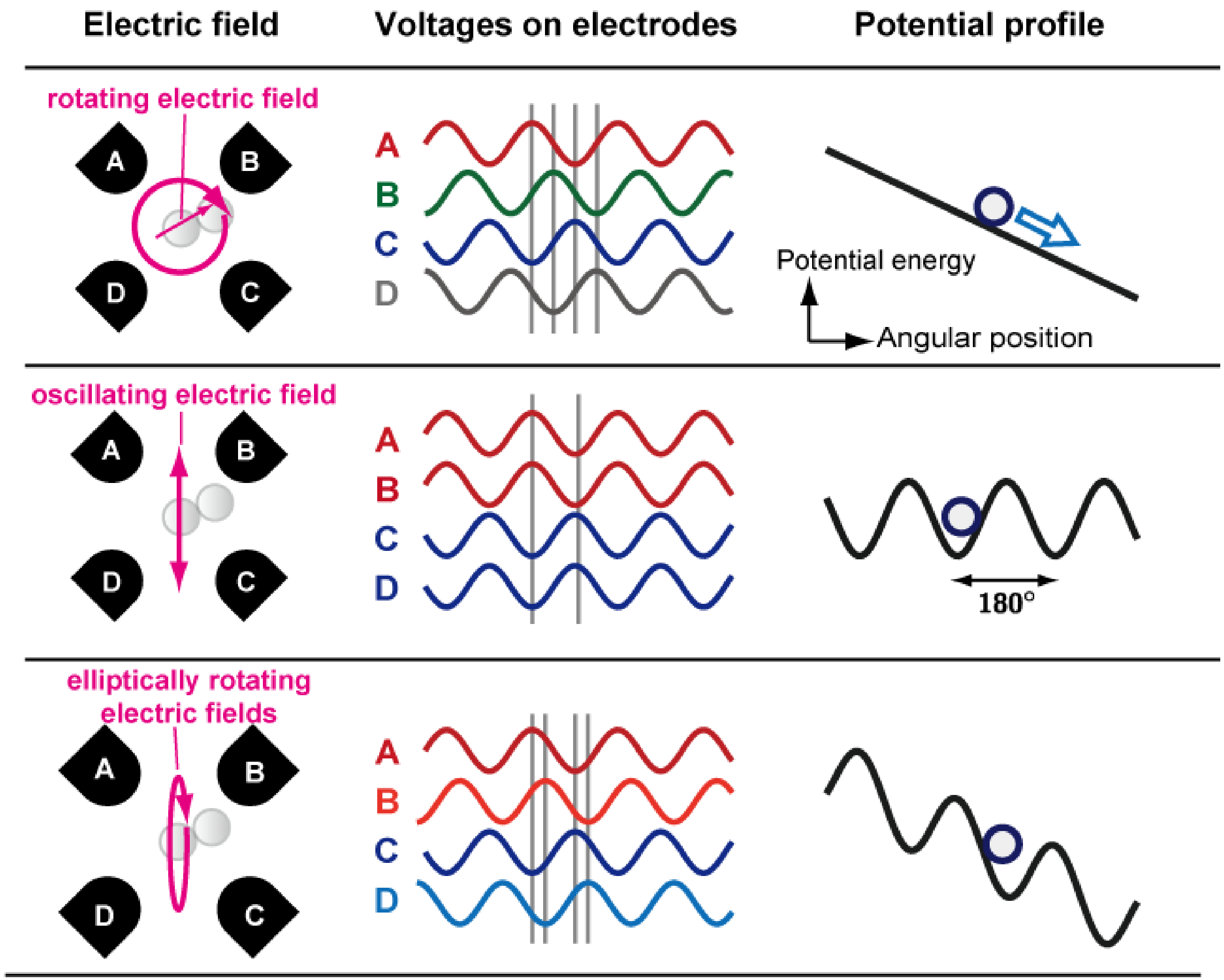




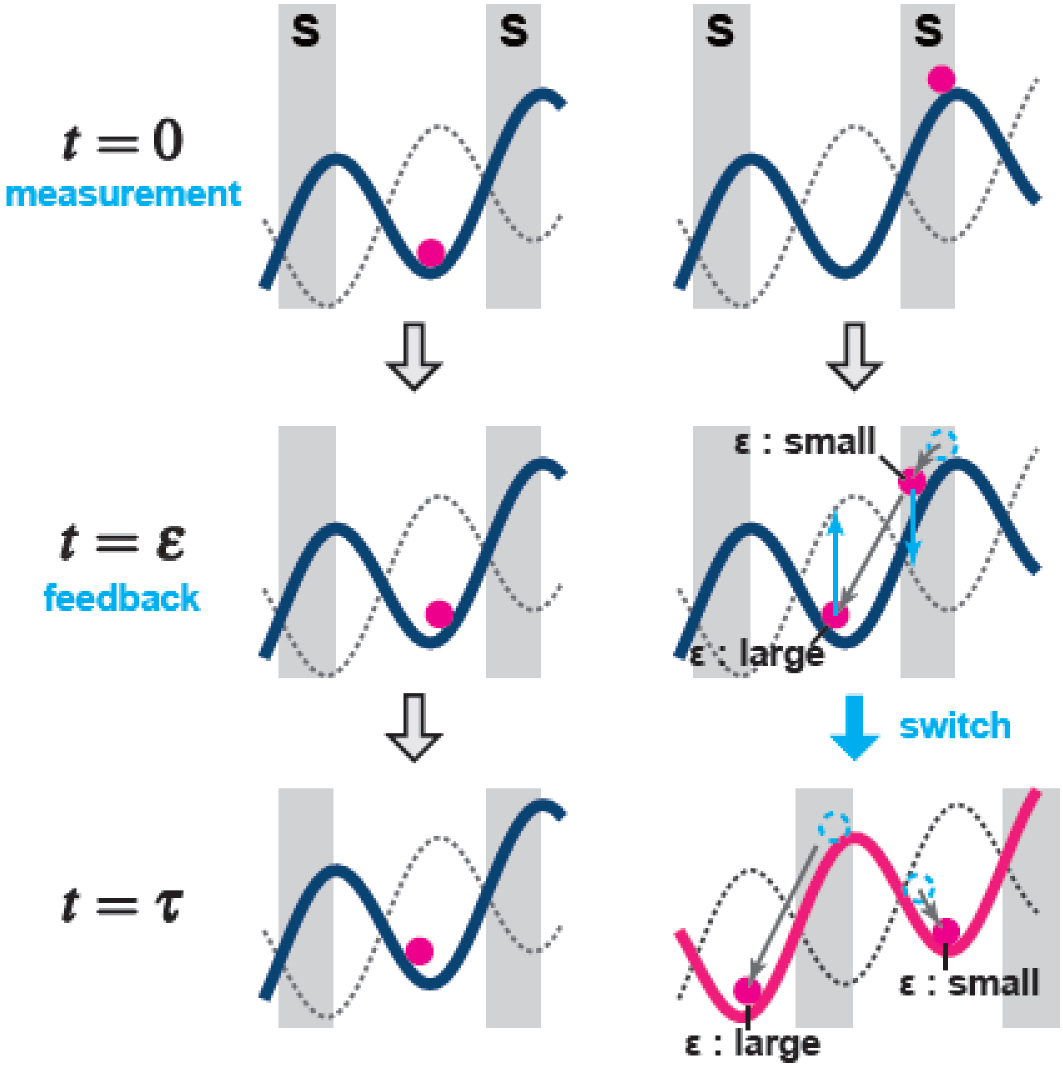
Peak-to-peak = 3.0 kT
Slope = 1.0 kT/rotation

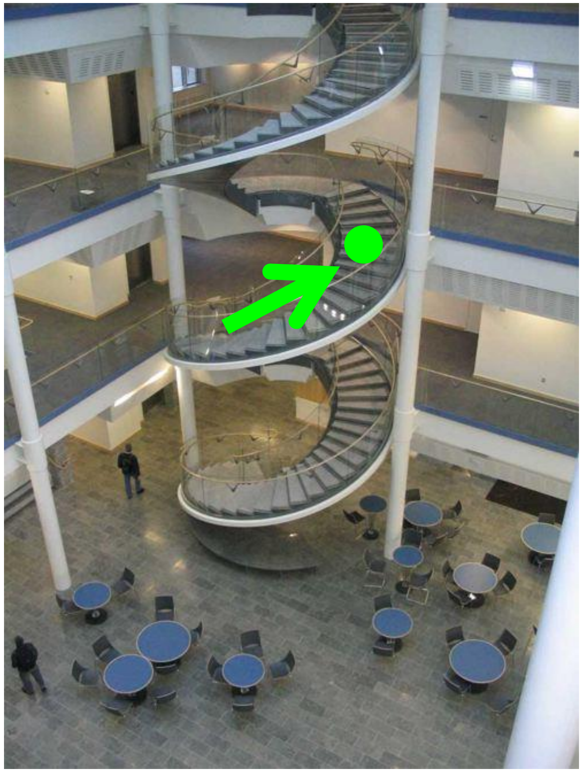
Realized a spiral-stairs-like potential

How To Make Potential

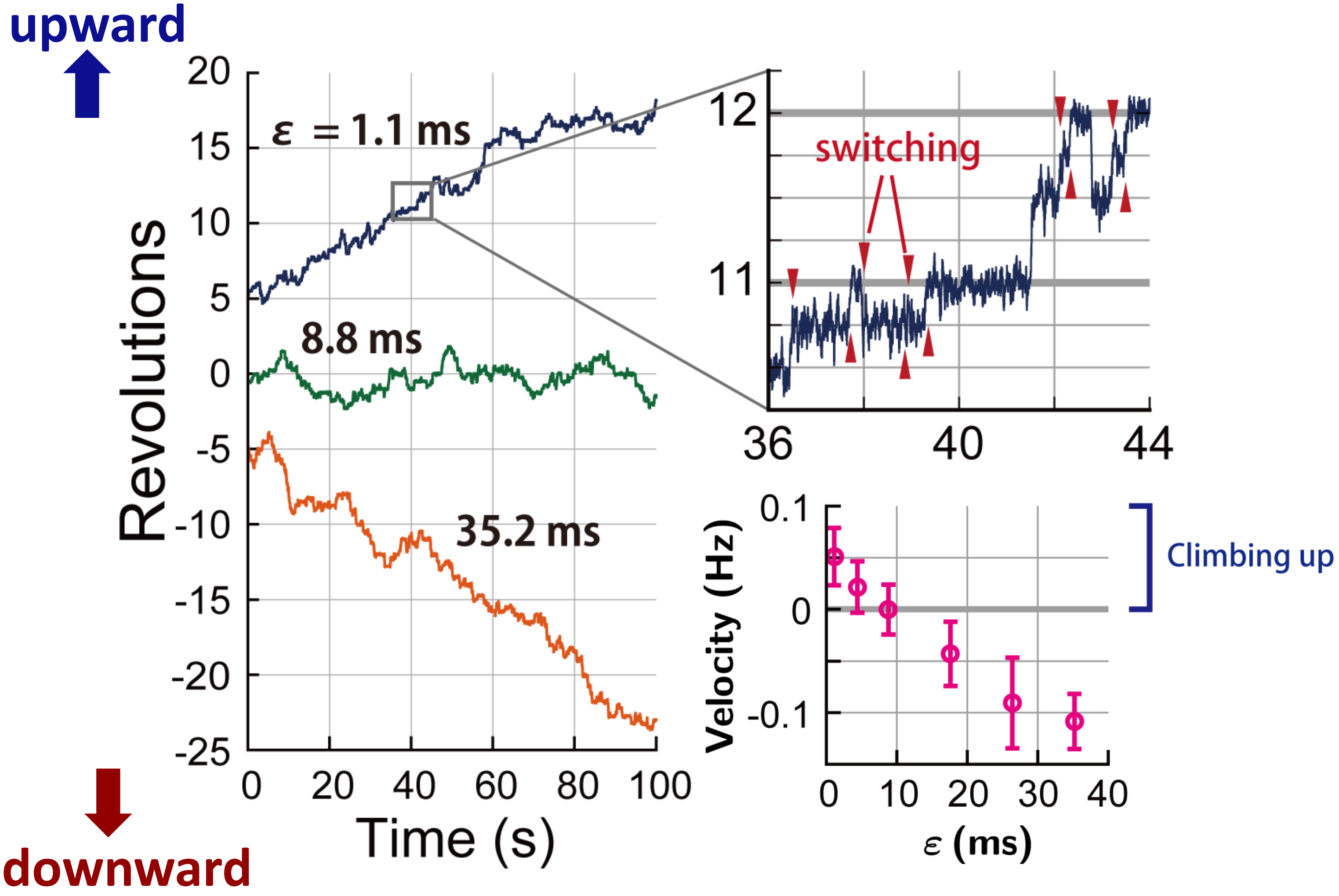


Feedback Protocol



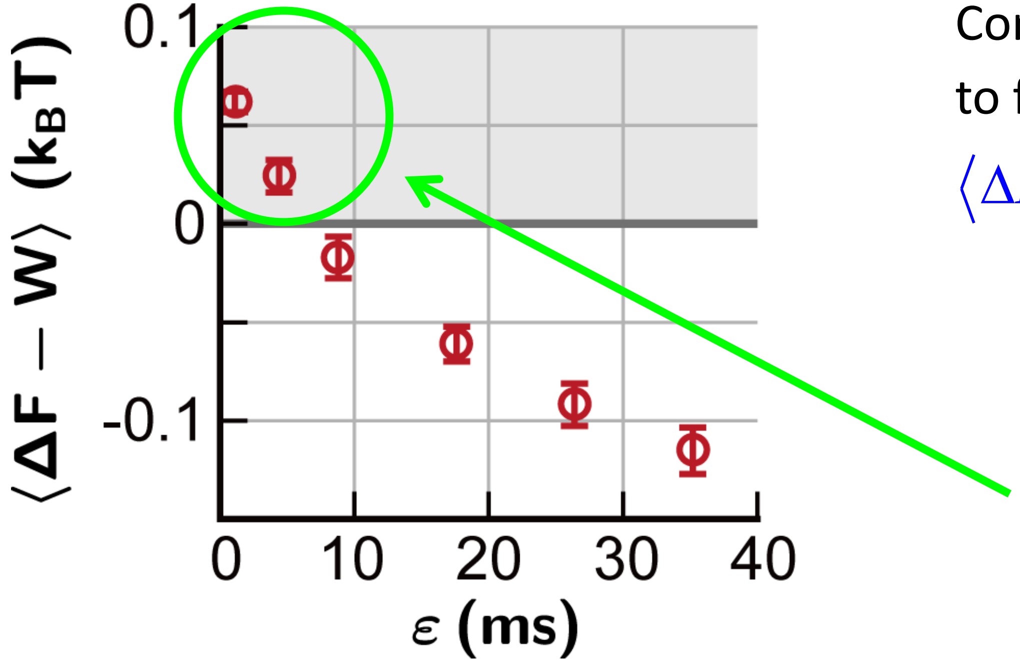


Experimental Result: Dynamics



S. Toyabe, TS, M. Ueda, E. Muneyuki, and M. Sano, Nature Physics **6**, 988 (2010).

Experimental Result: Energetics



Conversion rate from information to free energy is about 28%.

$$\langle \Delta F - W \rangle = 0.062 k_B T \qquad I = 0.22$$

$$\langle \Delta F - W \rangle \leq k_B T I$$

Extracted more work than the conventional bound.

Realized the Szilard-type Maxwell's demon!

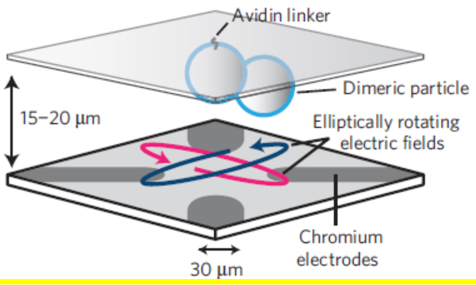
S. Toyabe, TS, M. Ueda, E. Muneyuki, and M. Sano, Nature Physics **6**, 988 (2010).

