The 6th KIAS School and Workshop on Quantum Information and Thermodynamics @ Busan, Dec. 7-10, 2022

Many-Body Localization

Fundamentals and Selected Topics

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Outline

1. **Thermalization in a closed quantum system**
   - Eigenstate Thermalization Hypothesis
   - Operator Spreading
2. **Many-body localization**
   - MBL vs. Anderson localization vs. thermalization
   - LIOM and the effective l-bit model
   - Instability of a MBL phase

Thermalization

Closed quantum system

- No symmetry assumed, for simplicity, but energy conservation
- Unitary dynamics (reversible!)

$$|\Psi(t)\rangle = U|\Psi(0)\rangle \qquad U = e^{-\frac{i}{\hbar}Ht}$$

- "Deterministic" time-evolution of a quantum state

t*-indep. *H

interacting

$N \to \infty,\ t \to \infty$

How does thermalization arise in a unitary time evolution?

(** $N \to \infty$, $t \to \infty$, **local observables)

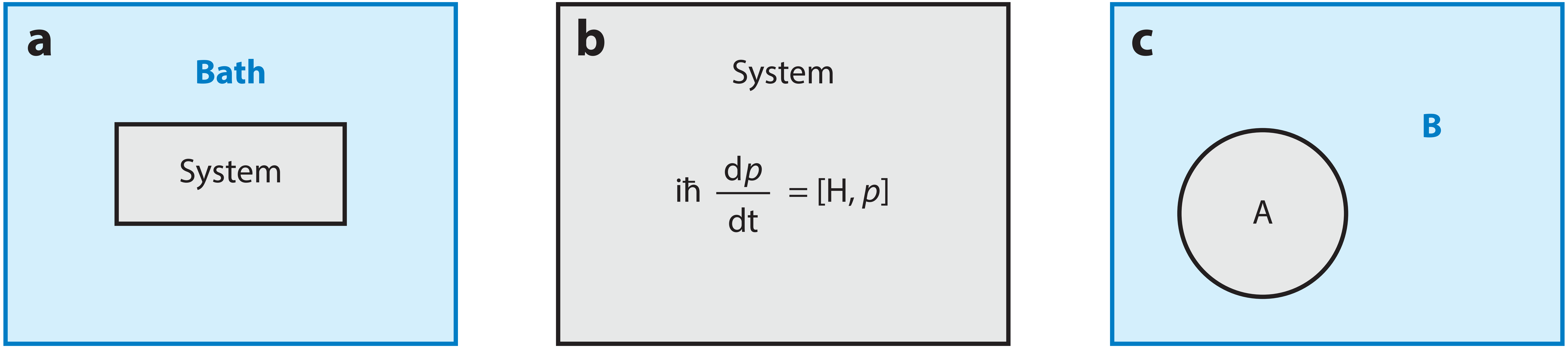


Figure 2

(*a*) Conventional quantum statistical mechanics assume that the system of interest is coupled to a reservoir (or bath) with which it can exchange energy and particles. (*b*) Here, we are interested in the statistical mechanics of a closed quantum system undergoing unitary time evolution. There is no external reservoir. (*c*) It can be useful to partition the closed quantum system into a (A) subsystem and (B) everything else. If the system quantum thermalizes, then the region (B) is able to act as a bath for the subsystem (A).

Figure from Nandkishore and Huse, Annual Review of Condensed Matter Physics 2015

Thermal Ensembles

- Microcanonical ensemble

$$I = [E - \Delta, E + \Delta],\ O_{\mathrm{MC}}(E) \equiv \frac{1}{\Omega} \sum_{E_\alpha \in I} \langle \alpha | \hat{O} | \alpha \rangle$$

- (Grand-) Canonical ensemble

$$\hat{\rho}_\beta \equiv \frac{1}{Z} \exp(-\beta \hat{H}),\ \langle \hat{O} \rangle = \mathrm{tr}[\hat{\rho}_\beta \hat{O}]$$

In a thermal system,
in the thermodynamic limit,

$$\frac{1}{\Omega} \sum_{E_\alpha \in I} \langle \alpha | \hat{O} | \alpha \rangle = \mathrm{tr}[\hat{\rho}_\beta \hat{O}]$$

Observables

- Not all Hermitian operators are observables.

$$\hat{H} = \sum_{\alpha} E_{\alpha} |\alpha\rangle\langle\alpha|$$

$$[\hat{H}, |\alpha\rangle\langle\alpha|] = 0$$

But, $|\alpha\rangle\langle\alpha|$ ***may not be measurable.***

- "Physical" observables are something that can be measured.
  - local spins, two-point correlator, etc.
  - We consider mostly ***local*** observables.