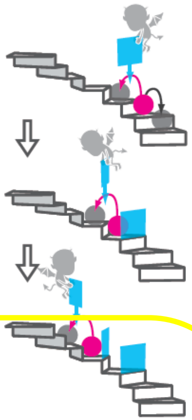
Experimental Demonstrations

- With a colloidal particle
  Toyabe, TS, Ueda, Muneyuki, & Sano, Nature Physics (2010)

Efficiency: 30%

Validation of $\left\langle e^{-\beta(W-\Delta F)} \right\rangle = \gamma$

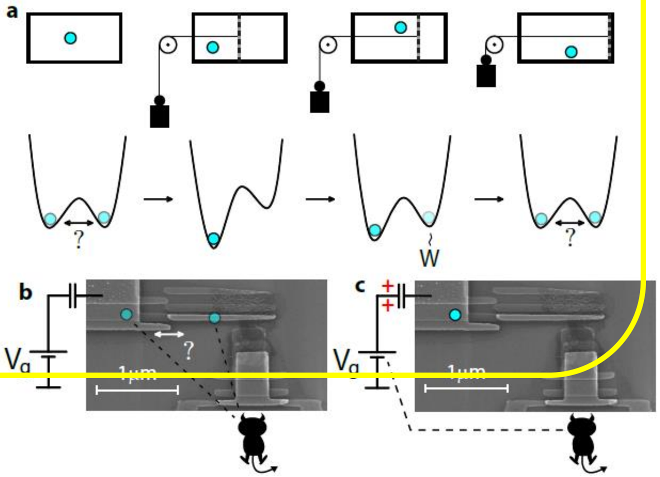




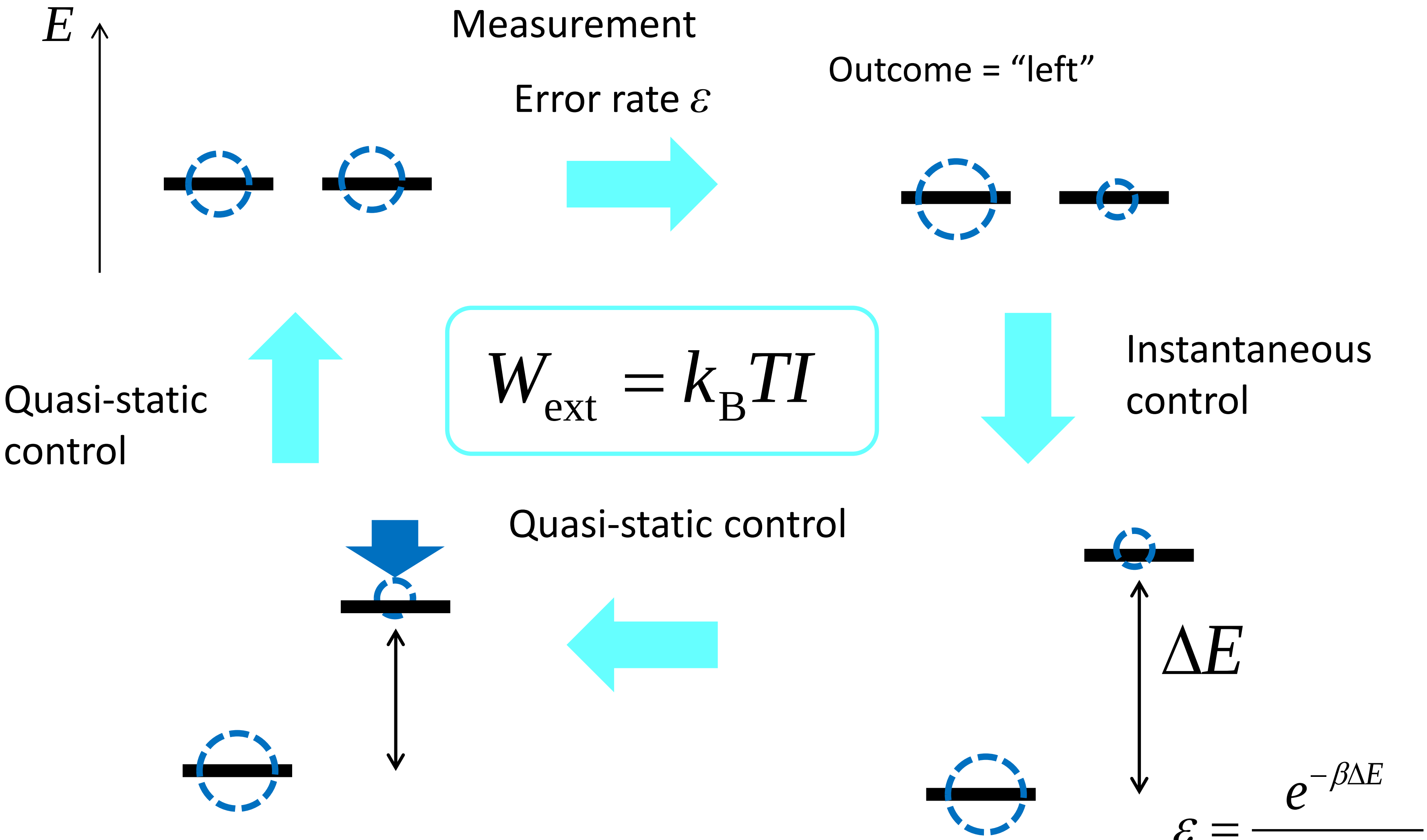
- With a single electron
  Koski, Maisi, TS, & Pekola, PRL (2014)

Efficiency: 75%

Validation of $\left\langle e^{-\beta(W-\Delta F)-I} \right\rangle = 1$



Szilard-like Engine with Two Sites



Measurement

Error rate ε

Outcome = “left”

Instantaneous control

Quasi-static control

Quasi-static control

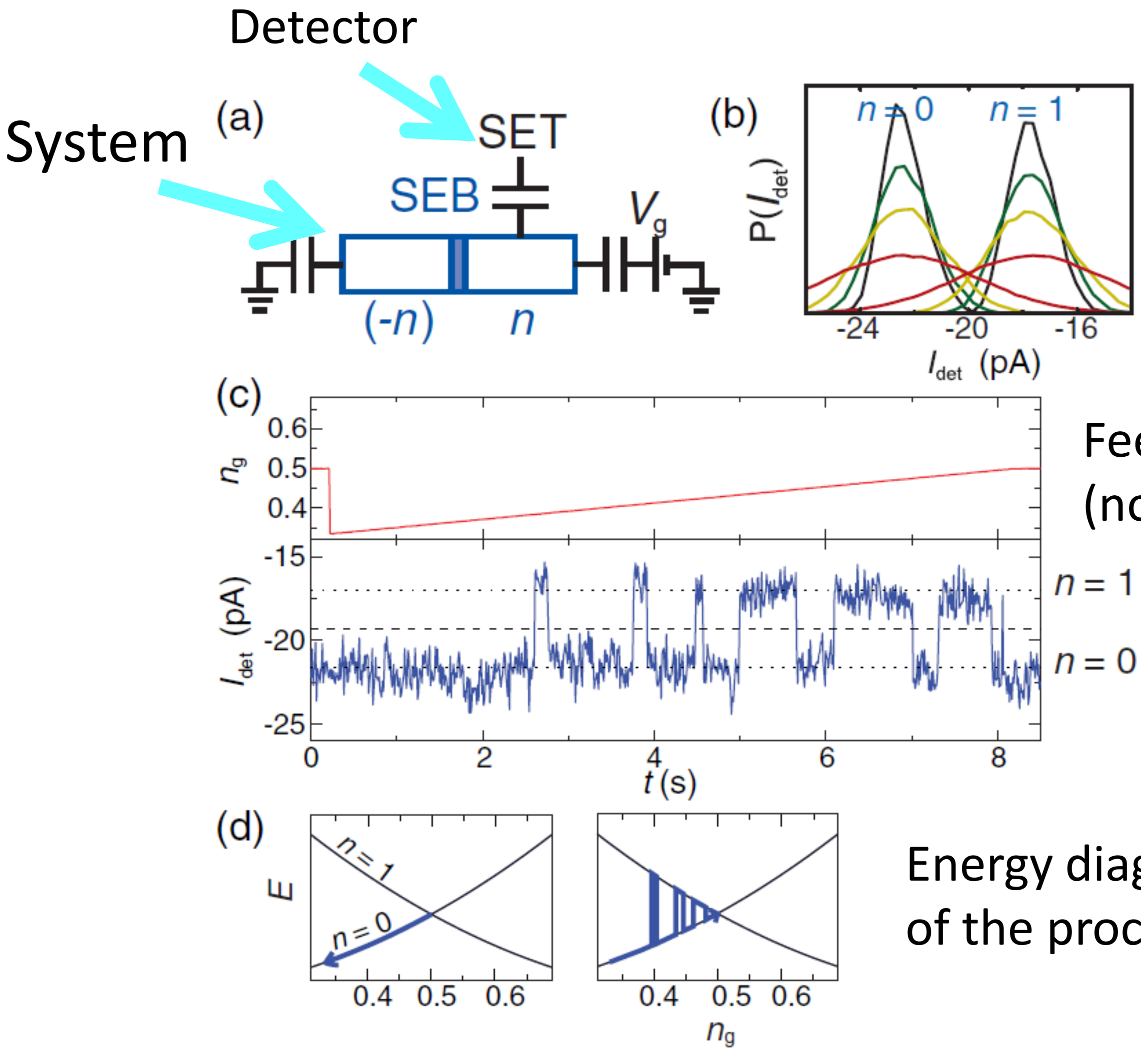
ΔE

$$W_{\mathrm{ext}} = k_{\mathrm{B}} T I$$

$$\varepsilon = \frac{e^{-\beta \Delta E}}{1 + e^{-\beta \Delta E}}$$

Horowitz & Parrondo, *EPL* **95** 10005 (2011)

Setup



Current distribution of the detector (with different filter cutoff frequencies)

Feedback protocol (normalized gate voltage)

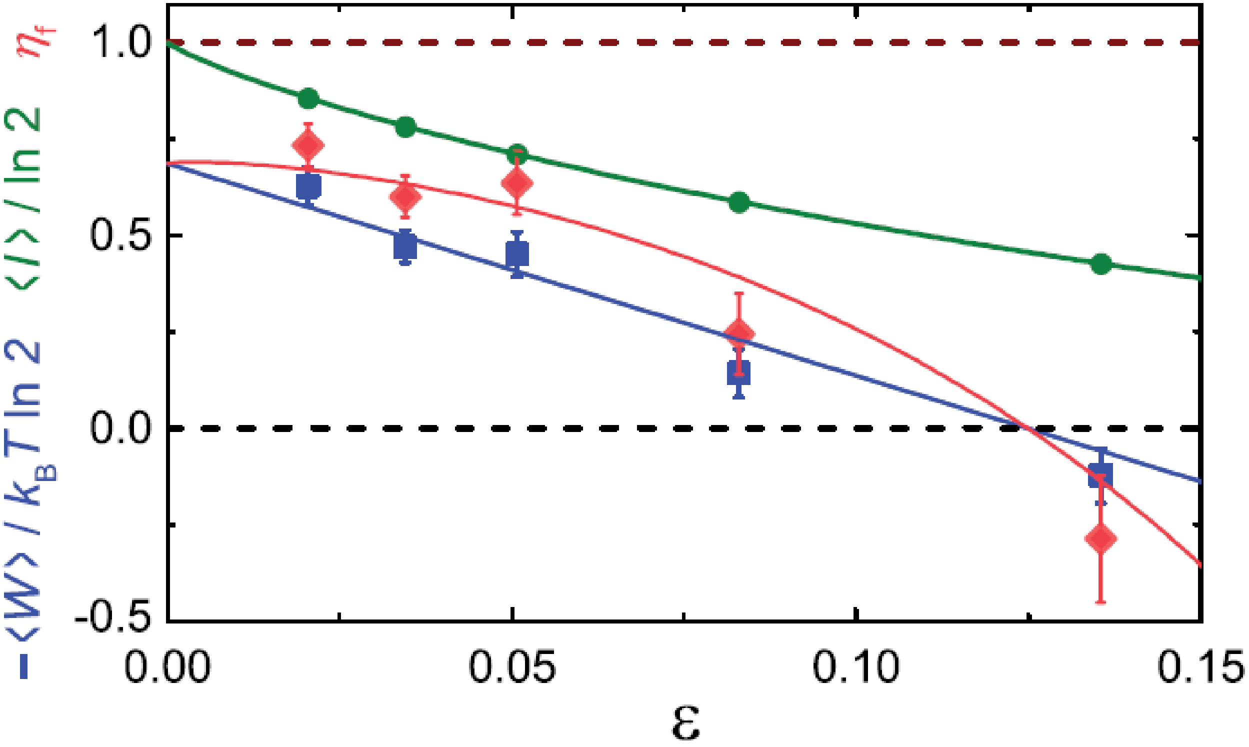
A measured trajectory of the system

Energy diagrams of the process

$$E = E_C(n - n_g)^2$$

Koski, Maisi, TS, & Pekola, PRL **113**, 030601 (2014)

Work, Information, Efficiency



Extractable work

$$W_{\text{ext}} \equiv -\langle W \rangle$$

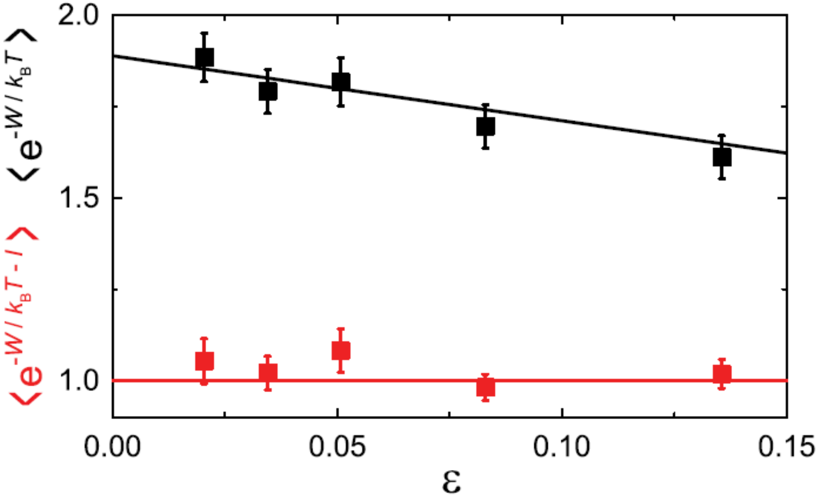
Efficiency

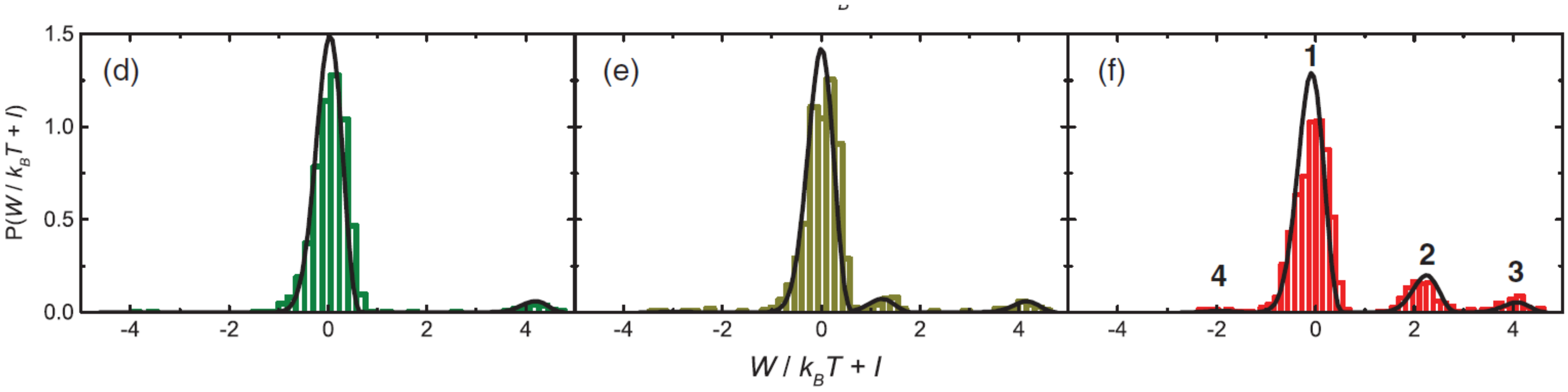
$$\eta_{\text{f}} \equiv \frac{W_{\text{ext}}}{k_{\text{B}} T \langle I \rangle} \leq 1$$

Koski, Maisi, TS, & Pekola, PRL **113**, 030601 (2014)

Validation of the Generalized Jarzynski Equality

$$\left\langle e^{-\beta(W-\Delta F)-I} \right\rangle = 1$$





Koski, Maisi, TS, & Pekola, PRL **113**, 030601 (2014)

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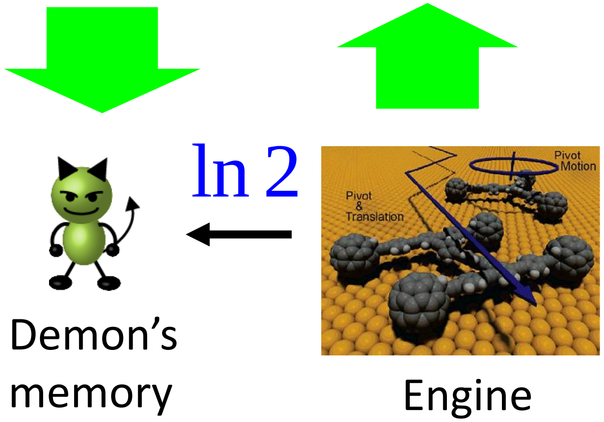
What Compensates for the Excess Work?

$$-k_B T \ln 2 + k_B T \ln 2 \le 0$$

When is this cost needed in principle?

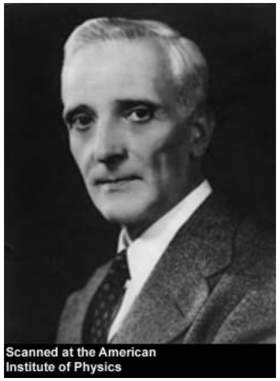


$\ln 2$

Demon's memory

Engine

Measurement process!



Scanned at the American Institute of Physics

Brillouin

Erasure process!

Bennett
&
Landauer

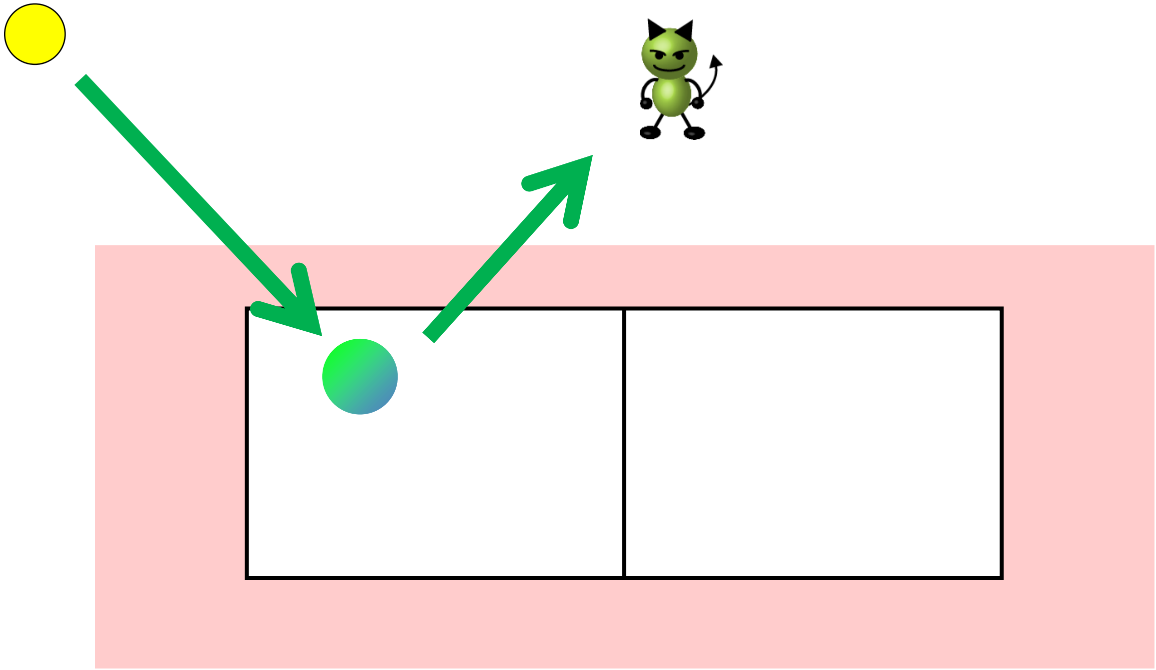
Brillouin’s Argument

A photon is used to measure the position of the particle.

Photon’s energy >> Background radiation

$$\hbar\omega >> k_{\mathrm{B}}T$$

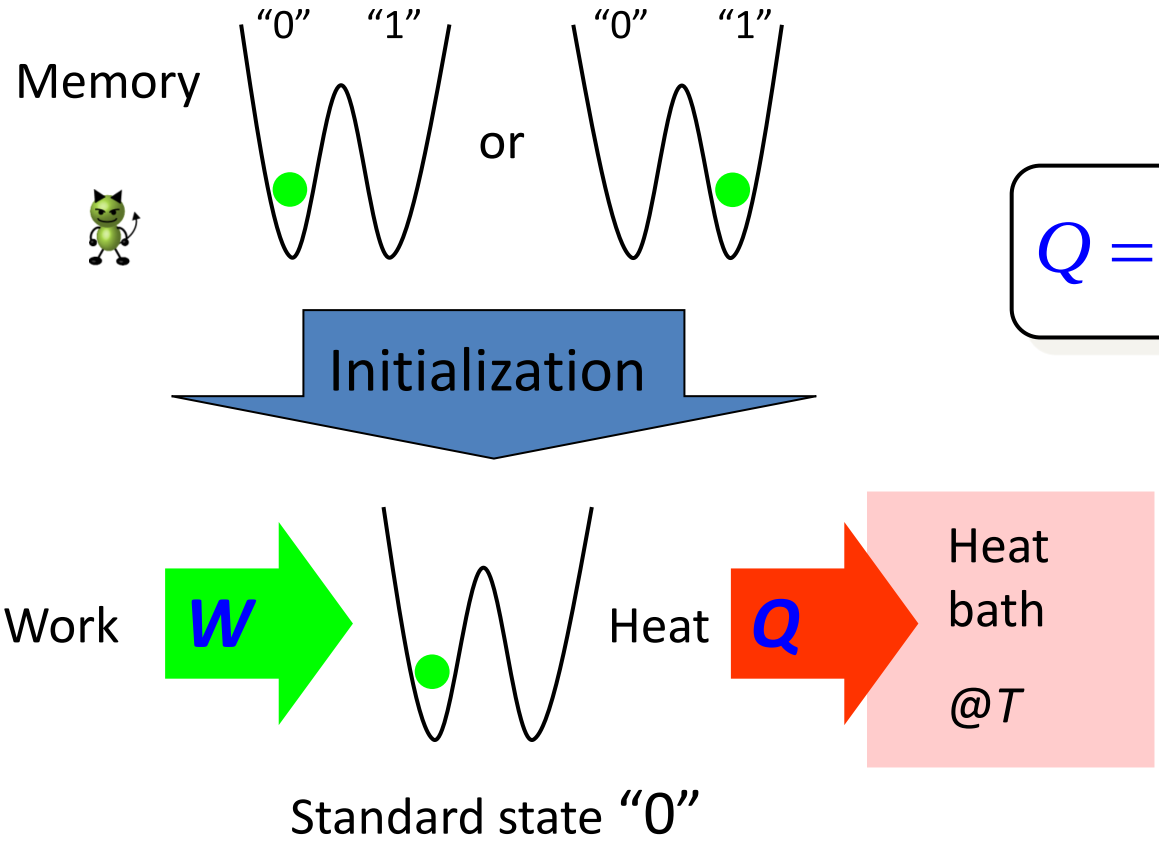
$$\Rightarrow \quad W_{\text{photon}} >> k_{\mathrm{B}}T \ln 2$$



L. Brillouin, J. Appl. Phys. **22**, 334 (1951).

Landauer's Principle for Information Erasure

R. Landauer, IBM J. Res. Dev. **5**, 183 (1961)



$$Q = W \geq k_B T \ln 2$$

Information erasure → Heat generation and work requirement

Proof of the Landauer Principle

$Q \leq T\Delta S$: Second law of thermodynamics

$\Delta S = -k_{\mathrm{B}} \boxed{\ln 2}$: Decrease of one bit of information entropy

$$Q \leq -k_{\mathrm{B}} T \ln 2 \quad \Leftrightarrow \quad Q_{\mathrm{out}} \geq k_{\mathrm{B}} T \ln 2$$

Absorbed heat

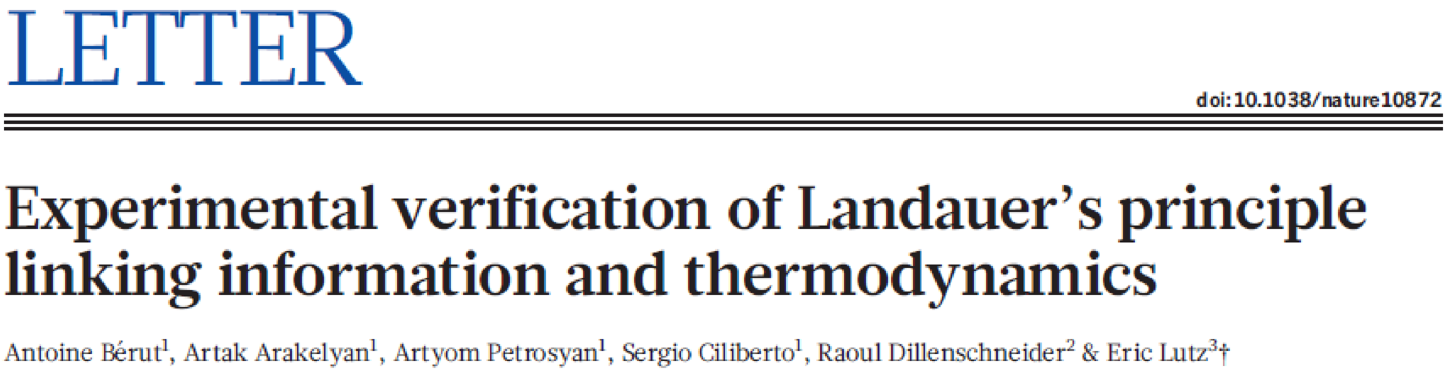
Heat generation

$\Delta E = 0$ (energy degeneration)

$$\Rightarrow \quad W = \Delta E + Q_{\mathrm{out}} \geq k_{\mathrm{B}} T \ln 2$$

K. Shizume, Phys. Rev. E **52**, 3495 (1995).
B. Piechocinska, Phys. Rev. A **61**, 062314 (2000).

Experimental Verification

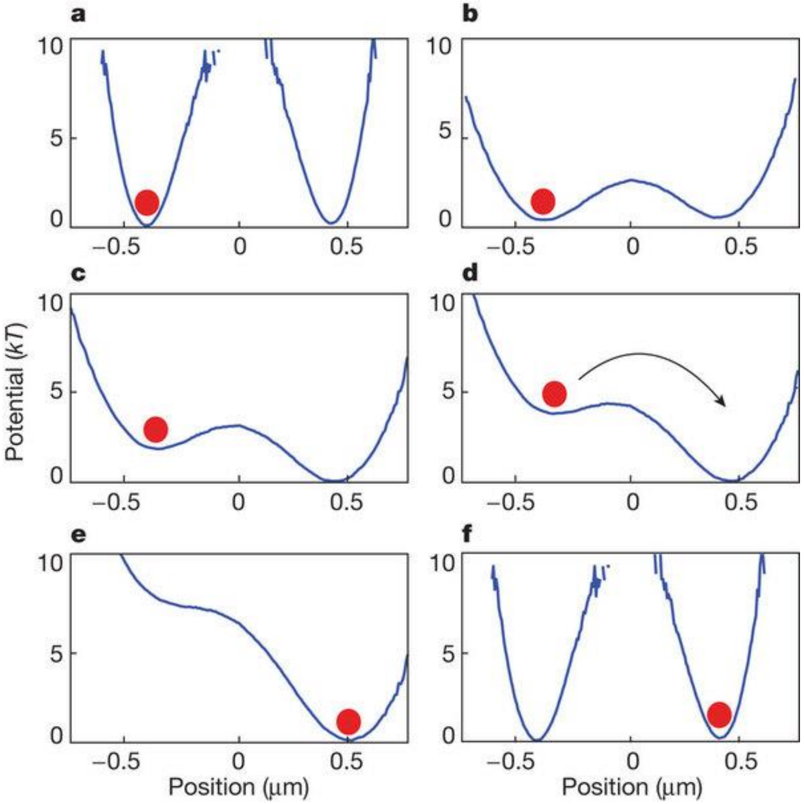


LETTER

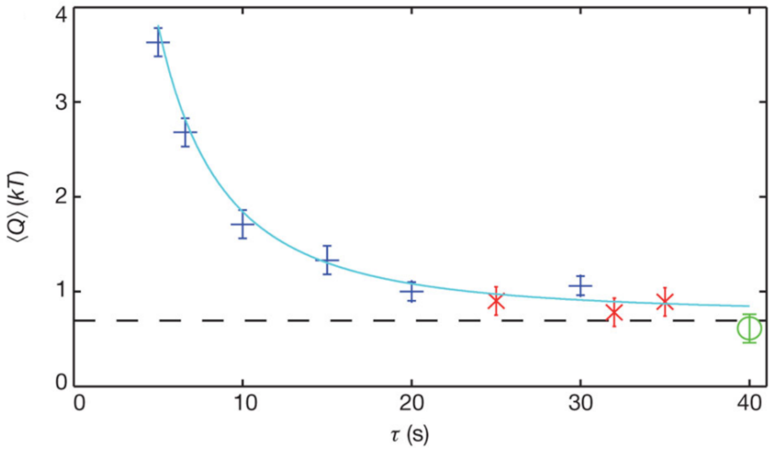
doi:10.1038/nature10872

Experimental verification of Landauer's principle linking information and thermodynamics

Antoine Bérut[1], Artak Arakelyan[1], Artyom Petrosyan[1], Sergio Ciliberto[1], Raoul Dillenschneider[2] & Eric Lutz[3]†



Nature **483**, 187–189 (2012)



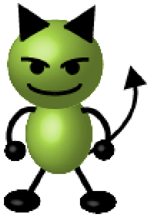
Bennett's Argument

C. H. Bennett, Int. J. Theor. Phys. **21**, 905 (1982).

For a cycle of the engine and the demon…

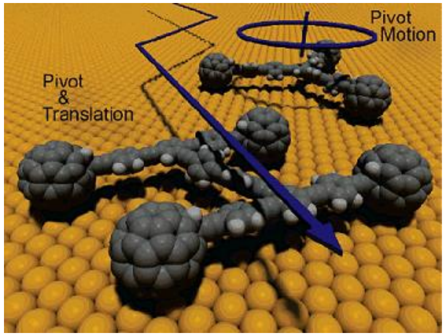
$$-k_{\mathrm{B}}T\ln 2 + k_{\mathrm{B}}T\ln 2 \leq 0$$

This cost is needed for the logically irreversible information erasure.