(dynamical) thermalization

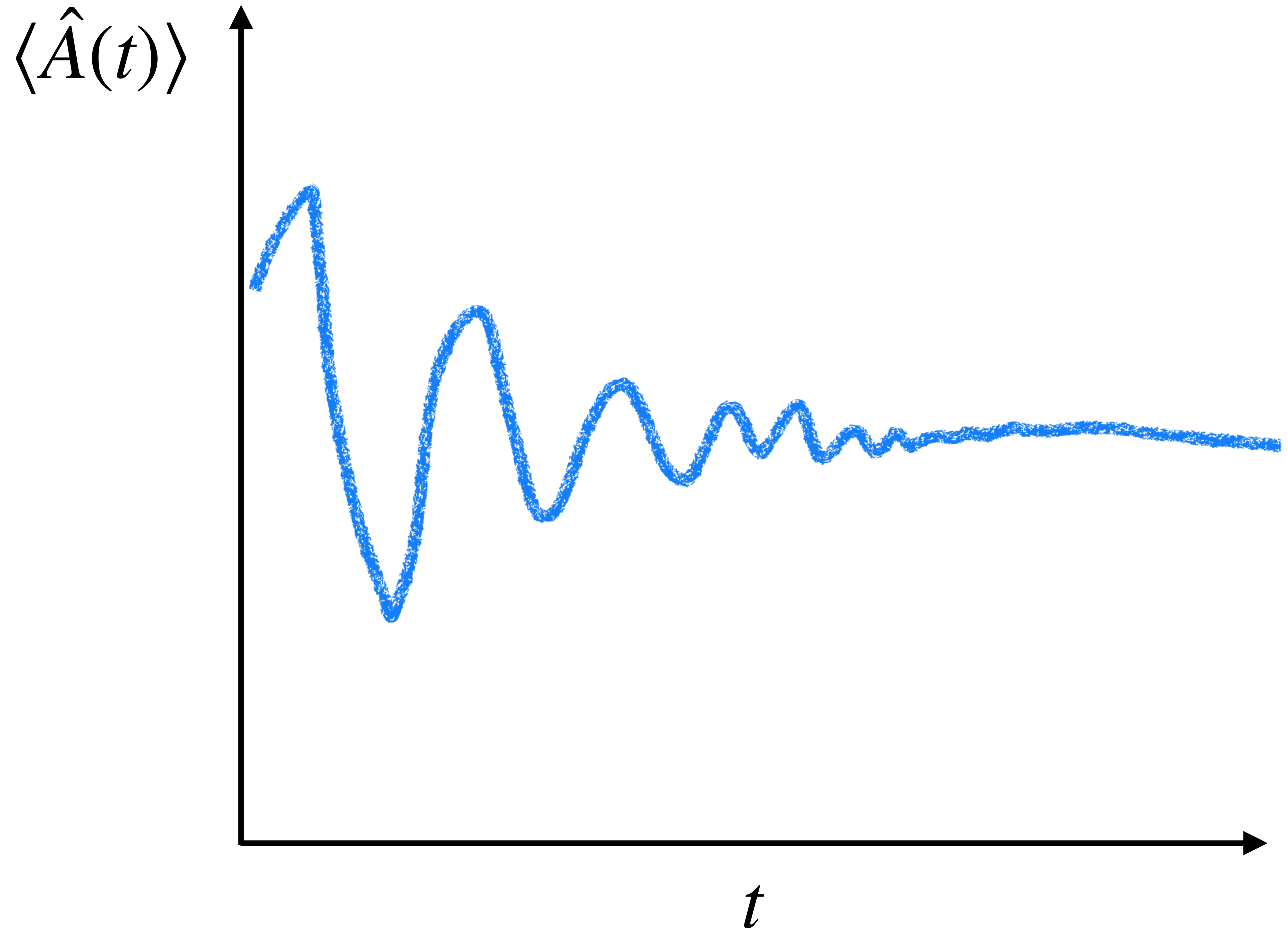
- Infinite-time average (relaxation)

$$\bar{A}_\infty \equiv \lim_{T\to\infty} \frac{1}{T}\int_0^T dt\,\langle\Psi(t)\,|\,\hat{A}\,|\,\Psi(t)\rangle$$

- Infinite-time average ?= ensemble average

$$\bar{A}_\infty = A_{\mathrm{MC}}(E) = \mathrm{tr}[\hat{\rho}_\beta \hat{A}]$$

What makes this hold true?