Demon's memory

$\ln 2$



Engine

Is this the final answer?

Asymmetric Memory

Standard state "0" of the memory with free energy F_0

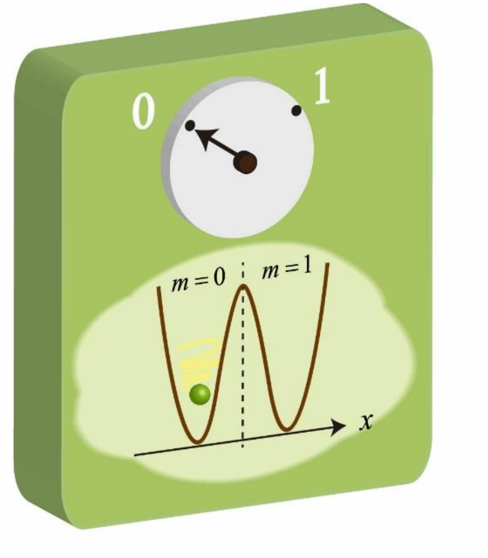
The memory stores measurement outcome "m" with probability p_m.

Free-energy difference: $\Delta F \equiv \sum_m p_m F_m - F_0$

Conditional free energy with m

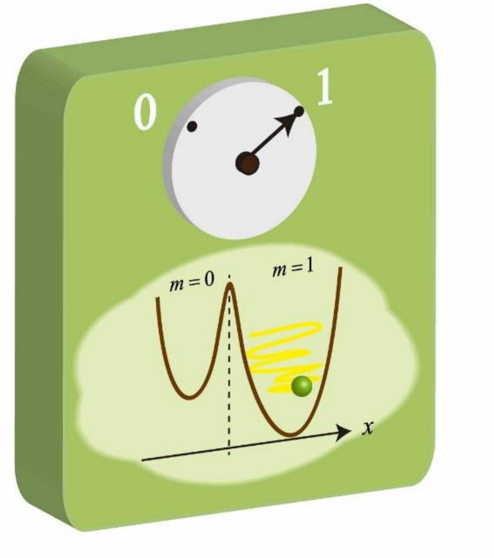
Symmetric memory

$$\Delta F = 0$$



Asymmetric memory

$$\Delta F \neq 0$$



Measurement and Information Erasure: Setup

Isothermal information processing at temperature *T*

Standard state “0” of the memory with free energy F_0

The memory interacts with a measured system, and stores measurement outcome “*k*” with probability p_k.

Free-energy difference $\Delta F \equiv \sum_k p_k F_k - F_0$

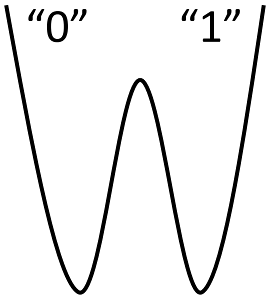
Measurement

The memory detaches from the measured system, and returns to the initial state “0” with unit probability

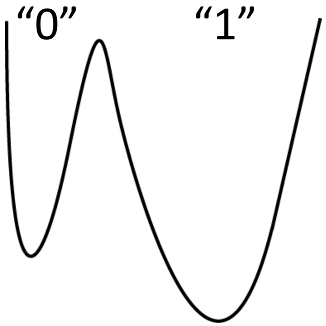
Free-energy difference $-\Delta F$

Erasure

Symmetric memory $\Delta F = 0$



Asymmetric memory $\Delta F \neq 0$



Determined the fundamental energy cost for the measurement and erasure

Minimal Energy Cost for Information Erasure

$$W_{\mathrm{eras}} \geq k_{\mathrm{B}} T H - \Delta F$$

$$H = -\sum_{k} p_k \ln p_k$$: Shannon information of the stored outcome

If the free-energy difference is given by $\Delta F = 0$,

$$W_{\mathrm{eras}} \geq k_{\mathrm{B}} T H$$: Landauer's principle

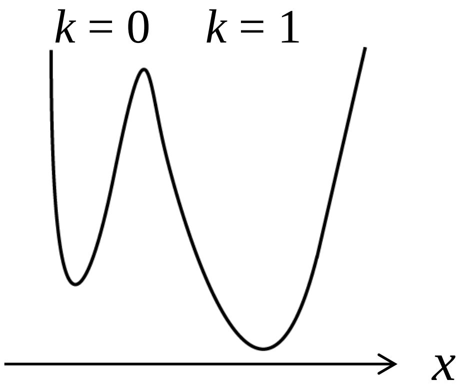
If $\Delta F \neq 0$, $W_{\mathrm{eras}} < k_{\mathrm{B}} T H$ is achievable.

TS and M. Ueda, PRL **102**, 250602 (2009)
O. J. E. Maroney, Phys. Rev. E **79**, 031105 (2009).

Decomposition of the Memory's Entropy (1)

$P[x] = \sum_k p_k P_k[x]$: statistical mixture

(The supports of $P_k(x)$ are assumed to be non-crossing)



$$H(\boldsymbol{P}) = \sum_k p_k H(\boldsymbol{P}_k) + H(\boldsymbol{p})$$

Total entropy **Thermal entropy** **Shannon Information**

$$H(\boldsymbol{P}) = -\sum_x P[x] \ln P[x]$$

$$H(\boldsymbol{P}_k) = -\sum_x P_k[x] \ln P_k[x]$$

$$H(\boldsymbol{p}) = -\sum_k p_k \ln p_k$$

T. M. Cover and J. A. Thomas,
"Elements of Information Theory"

Decomposition of the Memory’s Entropy (2)

Proof of the decomposition formula

(The supports of $P_k(x)$ are assumed to be non-crossing)

$$H(\boldsymbol{P}) = -\sum_x \left(\sum_k p_k P_k[x] \right) \ln \left(\sum_k p_k P_k[x] \right)$$

$$= -\sum_k \sum_x p_k P_k[x] \ln (p_k P_k[x])$$

$$= -\sum_k \sum_x p_k P_k[x] \ln P_k[x] - \sum_k p_k \ln p_k$$

$$= \sum_k p_k H(\boldsymbol{P}_k) + H(\boldsymbol{p})$$

T. M. Cover and J. A. Thomas,
“Elements of Information Theory”

Proof of the generalized Landauer principle

$P[x]=\sum_k p_k P_k[x]$: initial state $\Rightarrow$ $P[x]=\sum_k p'_k P_k[x]$: final state

For information erasure: $p'_0=1$

$P_k[x]=\exp\left(\beta(F_k-E_k[x])\right)$: local equilibrium state under the condition of *k*

$H(\boldsymbol{P})=\sum_k p_k H(\boldsymbol{P}_k)+H(\boldsymbol{p})$ $\Rightarrow$ $\Delta H(\boldsymbol{P})=\sum_k \Delta p_k H(\boldsymbol{P}_k)+\Delta H(\boldsymbol{p})$

$\Delta p_k=p'_k-p_k \quad \Delta H(\boldsymbol{p})=H(\boldsymbol{p}')-H(\boldsymbol{p})$

$$\Delta H(\boldsymbol{P})\geq\beta Q \Leftrightarrow \beta\sum_k \Delta p_k(\langle E_k\rangle-F_k)+\Delta H(\boldsymbol{p})\geq\beta Q$$

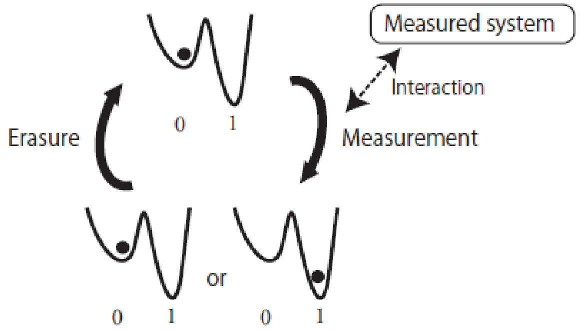
$$\Leftrightarrow W\geq -k_{\mathrm{B}}T\Delta H(\boldsymbol{p})-\Delta F$$

For information erasure: $W\geq k_{\mathrm{B}}TH(\boldsymbol{p})-\Delta F$

Minimal Energy Cost for Measurement

$$W_{\mathrm{meas}} \geq \underline{-k_{\mathrm{B}}TH + \Delta F} + \boxed{k_{\mathrm{B}}TI}$$

Cf. $$W_{\mathrm{eras}} \geq \underline{k_{\mathrm{B}}TH - \Delta F}$$



If $\Delta F = 0$ and $H = I$, then $W_{\mathrm{meas}} \geq 0$

Determined the fundamental lower bound based on statistical mechanics

- Valid for both quantum and classical cases
- R.H.S. depends on both the Shannon and mutual information

Assumed that the measurement is adiabatic in terms of the measured system

TS and M. Ueda, PRL **102**, 250602 (2009); **106**, 189901(E) (2011).

Trade-off Relation

There is no fundamental lower bound to the individual work needed for measurement or erasure,

but there **exists** the fundamental minimal work for the sum of them.

$$W_{\mathrm{meas}} \geq -k_{\mathrm{B}}TH + \Delta F + k_{\mathrm{B}}TI$$

Landauer's principle

$$+)\quad \boxed{W_{\mathrm{eras}} \geq k_{\mathrm{B}}TH} - \Delta F$$

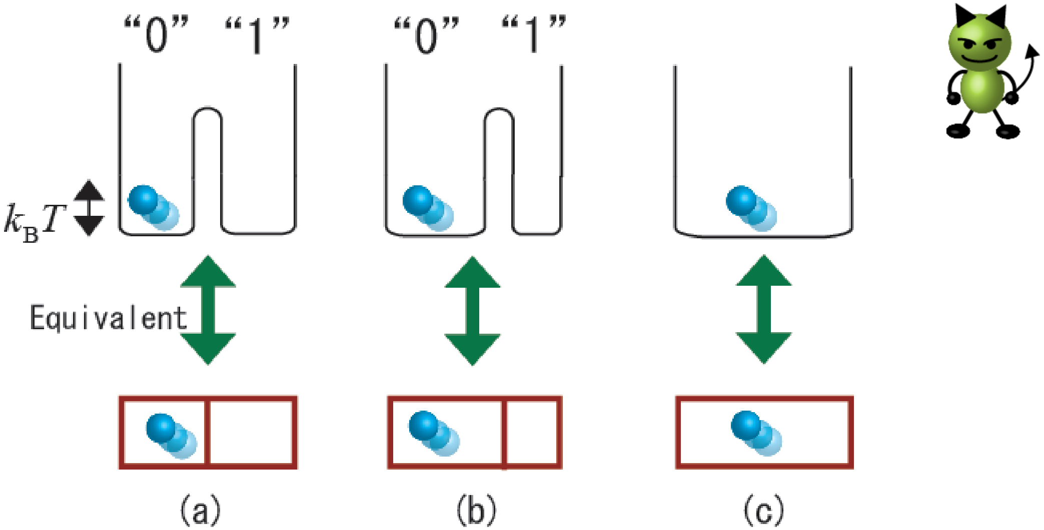
New term

$$W_{\mathrm{meas}} + W_{\mathrm{eras}} \geq k_{\mathrm{B}}TI$$

The lower bound depends neither on the Shannon information content nor on the free-energy difference, but **only on the mutual information content** between the measured system and the memory.

TS and M. Ueda, PRL **102**, 250602 (2009); **106**, 189901(E) (2011).

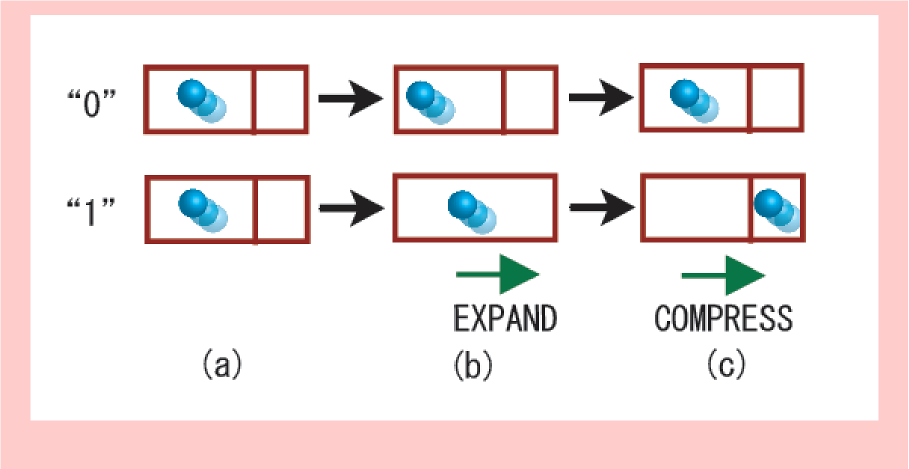
A Toy Model of Memory

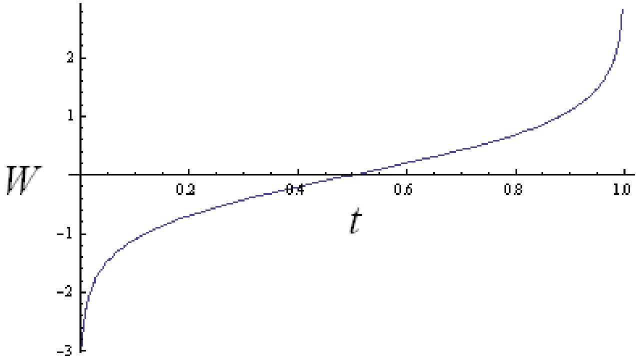


Ratio of the volumes $t:1-t$

M. M. Barkeshli, arXiv:cond-mat/0504323 (2005).
O. J. E. Maroney, Phys. Rev. E **79**, 031105 (2009).
TS and M. Ueda, PRL **102**, 250602 (2009); **106**, 189901(E) (2011).

Model of Measurement





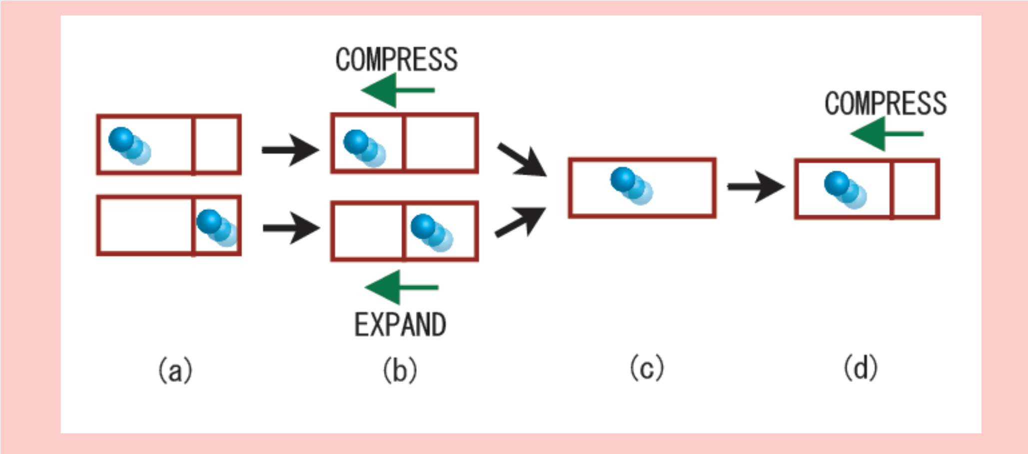
Cf. $W_{\mathrm{meas}} \geq -k_{\mathrm{B}}T(H-I)+\Delta F$

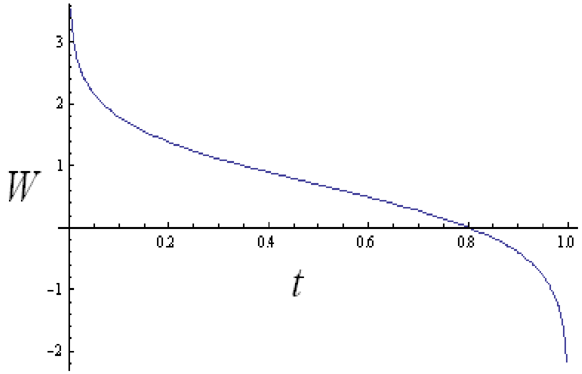
$$W_{\mathrm{meas}} = (k_{\mathrm{B}}T/2)\ln(t/(1-t))$$

Symmetric memory: $t = 1/2 \;\Rightarrow\; W_{\mathrm{meas}} = 0$

Asymmetric memory: $t = 4/5 \;\Rightarrow\; W_{\mathrm{meas}} = k_{\mathrm{B}}T\ln 2$

Model of Information Erasure





Cf. $W_{\mathrm{eras}} \geq k_{\mathrm{B}}TH - \Delta F$

$$W_{\mathrm{eras}} = k_{\mathrm{B}}T\ln 2 - (k_{\mathrm{B}}T/2)\ln(t/(1-t))$$

Symmetric memory: $t = 1/2 \Rightarrow W_{\mathrm{eras}} = k_{\mathrm{B}}T\ln 2$

Asymmetric memory: $t = 4/5 \Rightarrow W_{\mathrm{eras}} = 0$

Trade-off relation for arbitrary *t*

$$\boxed{W_{\mathrm{meas}} + W_{\mathrm{eras}} = k_{\mathrm{B}}T\ln 2}$$

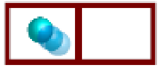
Entropy Balance

$P(x)=\sum_k p_k P_k(x)$: statistical mixture

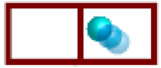
(The supports of $P_k(x)$ are assumed to be non-crossing)

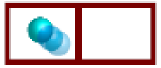
$$H(\boldsymbol{P})=\sum_k p_k H(\boldsymbol{P}_k)+H(\boldsymbol{p})$$

Total entropy **Thermal entropy** **Information**



or





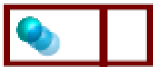
V: Volume of the box

$$\underbrace{\frac{1}{2}\left(\ln\left(\frac{1}{2}V\right)+\ln\left(\frac{1}{2}V\right)\right)}_{\text{Thermal entropy}}+\underbrace{\ln 2}_{\text{Information}}$$

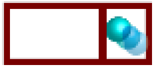
$$\underbrace{\ln\left(\frac{1}{2}V\right)}+\underbrace{0}$$

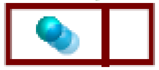
$-\ln 2$ of difference

Dissipation



or





$$\underbrace{\frac{1}{2}\left(\ln\left(\frac{4}{5}V\right)+\ln\left(\frac{1}{5}V\right)\right)}_{\text{Thermal entropy}}+\underbrace{\ln 2}_{\text{Information}}$$

No entropy change
(No dissipation)

$$\underbrace{\ln\left(\frac{4}{5}V\right)}+\underbrace{0}$$

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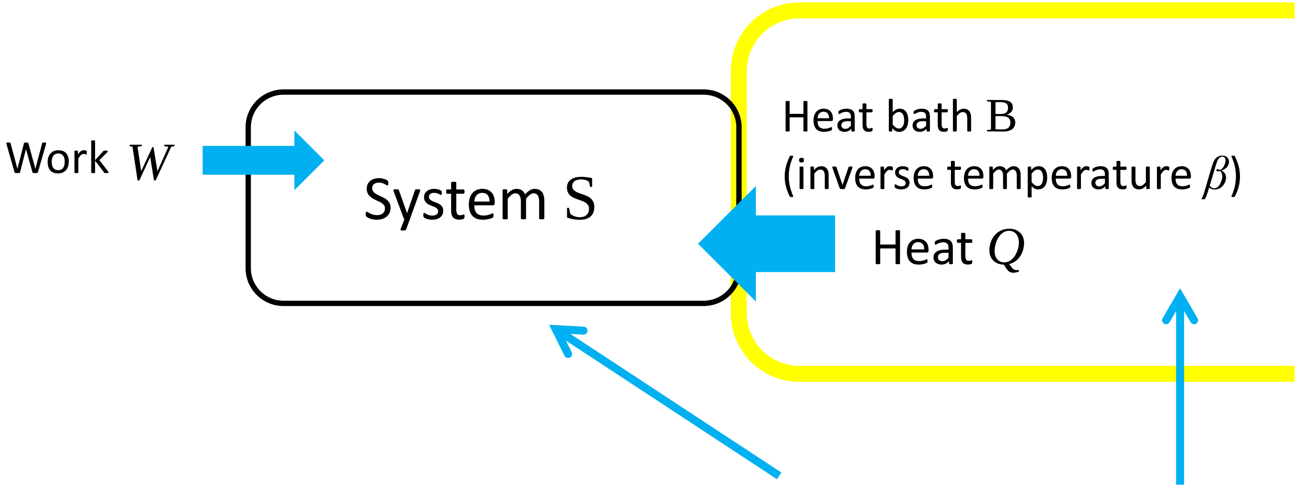
- **Entropy production**
- General framework of information thermodynamics
- Autonomous Maxwell’s demons
- Summary

Entropy Production

Stochastic dynamics of system S (e.g., Langevin system)



Work W

System S

Heat bath B
(inverse temperature β)

Heat Q

Entropy production in the total system:

$$\Delta S_{\mathrm{SB}} \equiv \underline{\Delta S_{\mathrm{S}}} - \beta \underline{\langle Q \rangle}$$

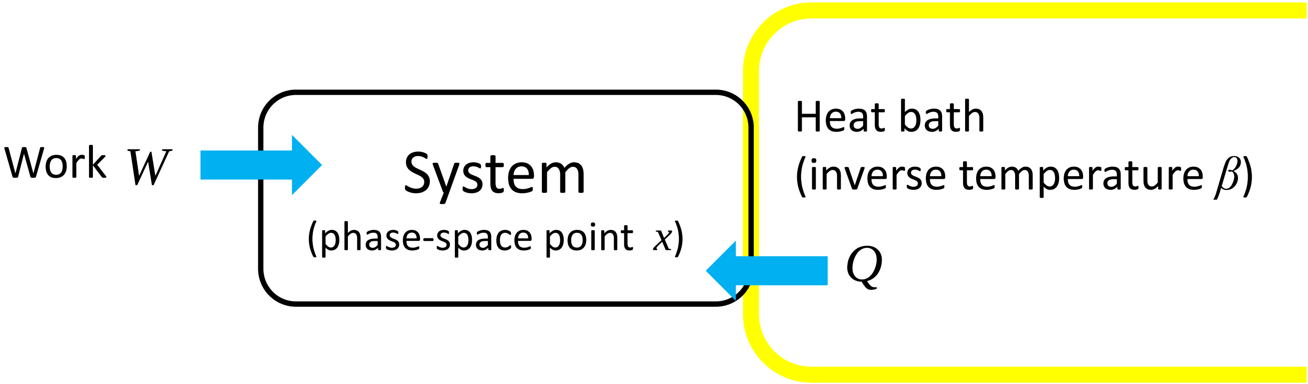
Change in the Shannon entropy of S

Averaged heat absorbed by S

If the initial and the final states are canonical distributions: $\Delta S_{\mathrm{SB}} = \beta\left(\langle W \rangle - \Delta F\right)$

Free-energy difference

Stochastic Entropy Production



Work W

System

(phase-space point x)

Heat bath

(inverse temperature β)

Q

Stochastic entropy production along a trajectory of the system from time 0 to τ

$$\Delta s_{\mathrm{SB}} \equiv \Delta s_{\mathrm{S}} - \beta Q$$

$$\Delta s_{\mathrm{S}} \equiv s_{\mathrm{S}}[x(\tau),\tau] - s_{\mathrm{S}}[x(0),0] \qquad s_{\mathrm{S}}[x,t] \equiv -\ln P[x,t]$$

$$\left\langle \Delta s_{\mathrm{S}} \right\rangle = \Delta S_{\mathrm{S}}$$

$P[x,t]$: probability distribution at time t

If the initial and the final states are canonical distributions: $\Delta s_{\mathrm{SB}} = \beta\left(W - \Delta F\right)$

Integral Fluctuation Theorem and Jarzynski Equality

Integral fluctuation theorem

$$\left\langle e^{-\Delta s_{\mathrm{SB}}} \right\rangle = 1$$

Seifert, PRL (2005), …

for any initial and final distributions

Second law can be expressed by an **equality** with full cumulants

⇒ The second law of thermodynamics (Clausius inequality)

$$\left\langle \Delta s_{\mathrm{SB}} \right\rangle \geq 0 \quad \Leftrightarrow \quad \Delta S_{\mathrm{S}} \geq \beta \langle Q \rangle$$

Jarzynski equality

Jarzynski, PRL (1997)

$$\Delta s_{\mathrm{SB}} = \beta(W - \Delta F) \quad \Rightarrow \quad \left\langle e^{-\beta W} \right\rangle = e^{-\beta \Delta F} \quad \Rightarrow \quad \langle W \rangle \geq \Delta F$$

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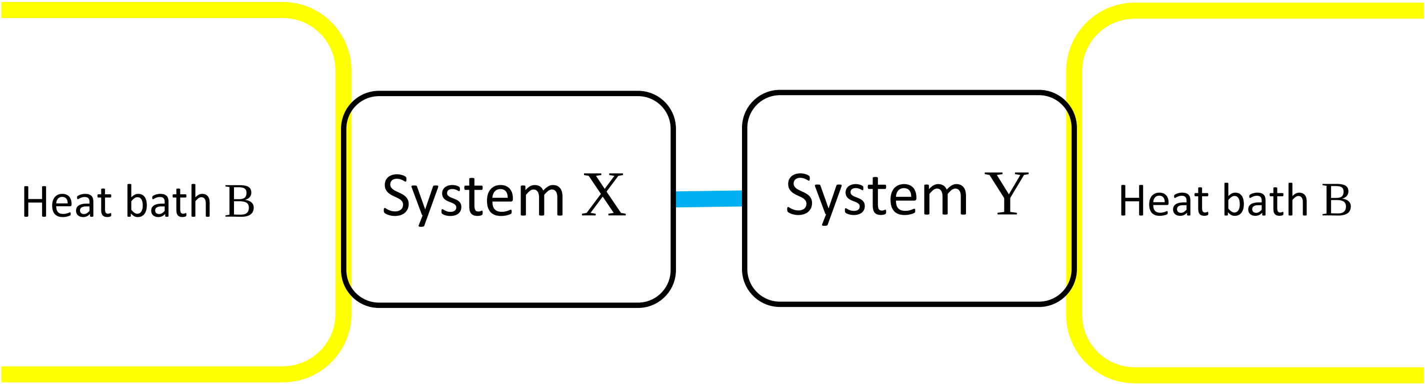
The Setup

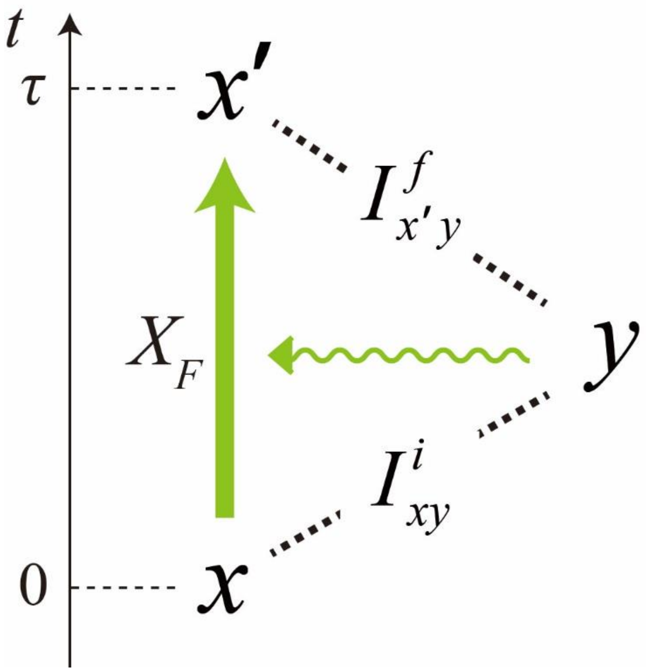


Heat bath B

System X

System Y

Heat bath B



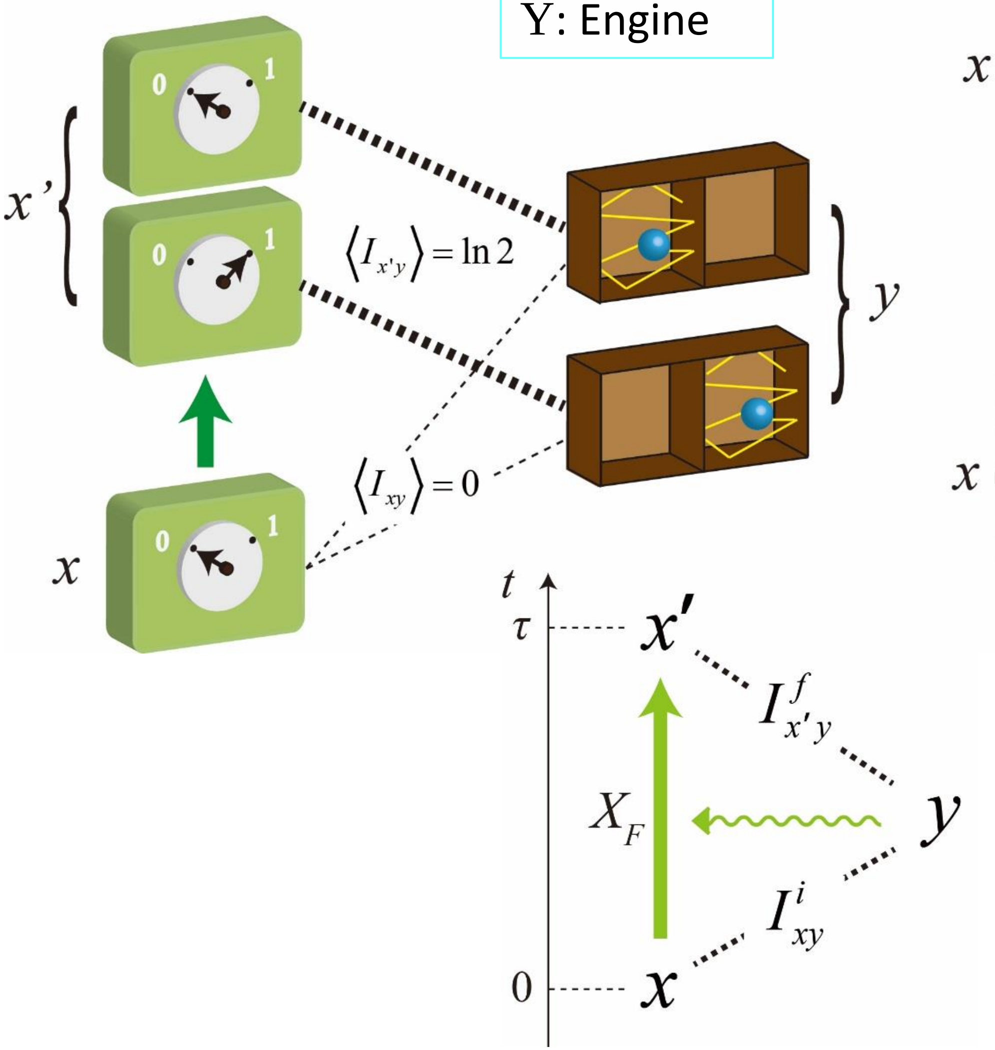
X and Y interact with each other only through information exchange