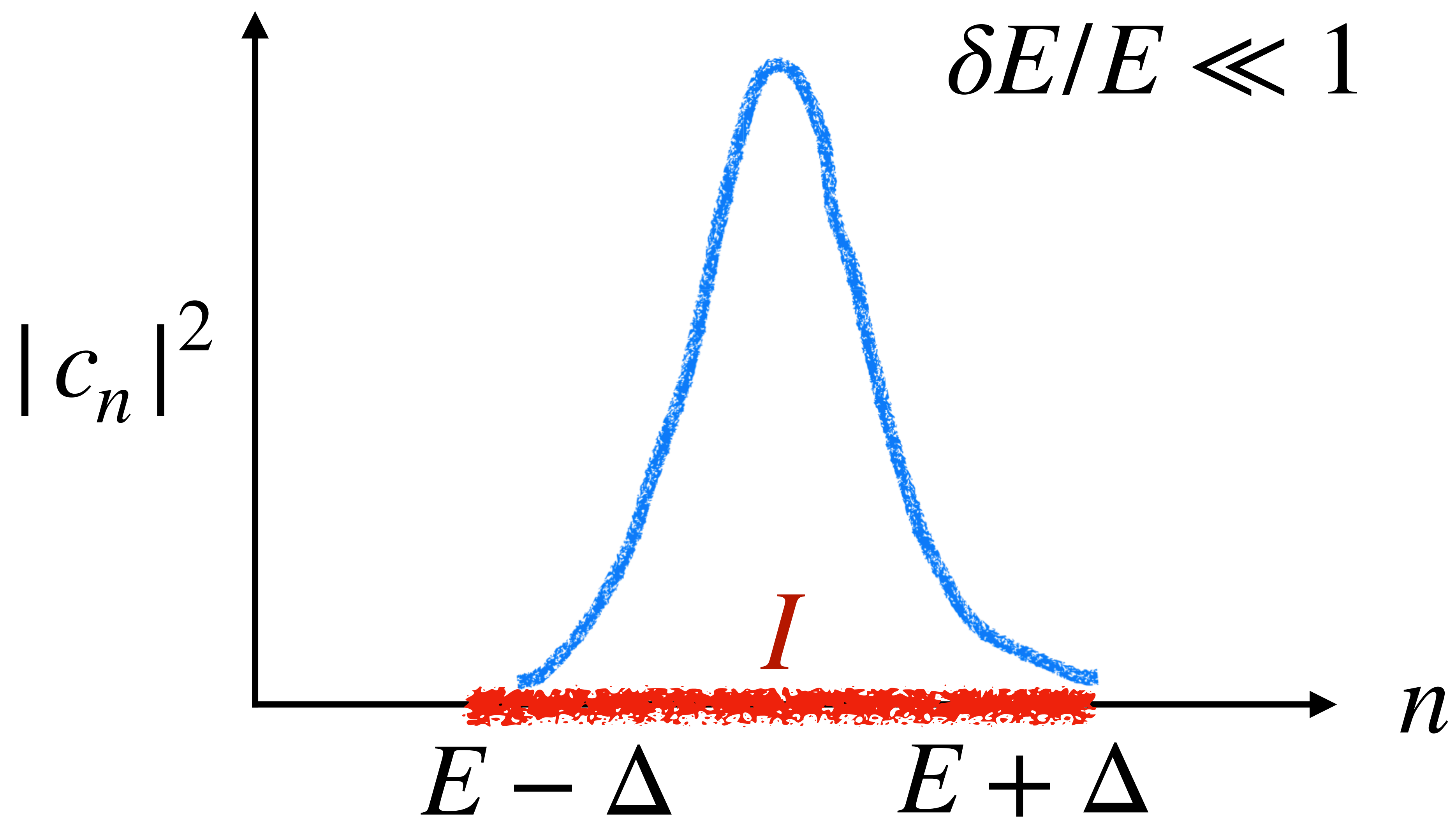


Generic initial state

- Start with an initial state for a targeted E.

$$|\Psi(0)\rangle = \sum_{\alpha} c_\alpha |\alpha\rangle$$

$$|\Psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\Psi(0)\rangle = \sum_{\alpha} c_\alpha e^{-\frac{i}{\hbar}E_\alpha t}|\alpha\rangle$$