Time evolution of X
under the influence of Y with
initial and final correlations

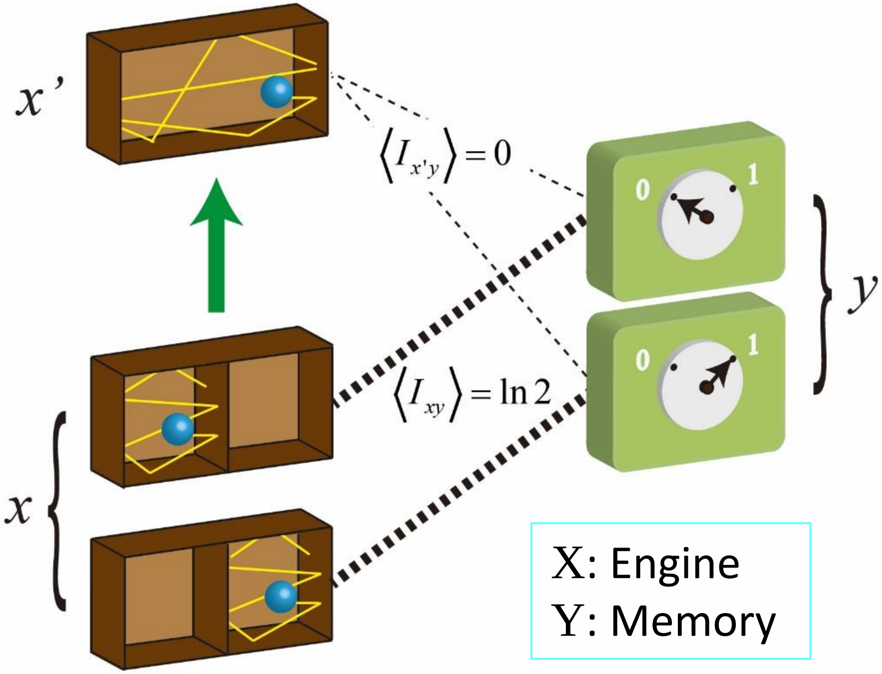
Special Cases: Measurement and Feedback

Measurement

X: Memory
Y: Engine



Feedback

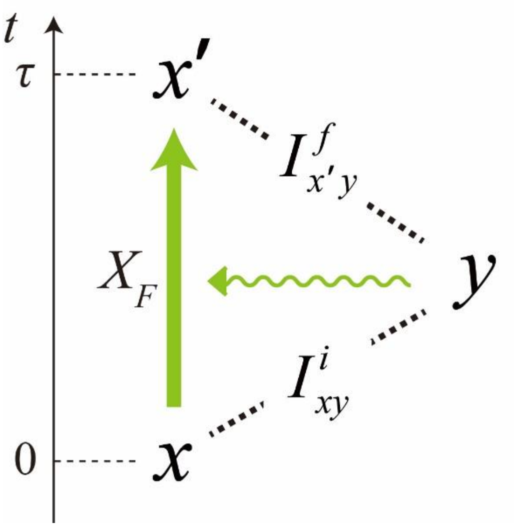


X: Engine
Y: Memory

Entropy and Mutual Information

Entropy increase in XB

$$\Delta S_{\mathrm{XB}} \equiv \Delta S_{\mathrm{X}} - \beta Q_{\mathrm{X}}$$



Initial correlation

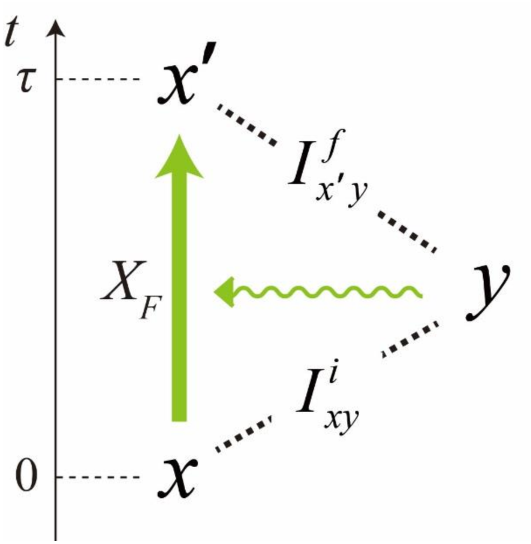
$$I(X:Y) = \int dxdyP[x,y]\ln\frac{P[x,y]}{P[x]P[y]}$$

Final correlation

$$I(X':Y) = \int dx'dyP[x',y]\ln\frac{P[x',y]}{P[x']P[y]}$$

Decomposition of Total Entropy

$$\Delta S_{\mathrm{XYB}} \equiv \Delta S_{\mathrm{XY}} - \beta Q_{\mathrm{X}}$$
$$= \Delta S_{\mathrm{XB}} - \Delta I$$



$$\Delta S_{\mathrm{XYB}} \equiv \Delta S_{\mathrm{XY}} - \beta Q_{\mathrm{X}}$$

$$\Delta S_{\mathrm{XB}} \equiv \Delta S_{\mathrm{X}} - \beta Q_{\mathrm{X}}$$

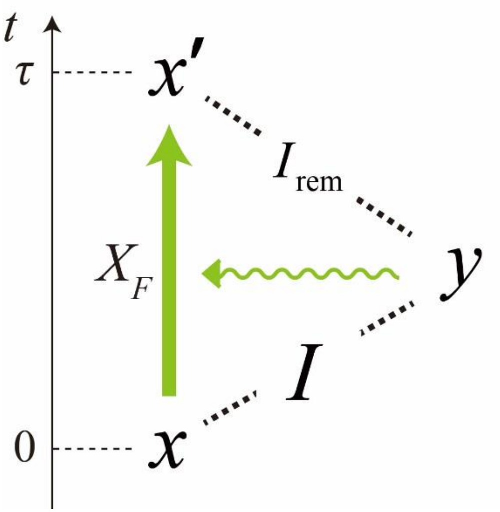
$$\Delta I \equiv I(X':Y) - I(X:Y)$$

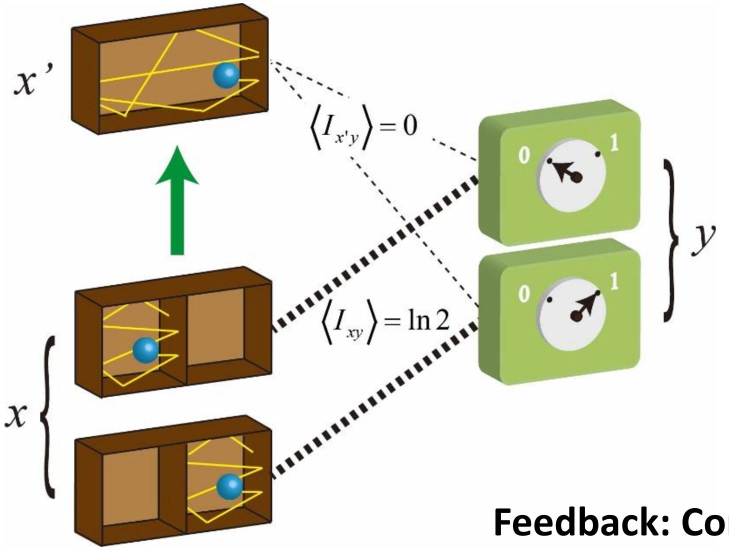
Second law:

$$\Delta S_{\mathrm{XYB}} \geq 0 \quad \Leftrightarrow \quad \Delta S_{\mathrm{XB}} \geq \Delta I$$

TS and M. Ueda, Phys. Rev. Lett. **109**, 180602 (2012).

Special Case 1: Feedback Control





X: Engine
Y: Memory

Feedback: Control protocol depends on the measurement outcome

$$\boxed{\Delta I \equiv I_{\mathrm{rem}} - I}$$

$$\Delta S_{\mathrm{XB}} \geq \underline{-(I - I_{\mathrm{rem}})}$$

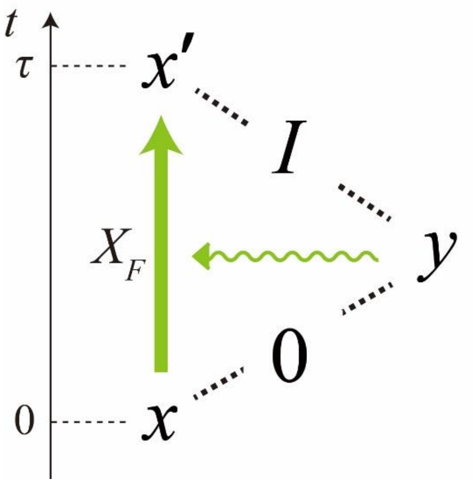
(Upper bound of) the correlation that is used by feedback

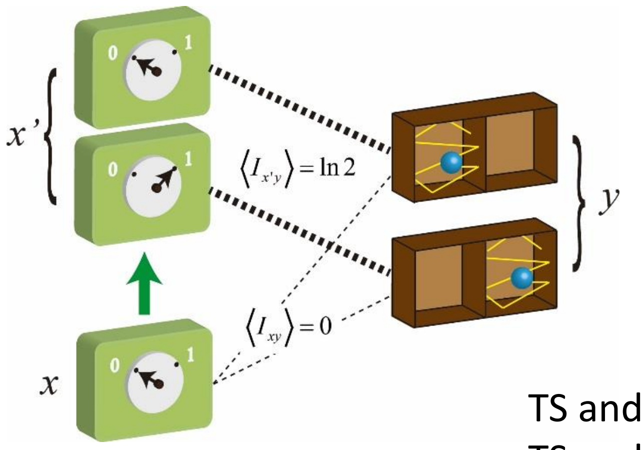
$$\underset{\text{Extractable work}}{W_{\mathrm{ext}}} \leq -\Delta F + k_{\mathrm{B}} T I$$

TS and M. Ueda, PRL **100**, 080403 (2008).
TS and M. Ueda, PRL **104**, 090602 (2010).

Special Case 2: Measurement

X: Memory
Y: Engine





TS and M. Ueda, PRL **102**, 250602 (2009)
TS and M. Ueda, PRL **109**, 180602 (2012)

$$\Delta I \equiv I \quad \Rightarrow \quad \Delta S_{\mathrm{XB}} \ge I$$

$$\Rightarrow \quad \beta W_{\mathrm{X}} \ge \underbrace{\beta \Delta E_{\mathrm{X}} - \Delta S_{\mathrm{X}}} + I$$

Work cost for measurement

Nonequilibrium free energy of the memory

Additional energy cost to obtain information

Measurement & Erasure

Measurement cost

$$W_{\mathrm{meas}} \geq \left(\Delta E - k_{\mathrm{B}} T \Delta S\right) + k_{\mathrm{B}} T I$$

Erasure cost

$$W_{\mathrm{eras}} \geq -\left(\Delta E - k_{\mathrm{B}} T \Delta S\right)$$

$+\,)$

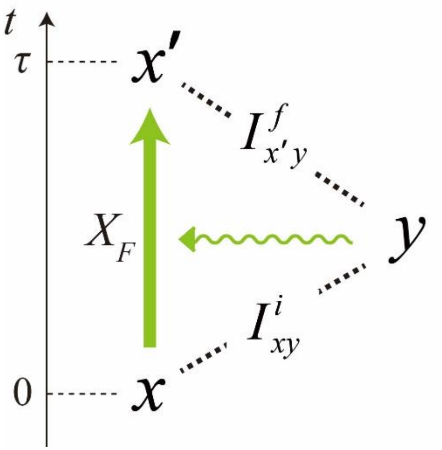
$$W_{\mathrm{meas}} + W_{\mathrm{eras}} \geq k_{\mathrm{B}} T I$$

Fundamental trade-off!

TS and M. Ueda, PRL **102**, 250602 (2009)
TS and M. Ueda, PRL **109**, 180602 (2012)

Second Law of Information Thermodynamics: Summary

$$\begin{aligned}\Delta S_{\mathrm{XYB}} &\equiv \Delta S_{\mathrm{XY}} - \beta Q_{\mathrm{X}} \\ &= \Delta S_{\mathrm{XB}} - \Delta I\end{aligned}$$



$$\Rightarrow \Delta S_{\mathrm{XB}} \geq \Delta I$$

Feedback:

$$\Delta S_{\mathrm{XB}} \geq -I$$

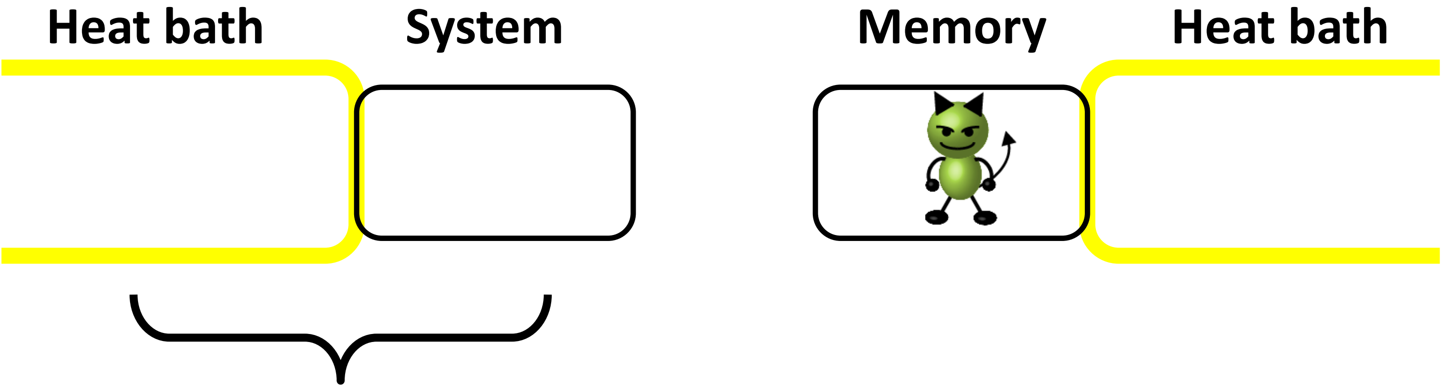
Measurement:

$$\Delta S_{\mathrm{XB}} \geq I$$

Unified formulation of measurement and feedback

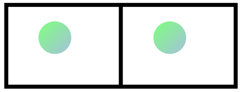
TS and M. Ueda, PRL **109**, 180602 (2012).

Paradox of Maxwell's Demon

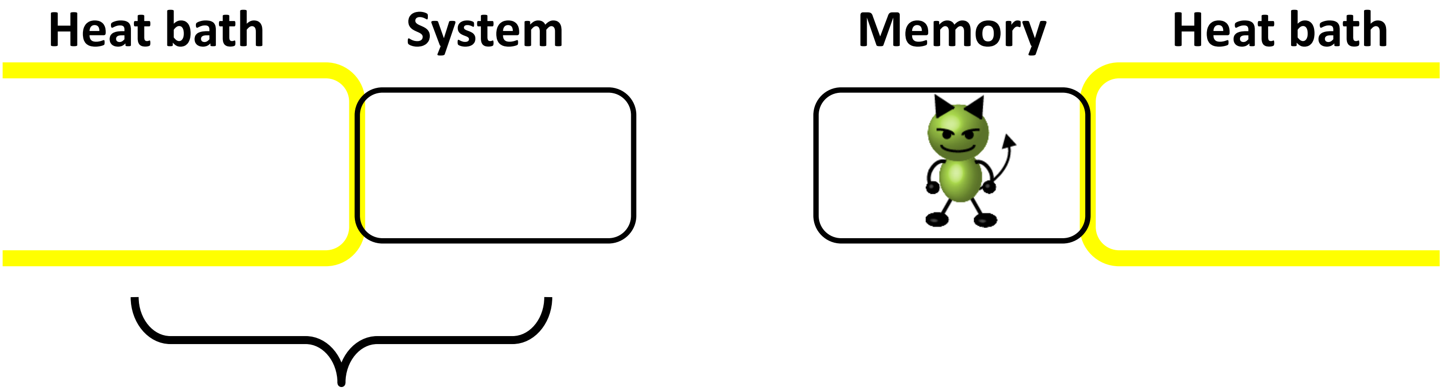


What compensates for the entropy decrease here?