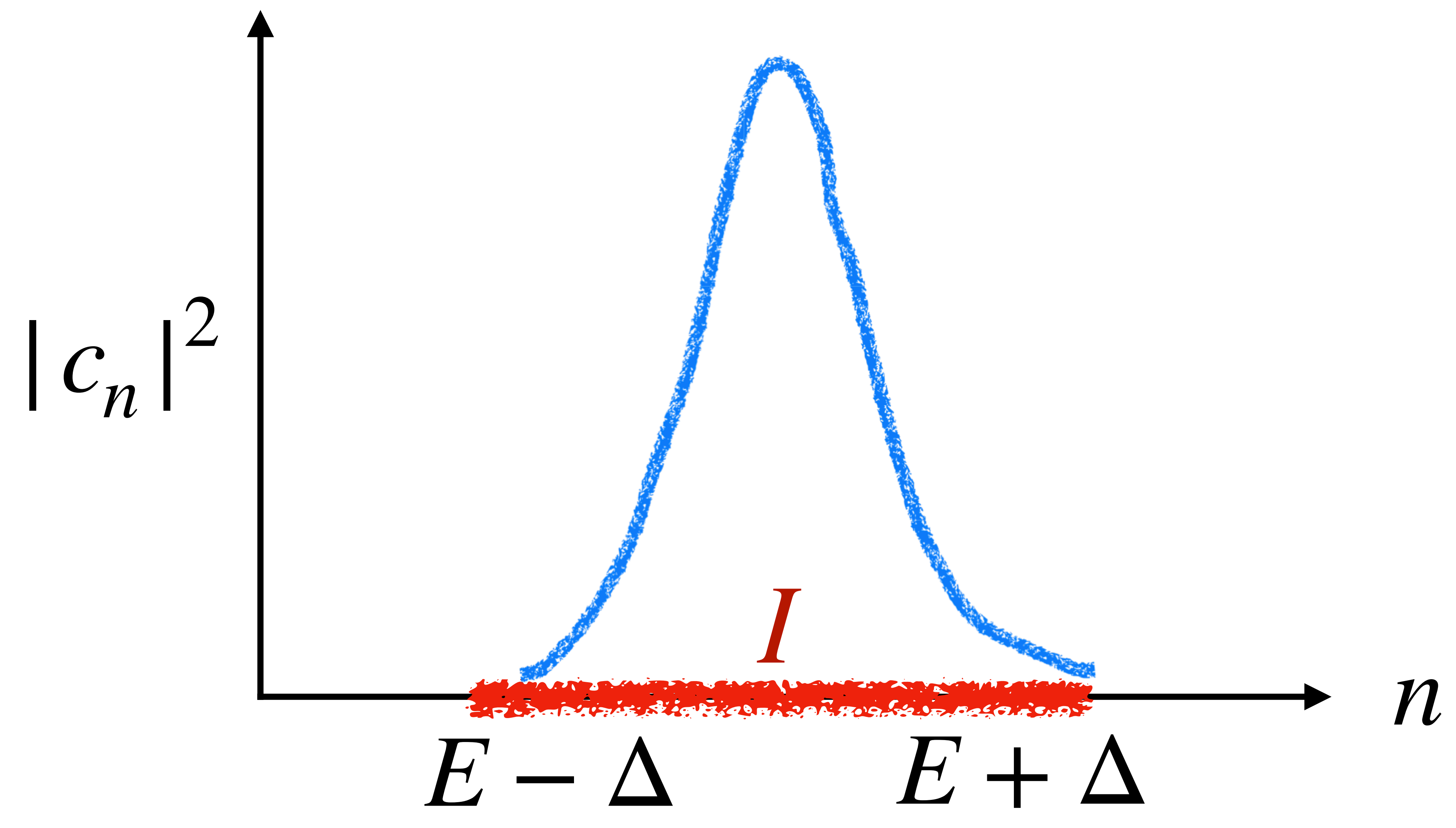
- Time-evolution of an observable

$$\langle\Psi(t)|\hat{A}|\Psi(t)\rangle = \sum_{\alpha,\beta} c_\alpha^* c_\beta e^{\frac{i}{\hbar}(E_\alpha - E_\beta)t}\langle\alpha|\hat{A}|\beta\rangle \xrightarrow[N\to\infty]{t\to\infty} \sum_\alpha |c_\alpha|^2 A_{\alpha\alpha}$$

Diagonal hypothesis

- Time-evolution of an observable

$$\langle \Psi(t) | \hat{A} | \Psi(t) \rangle = \sum_{\alpha,\beta} c_\alpha^* c_\beta e^{\frac{i}{\hbar}(E_\alpha - E_\beta)t} A_{\alpha\beta}$$



$N \to \infty$ $\quad$ $t \to \infty$

If only diagonal terms survive,

$$\bar{A}_\infty \to \sum_\alpha |c_\alpha|^2 A_{\alpha\alpha} \approx \frac{1}{\Omega} \sum_{E_\alpha \in I} A_{\alpha\alpha}$$

thermalization!

Random Matrix Theory

- Assumption (Deutsch 1991, extended Berry's conjecture):

  The eigenstates of ergodic Hamiltonian is essentially random vectors.

$$O_{mn} \approx \bar{O}\delta_{mn} + \sqrt{\frac{\overline{O^2}}{D}} R_{mn}$$

- Random level repulsion: Wigner-Dyson distribution

$$P(\omega) = A_\beta \omega^\beta \exp(-B_\beta \omega^2)$$

$\beta = 1$ **(GOE): w. time-reversal symmetry**

$\beta = 2$ **(GUE): w.o. time-reversal symmetry**

Eigenstate Thermalization Hypothesis

- Jensen & Shankar (1985), Srednicki (1994), Deutsch (1991), ...
- "ALL eigenstates are thermal." (strong-ETH)

$$\boxed{A_{\alpha\beta} = A_{\mathrm{MC}}(\bar{E})\delta_{\alpha\beta} + e^{-S(\bar{E})/2} f(\bar{E}, \omega)\, R_{\alpha,\beta}}$$

$S(\bar{E})$: thermodynamic entropy

$f(\bar{E}, \omega)$: a slowly varying function

$R_{\alpha\beta}$: a "pseudo"-random variable

$\bar{E} = (E_\alpha + E_\beta)/2$

$\omega = E_\beta - E_\alpha$

Entanglement of Eigenstates

- ETH implies the extensive entanglement of an eigenstate.