For Szilard engine, $\langle \Delta s_{\mathrm{SB}} \rangle = -\ln 2$



Revisiting the Paradox of Maxwell’s Demon

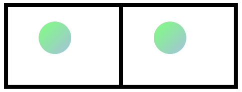


Heat bath **System** **Memory** **Heat bath**

What compensates for the entropy decrease here?

Mutual-information change compensates for it.

For Szilard engine, $\Delta S_{\mathrm{SB}} = -\ln 2$



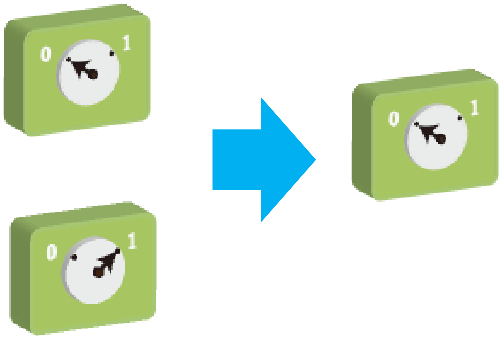
$$\Delta S_{\mathrm{SMB}} = \Delta S_{\mathrm{SB}} + I = -\ln 2 + \ln 2 = 0$$

Logical vs Thermodynamic Reversibility

Logically reversible:

$$\Delta S_{\mathrm{S}} = 0$$

Zero entropy change in the logical states
(Assume: symmetric memory)



Thermodynamically reversible:

$$\Delta S_{\mathrm{S}} - \beta Q = 0$$

Zero entropy production in the system and the bath

They are fundamentally distinct concepts!

Key Observations

- Maxwell’s demon is consistent with the second law for measurement and feedback processes *individually*
  - The mutual information term in the entropy production is the key

- We don’t necessarily need to consider the information erasure to understanding the consistency

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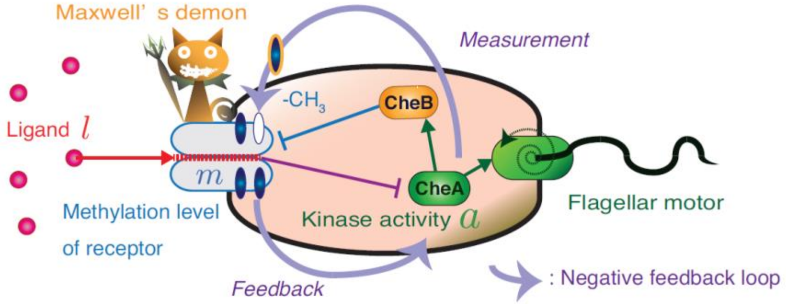
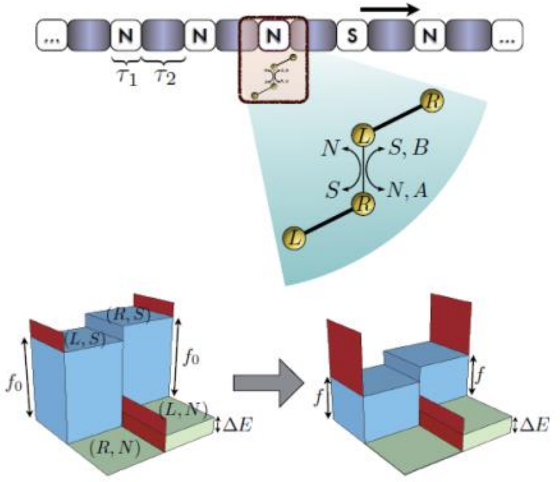
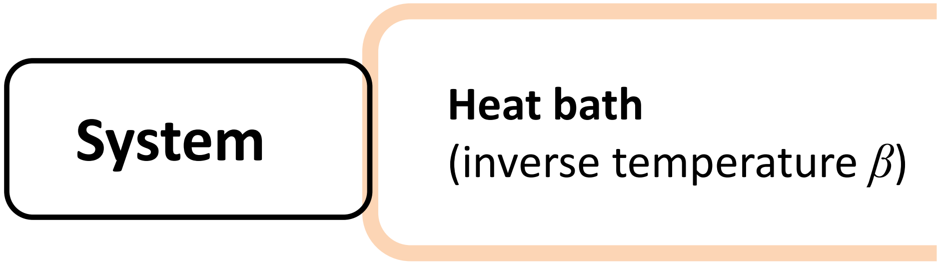
Thermodynamics of *autonomous* information processing

- Measurement and feedback are continuous in time
- No control by an external agent
- Relevant to biological information processing and artificial nanomachines





Reminder: Second law of thermodynamics



System

Heat bath
(inverse temperature β)

System's Shannon entropy:

$$S \equiv -\sum_x P(x) \ln P(x)$$

Heat absorption by the system: Q

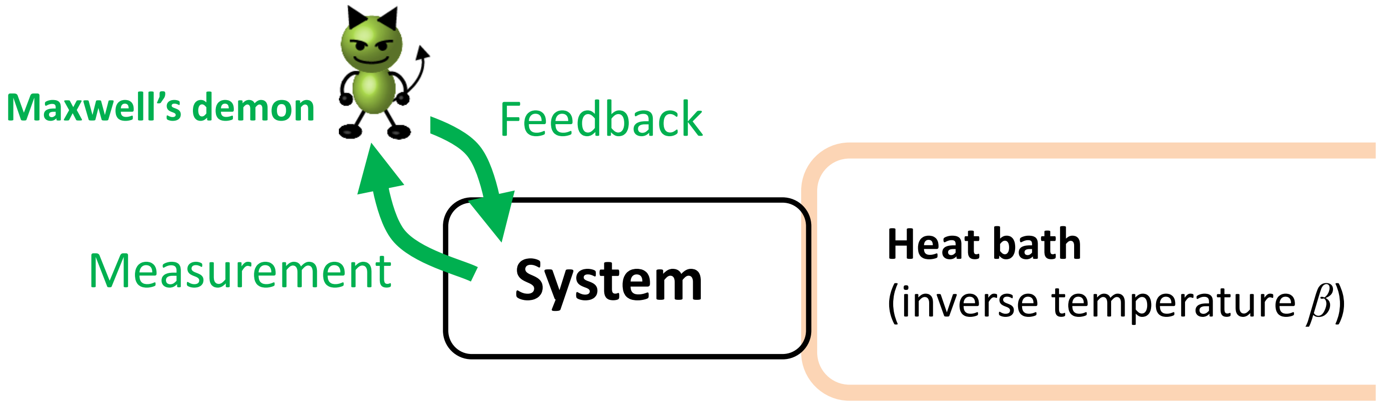
Entropy production rate

$$\dot{\Sigma} \equiv \dot{S} - \beta\dot{Q} \geq 0$$

Second law

Time integral: $\Sigma \equiv \int_0^\tau \dot{\Sigma} dt = \Delta S - \beta Q \geq 0$

Second law with single feedback



Maxwell's demon

Feedback

Measurement

System

Heat bath
(inverse temperature β)

Mutual information $I \equiv \sum_{x,y} P(x,y) \ln \frac{P(x,y)}{P(x)P(y)} \geq 0$

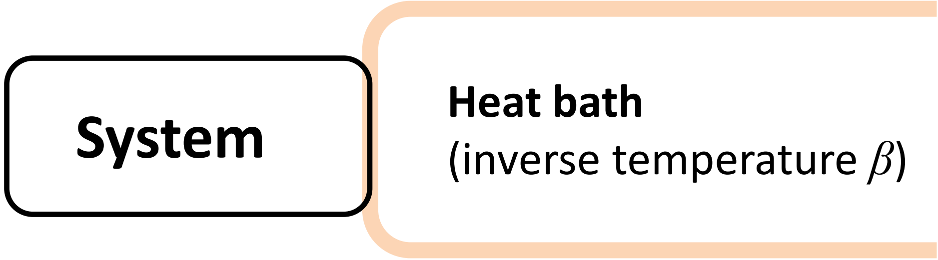
x: system's state
y: demon's outcome

Generalized second law: $\Sigma \equiv \Delta S - \beta Q \geq -I$

⇨ Bound on the extractable work $W_{\mathrm{ext}} \leq -\Delta F + k_{\mathrm{B}} T I$

TS and M. Ueda, PRL **100**, 080403 (2008)
TS and M. Ueda, PRL **104**, 090602 (2010)
TS and M. Ueda, PRL **109**, 180602 (2012)

Fluctuation theorem



Stochastic Shannon entropy:

$$s(x) \equiv -\ln P(x)$$

Stochastic heat: q

$$\langle \dot{s} \rangle = \dot{S}$$

$$\langle \dot{q} \rangle = \dot{Q}$$

$$\langle \dot{\sigma} \rangle = \dot{\Sigma}$$

Stochastic entropy production rate: $\dot{\sigma} \equiv \dot{s} - \beta \dot{q}$

Time integral: $\sigma \equiv \int \dot{\sigma} dt = \Delta s - \beta q$

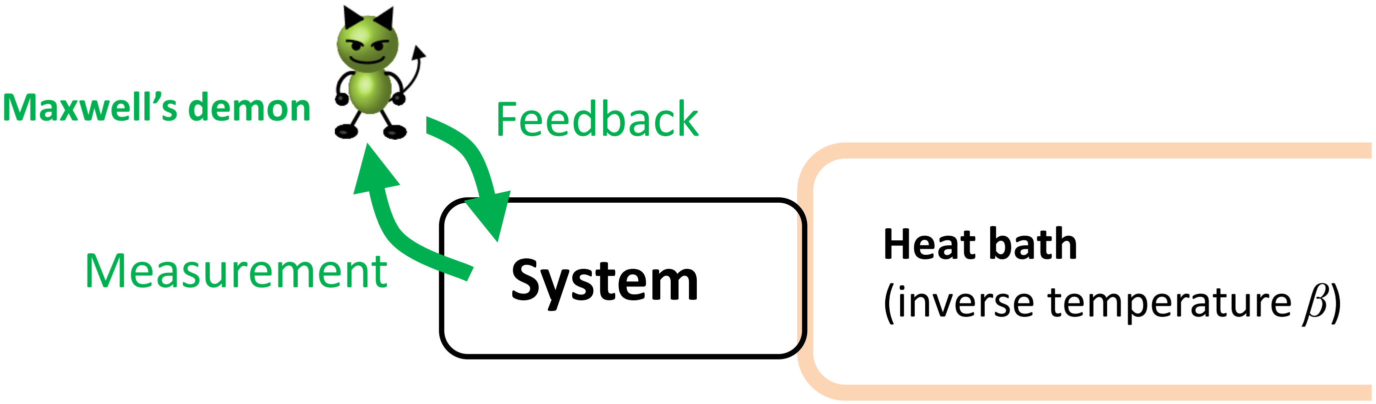
Fluctuation theorem

$$\langle \exp(-\sigma) \rangle = 1 \quad \Rightarrow$$

Second law:

$$\langle \sigma \rangle \geq 0 \quad \Leftrightarrow \quad \Sigma \geq 0$$

Fluctuation theorem with single feedback



Maxwell's demon

Feedback

Measurement

System

Heat bath
(inverse temperature β)

Stochastic **mutual information:** $i(x:y) \equiv \ln \dfrac{P(x,y)}{P(x)P(y)}$ $\qquad \langle i \rangle = I$

Generalized fluctuation theorem: $\langle \exp(-\sigma - i) \rangle = 1$

TS and M. Ueda, PRL **104**, 090602 (2010)

Reproduce the generalized second law:

$\Rightarrow \quad \langle \sigma \rangle \geq -\langle i \rangle \quad \Leftrightarrow \quad \Sigma \geq -I$

Main Questions:

- How to quantify *continuous* information flow?
- How to relate it to the second law?
- How is it relevant to biophysics?

Two Approaches to Continuous information flow

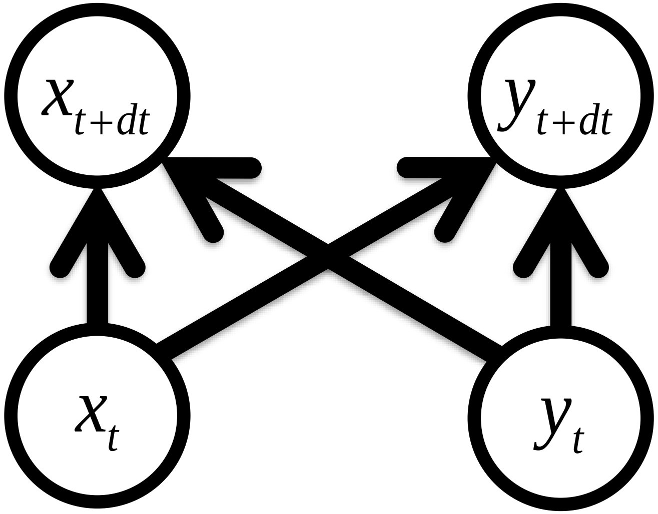
- **“Information flow”** approach
  - ✓ Not applicable to non-Markovian dynamics
  - ✓ Second law is stronger in Markovian dynamics

- **“Transfer entropy”** approach
  - ✓ Applicable to non-Markovian dynamics
  - ✓ Second law is weaker in Markovian dynamics

Setup: Bipartite Markov jump process

Master equation

$$\frac{d}{dt}P_t(x,y)=\sum_{x',y'}W_{x'x}^{y'y}P_t(x',y')$$



Bipartite property: No simultaneous transition!