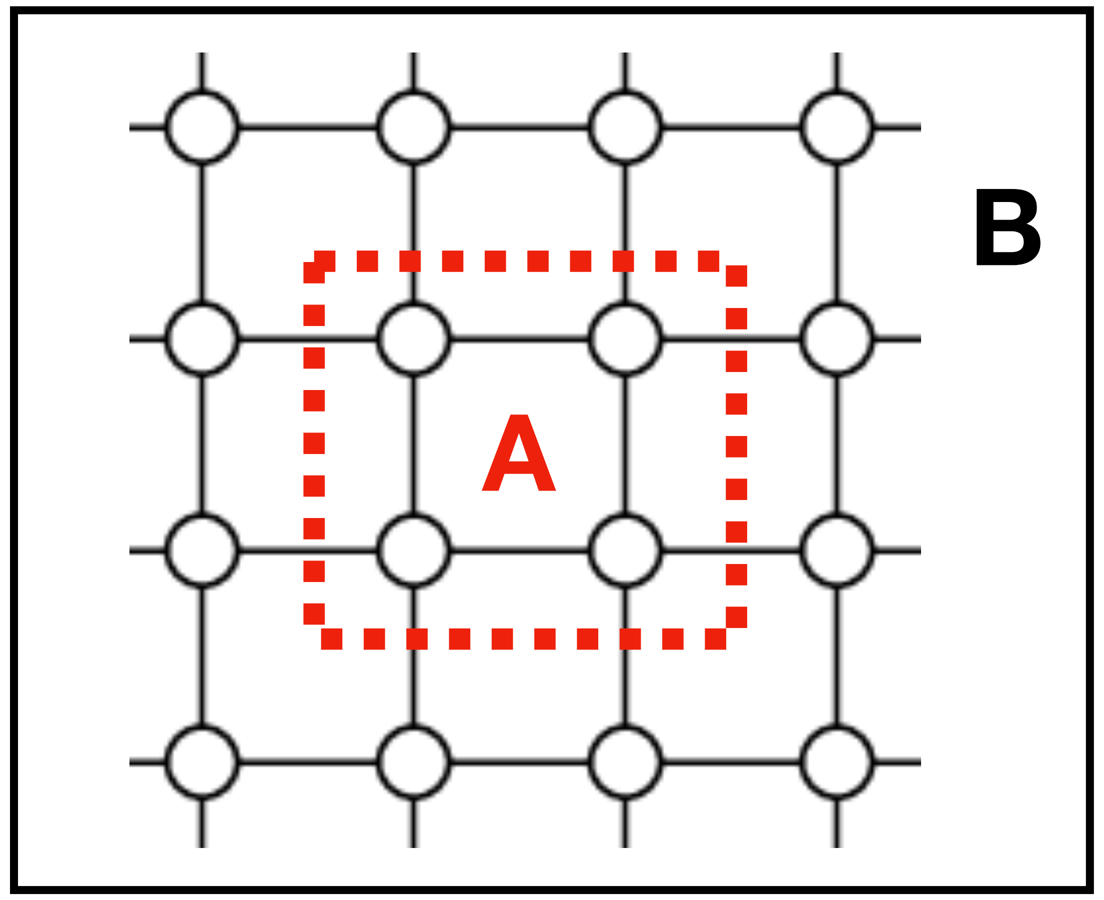
Eigenstate density matrix

$$\rho_E = |E\rangle\langle E|$$

ETH

Microcanonical density matrix

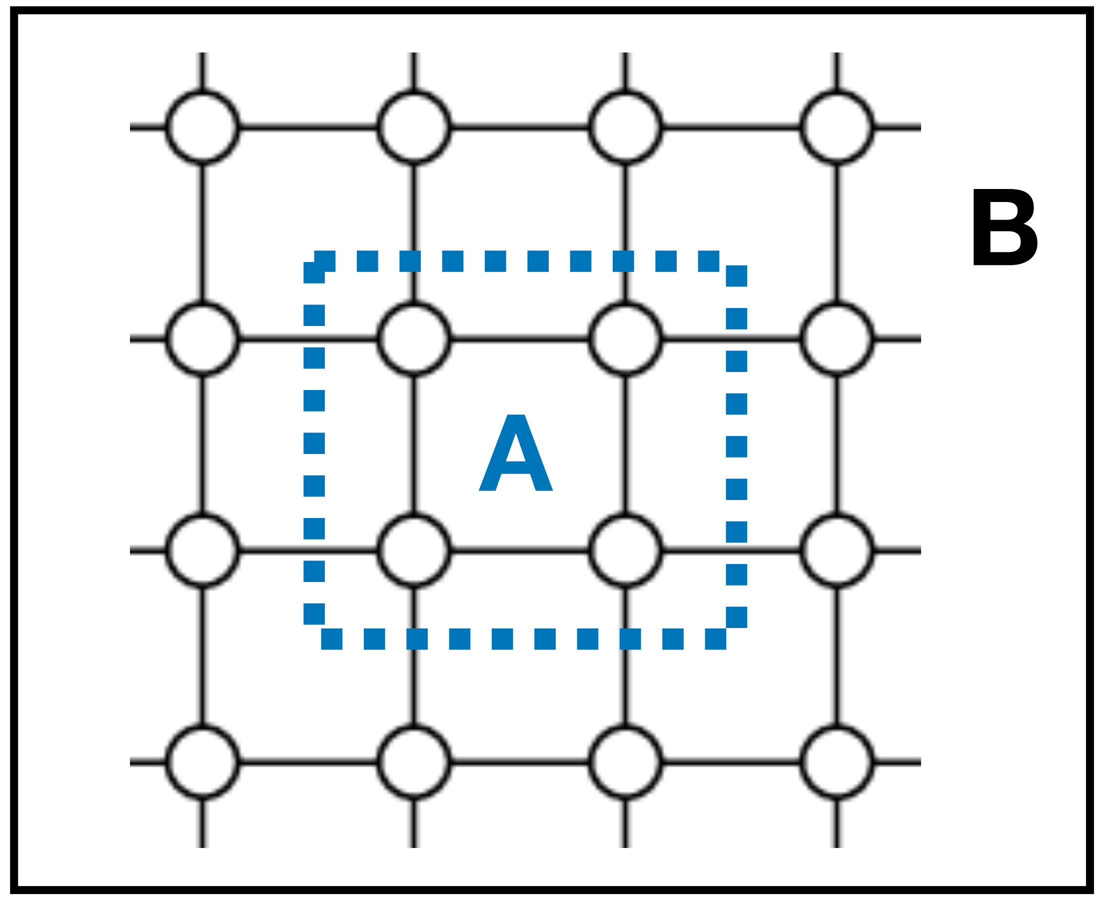
$$\rho_{\mathrm{MC}} = \frac{1}{\Omega} \sum_{E_\alpha \in I} |\alpha\rangle\langle\alpha|$$



Subsystem A:

$$\hat{\rho}_A = \mathrm{tr}_B[\hat{\rho}]$$

$$\boxed{\hat{\rho}_{E,A} \approx \hat{\rho}_{\mathrm{MC},A}}$$



Volume-law of entanglement entropy

- The von Neumann entropy

$$S = -\,\mathrm{tr}[\hat{\rho}\ln\hat{\rho}] \xrightarrow{\rho = |E\rangle\langle E|} \text{"0" (a pure state)}$$

- "Bipartite" entanglement entropy for $|A| \ll |B|$:
  The subsystem A thermalizes with the same energy density.

$$\hat{\rho}_A \approx \hat{\rho}_{\mathrm{MC},A} = \hat{\mathrm{I}}/d^{V_A} \longrightarrow S_A = -\,\mathrm{tr}[\hat{\rho}_A \ln \hat{\rho}_A] \propto V_A$$