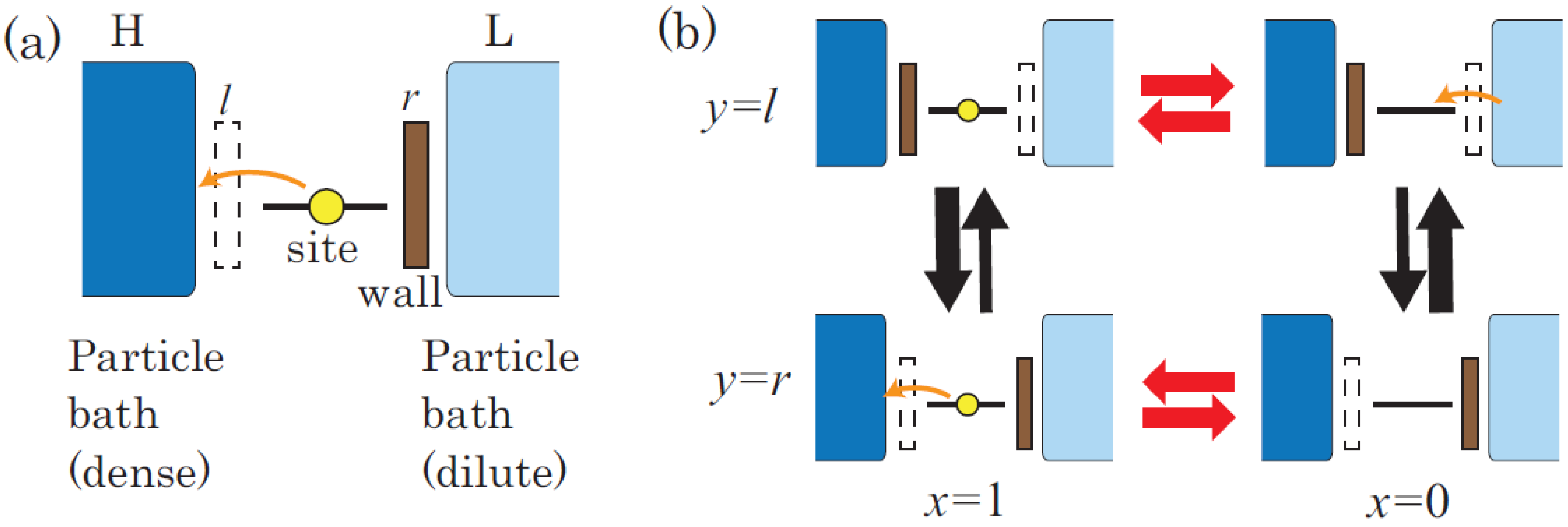
$$W_{x'x}^{y'y}=0 \quad \text{if } x\neq x' \text{ and } y\neq y'$$

Probability flux: $J_{x'\to x}^{y}\equiv W_{x'x}^{yy}P_t(x',y)$ $\quad J_{x}^{y'\to y}\equiv W_{xx}^{y'y}P_t(x,y')$

Various biochemical phenomena can be described by this setup

4-state model

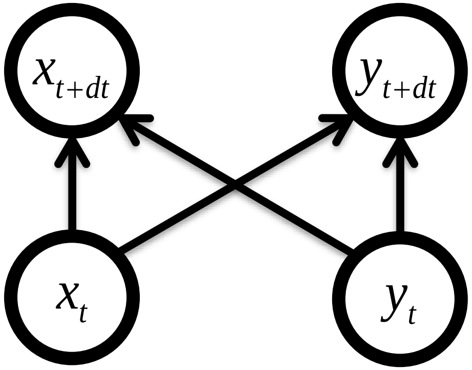
Simple example of autonomous measurement and feedback



Particle = Engine
Wall = Demon

Information flow (Learning rate)

Partial derivative of mutual information



$$\dot{I}_X \equiv \frac{I(X_{t+dt}:Y_t)-I(X_t:Y_t)}{dt}$$

$$\dot{I}_Y \equiv \frac{I(X_t:Y_{t+dt})-I(X_t:Y_t)}{dt}$$

$$\Rightarrow \quad \dot{I}_X + \dot{I}_Y = \dot{I}_{XY} \equiv \frac{dI(X_t:Y_t)}{dt}$$

In steady states, $\dot{I}_X + \dot{I}_Y = 0$

$\dot{I}_Y > 0$: **Y is the demon** that obtains information about X

$\dot{I}_X < 0$: **X is the engine** that is feedback-controlled by Y

Second Law with information flow

Total entropy production: $\dot{\Sigma}_{\mathrm{tot}} \equiv \dot{S}_{XY} - \beta \dot{Q}_{XY} \geq 0$

Entropy increase in subsystems: $\dot{\Sigma}_{X} \equiv \dot{S}_{X} - \beta \dot{Q}_{X}$ $\quad$ $\dot{\Sigma}_{Y} \equiv \dot{S}_{Y} - \beta \dot{Q}_{Y}$

Decomposition of the total entropy production:

$$\begin{aligned} \dot{\Sigma}_{\mathrm{tot}} &= \dot{\Sigma}_{X} + \dot{\Sigma}_{Y} - \dot{I}_{XY} \\ &= \boxed{(\dot{\Sigma}_{X} - \dot{I}_{X})} + \boxed{(\dot{\Sigma}_{Y} - \dot{I}_{Y})} \end{aligned}$$

Generalized second laws: $\boxed{\dot{\Sigma}_{X} - \dot{I}_{X}} \geq 0$ $\quad$ $\boxed{\dot{\Sigma}_{Y} - \dot{I}_{Y}} \geq 0$

$\dot{I}_{X} < 0$ ➡ $\dot{\Sigma}_{X} < 0$ is possible (Y is Maxwell's demon)

Allahverdyan, Janzing, Mahler, J. Stat. Mech. P09011 (2009)
Horowitz & Esposito, Phys. Rev. X **4**, 031015 (2014)

Towards the fluctuation theorem with information flow

How to define the *stochastic* information flow?

$$\frac{d}{dt}\underbrace{i_t(x_t:y_t)}_{\text{Stochastic mutual information}} = i_X + i_Y \qquad \text{such that} \quad \langle i_X \rangle = \dot{I}_X \quad \langle i_Y \rangle = \dot{I}_Y$$

Not so trivial than it looks like, but can be done with

$$i_X \equiv \underbrace{\frac{i_t(x_{t+dt}:y_t) - i_t(x_t:y_t)}{dt}}_{\text{Induced by state jumps of } X} + \underbrace{\frac{1}{P_t(x_t,y_t)}\sum_{x'} J^{y_t}_{x'\to x_t} - \frac{1}{P_t(x_t)}\sum_{y,x'} J^{y}_{x'\to x_t}}_{\text{Induced by the time evolution of the probability distributions, without state jumps of } X.}$$

Induced by state jumps of X

Induced by the time evolution of the probability distributions,
without state jumps of X.
Vanishes with the ensemble average!

N. Shiraishi & T. Sagawa, Phys. Rev. E **91**, 012130 (2015)

Fluctuation theorem with information flow

Integrated stochastic information flow: $\Delta i_X \equiv \int i_X dt$

Generalized fluctuation theorem:

$$\left\langle \exp\left(-\sigma_X + \Delta i_X\right)\right\rangle = 1$$

N. Shiraishi & T. Sagawa, Phys. Rev. E **91**, 012130 (2015)

Cf. Langevin case: Rosinberg & Horowitz, EPL **116** 10007 (2016)

Reproduce the generalized second law:

$$\langle \sigma_X \rangle - \langle \Delta i_X \rangle \geq 0 \quad \Longleftrightarrow \quad \Sigma_X - \Delta I_X \geq 0$$

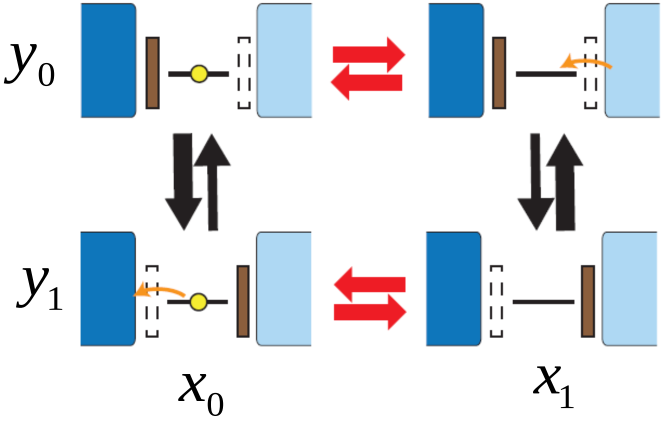
Information affinity

Suppose the steady state: $\dot{I} \equiv \dot{I}_Y = -\dot{I}_X$

F_X, F_Y : conventional thermodynamic affinity

J_X, J_Y : their conjugate current

$$\dot{\Sigma}_X = J_X F_X \qquad \dot{\Sigma}_Y = J_Y F_Y$$



Information affinity:

$$F_I \equiv i(x_1 : y_0) - i(x_0 : y_0) + i(x_0 : y_1) - i(x_1 : y_1) = \ln \frac{P(x_1 : y_0)P(x_0 : y_1)}{P(x_0 : y_0)P(x_1 : y_1)}$$

$$\dot{I} = J_I F_I$$

J_I : information driven current

Generalized second law is rewritten as:

$$\dot{\Sigma}_X + \dot{I} = J_X F_X + J_I F_I \geq 0 \qquad \dot{\Sigma}_Y - \dot{I} = J_Y F_Y - J_I F_I \geq 0$$

Horowitz & Esposito, Phys. Rev. X **4**, 031015 (2014)

Onsager reciprocity with information affinity

Expand the currents in the *linear* regime:

$$J_X = L_{XX} F_X + L_{XI} F_I \qquad J_I = L_{IX} F_X + L_{II} F_I$$

Reciprocity: $L_{IX} = L_{XI}$

Yamamoto, Ito, Shiraishi, Sagawa,
Phys. Rev. E **94**, 052121 (2016)

- ✓ Even with the information affinity: Quite nontrivial!
- ✓ The proof is based on the Schnakenberg's network theory

  J. Schnakenberg, Rev. Mod. Phys. **48**, 571 (1976)
- ✓ Fluctuation-dissipation theorem fails in the presence of the information affinity.

In the linear regime, $\dot{\Sigma}_X + \dot{I} = L_{XX} F_X^2 + 2L_{XI} F_X F_I + L_{II} F_I^2$

Generalized second law can be expressed as

$$L_{XX} \geq 0 \qquad L_{II} \geq 0 \qquad L_{XX} L_{II} - L_{XI}^2 \geq 0$$

Application: Efficiency at maximum power

Is there any universal bound on the efficiency at maximum power?

Seminal works:
F. Curzon & B. Ahlborn, Am. J. Phys. **43**, 22 (1975)
C. Van den Broeck, Phys. Rev. Lett. **95**, 190602 (2005)

Power = Work per unit time

Extension to information thermodynamics,
based on the Onsager reciprocity with information affinity

Yamamoto, Ito, Shiraishi, Sagawa, Phys. Rev. E **94**, 052121 (2016)

Information-thermodynamic efficiency: $\eta \equiv -\frac{J_X F_X}{J_I F_I} \leq 1$

Generalized second law

Power: $P \equiv -J_X F_X$

In the linear regime, the efficiency at maximum P bounded as

$$\boxed{\eta \leq \frac{1}{2}}$$

The same form as the conventional bound!

The equality is achieved if and only if the tight-coupling condition $L_{XX} L_{II} - L_{XI}^2 = 0$

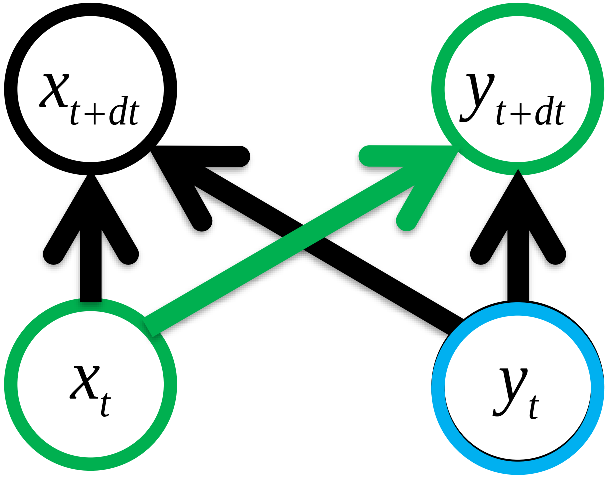
Two Approaches to Continuous information flow

- **“Information flow”** approach
  - ✓ Not applicable to non-Markovian dynamics
  - ✓ Second law is stronger in Markovian dynamics

- **“Transfer entropy”** approach
  - ✓ Applicable to non-Markovian dynamics
  - ✓ Second law is weaker in Markovian dynamics

Transfer entropy

Directional information transfer between two systems



Transfer entropy:
Directional information transfer
from X to Y
during time t and $t + dt$

Conditional mutual information

$$\dot{T}_{X\to Y} \equiv \frac{1}{dt} I\left(X_t : Y_{t+dt} \mid \underline{Y_t}\right)$$

$$\equiv \frac{1}{dt} \sum_{x_t y_t y_{t+dt}} P(x_t, y_t, y_{t+dt}) \ln \frac{P(x_t, y_{t+dt} \mid y_t)}{P(x_t \mid y_t) P(y_{t+dt} \mid y_t)}$$

$$\geq 0$$

T. Schreiber, PRL **85**, 461 (2000)

Fluctuation theorem and the second law

***Stochastic* transfer entropy** $\dot{\tau}_{X \to Y} \equiv \frac{1}{dt} \ln \frac{p(x_t, y_{t+dt} \mid y_t)}{p(x_t \mid y_t) p(y_{t+dt} \mid y_t)}$ $\dot{T}_{X \to Y} = \langle \dot{\tau}_{X \to Y} \rangle$

$\Delta i \coloneqq i_{\text{fin}} - i_{\text{ini}}$: Difference of the initial and final stochastic mutual information

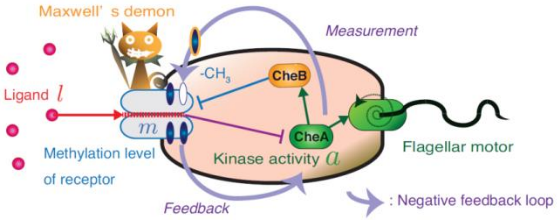
Fluctuation theorem: $\langle e^{-\sigma_X - \tau_{X \to Y} + \Delta i} \rangle = 1$

Second law: $\langle \sigma_X \rangle \geq -\langle \tau_{X \to Y} \rangle + \langle \Delta i \rangle$

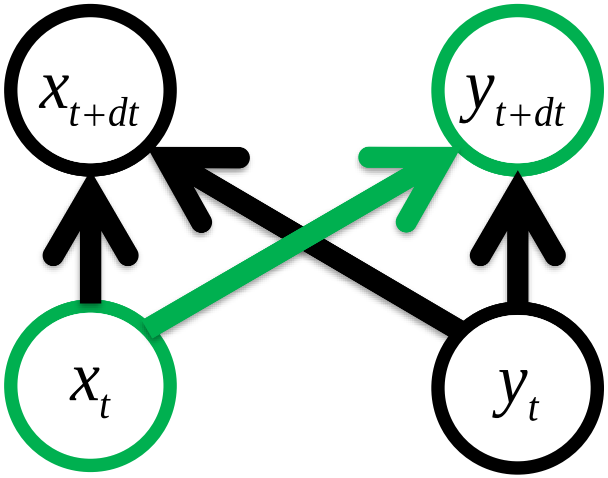
⇔ $\Sigma_X \geq -T_{X \to Y} + \Delta I$

S. Ito & T. Sagawa, PRL **111**, 180603 (2013).

Application to biochemical signal transduction:
S. Ito & T. Sagawa, Nat. Commu. **6**, 7498 (2015)



Transfer entropy vs. Information flow



D. Hartich, A. C. Barato, U. Seifert,
Phys. Rev. E **93**, 022116 (2016)

$$\dot{I}_Y \le \dot{T}_{X\to Y}$$ generally holds

Transfer entropy gives an upper bound of information flow!

Sensory capacity: $$C \equiv \frac{\dot{I}_Y}{\dot{T}_{X\to Y}} \le 1$$

Characterizes a kind of efficiency of information gain

When is the sensory capacity maximized?

Sufficient statistics

E.g. Coin toss

	Probability
H	θ
T	$1-\theta$

Estimation of θ :

$$p(\theta|\underline{\text{H H T H T T T}})$$

$$\Downarrow$$

$$p(\theta|\underbrace{\text{The number of head}}_{\text{Sufficient statistics}})$$

In our setup, y_{t+dt} is sufficient statistics of x_{t+dt} if $P(x_{t+dt} \mid y_{t+dt}, \boxed{y_t}) = P(x_{t+dt} \mid y_{t+dt})$

Our theorem: **Sufficient statistics** → $C = 1$

T. Matsumoto, T. Sagawa, *Phys. Rev. E* **97**, 042103 (2018).

Special case: **Kalman-Bucy filter** is sufficient statistics and gives $C = 1$

Horowitz & Sandberg, New J. Phys. 16 125007 (2014)

Outline

- Introduction
- An introduction to information theory
- Second law of information thermodynamics
- Experimental demonstrations of Maxwell’s demon
- Thermodynamics of measurement and erasure
- Entropy production
- General framework of information thermodynamics
- Autonomous Maxwell’s demons
- **Summary**

Summary

- ✓ Second law under feedback control
- ✓ Experimental demonstrations
- ✓ Thermodynamics of measurement and erasure
- ✓ Paradox of Maxwell's demon
- ✓ Autonomous Maxwell's demons

Review:
J. M. R. Parrondo, J. M. Horowitz, & T. Sagawa, Nature Physics **11**, 131-139 (2015).

Thank you for your attention!