- Every eigenstate obeying ETH satisfies the volume-law of EE.

ETH: off-diagonal terms

- Another implication of ETH: strong sensitivity to external perturbation

$$A_{\alpha\beta} = A_{\mathrm{MC}}(\bar{E})\delta_{\alpha\beta} + \underline{e^{-S(\bar{E})/2} f(\bar{E}, \omega)\, R_{\alpha,\beta}}$$

- Small perturbation -> exponentially large mixing in original eigenstates

(typical level spacing << off-diagonal element)

$$\Delta / J \sim e^{-S(E)} \ll e^{-S(E)/2}$$

"many-body resonances"

Growth of Entanglement Entropy

- Quantum quench from a non-entangled initial state

$$S(t) \propto t$$ ***(nonintegrable, quantum Ising chain; H. Kim and D. Hose, PRL 2013)***

- Lieb-Robinson Bound: upper bound on information propagation

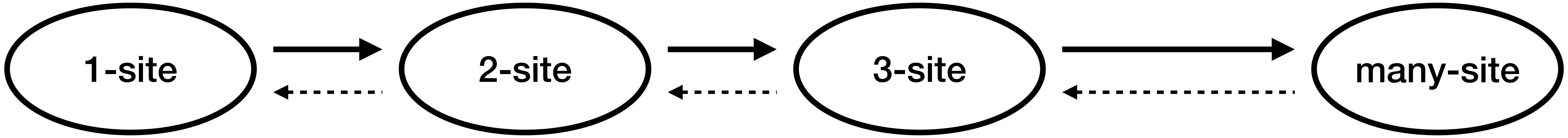
$$\|[A(t), B]\| \leq c\|A\|\|B\|e^{\lambda\left(t - \frac{r}{v_{LR}}\right)}$$

Operator Spreading

- In the Heisenberg picture,

$$\hat{A}(t) = \hat{U}(t)^{\dagger}\hat{A}\hat{U}(t) \qquad\qquad [U, A] \neq 0$$

- A is an observable few-site operator, but it spreads over many more sites.



e.g. a spin-1/2 system

$$\hat{A}(0) = \hat{X}_{100} \longrightarrow \hat{A}(t) = \cdots \hat{Y}_{80}\hat{X}_{97}\hat{Y}_{101}\hat{Z}_{110}\hat{X}_{121}\cdots + \cdots$$

A measure of operator spreading

- Out-of-time-ordered correlation (OTOC) [Larkin & Ovchinnikov 1969, Kitaev 2014]

$$\boxed{C_{ij}(t) = \langle [\hat{A}_i(t), \hat{B}_j]^\dagger [\hat{A}_i(t), \hat{B}_j] \rangle}$$

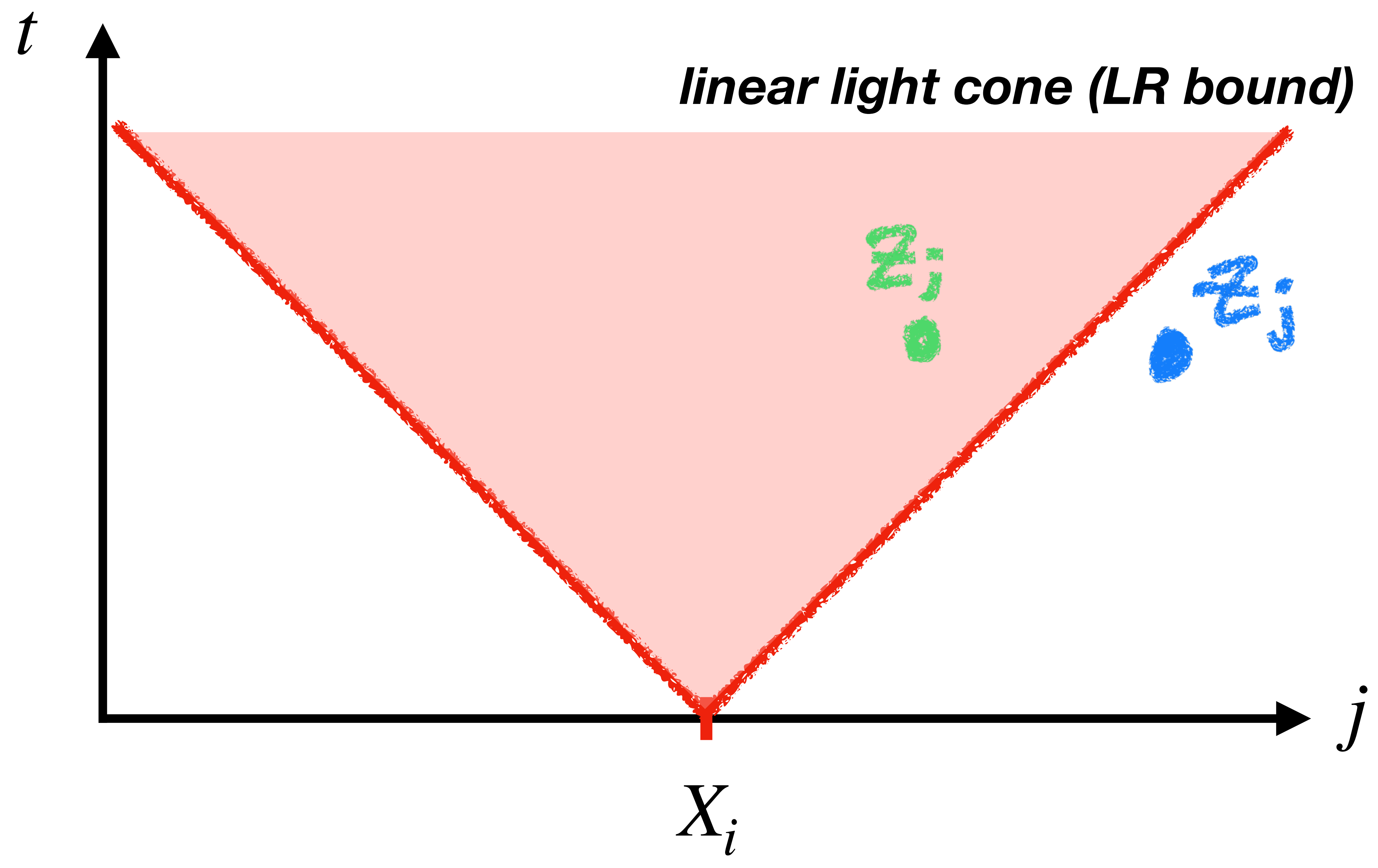
- How does it measure the operator spreading?

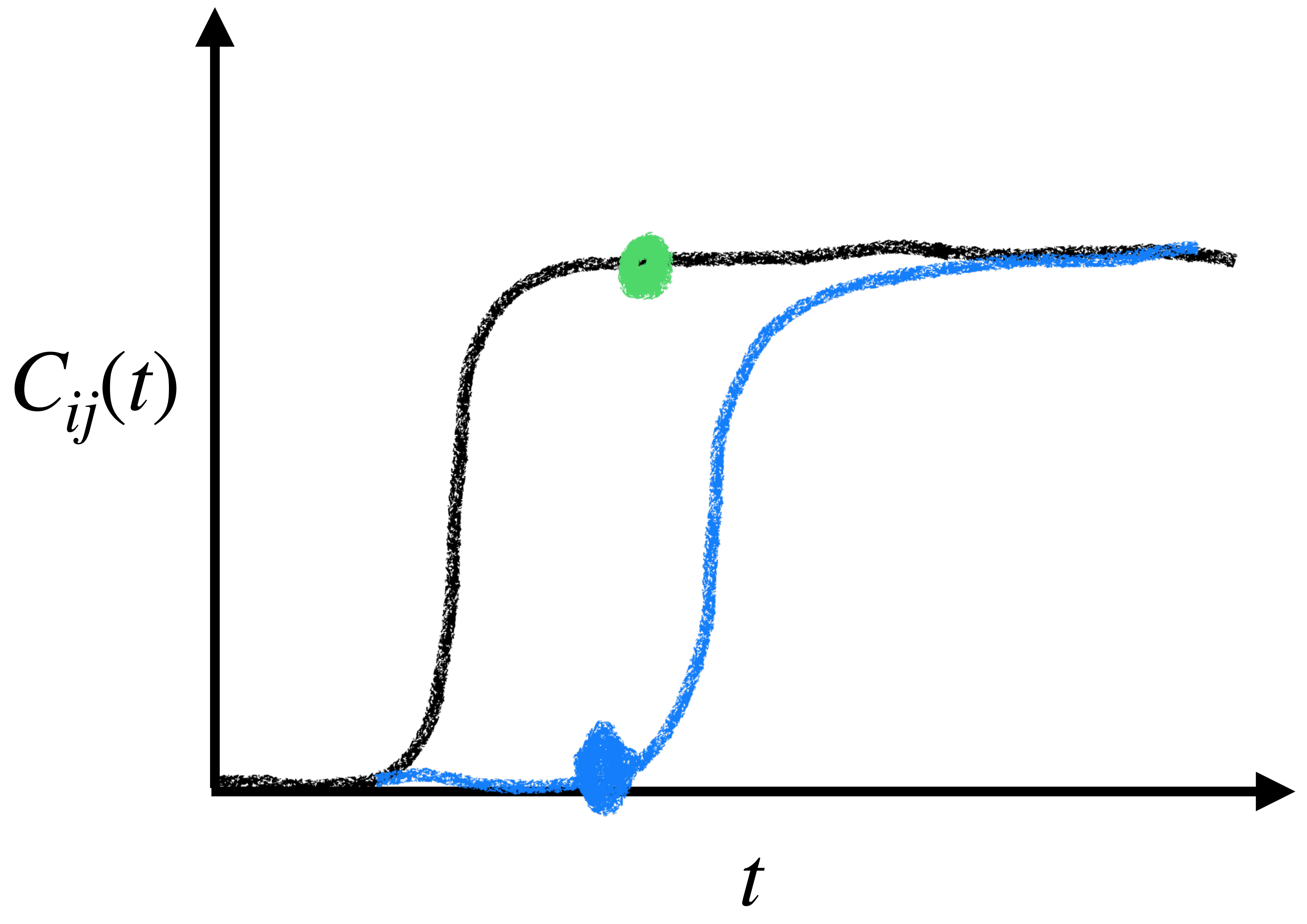
e.g. $T = \infty$ $\quad \hat{B} = \hat{Z}_j \quad \hat{A}(t) = \hat{U}^\dagger(t)\hat{X}_i\hat{U}(t) = \sum_{\hat{S}} a_{\hat{S}}(t)\hat{S}$

$$\longrightarrow \quad C_{ij}(t) = \sum_{\hat{S}} |a_{\hat{S}}(t)|^2 \mathrm{tr}\left([\hat{S}_j, \hat{Z}_j]^\dagger [\hat{S}_j, \hat{Z}_j]\right) = \sum_{\hat{S}_j = \{\hat{X}_j, \hat{Y}_j\}} |a_{\hat{S}}(t)|^2$$

Light cone

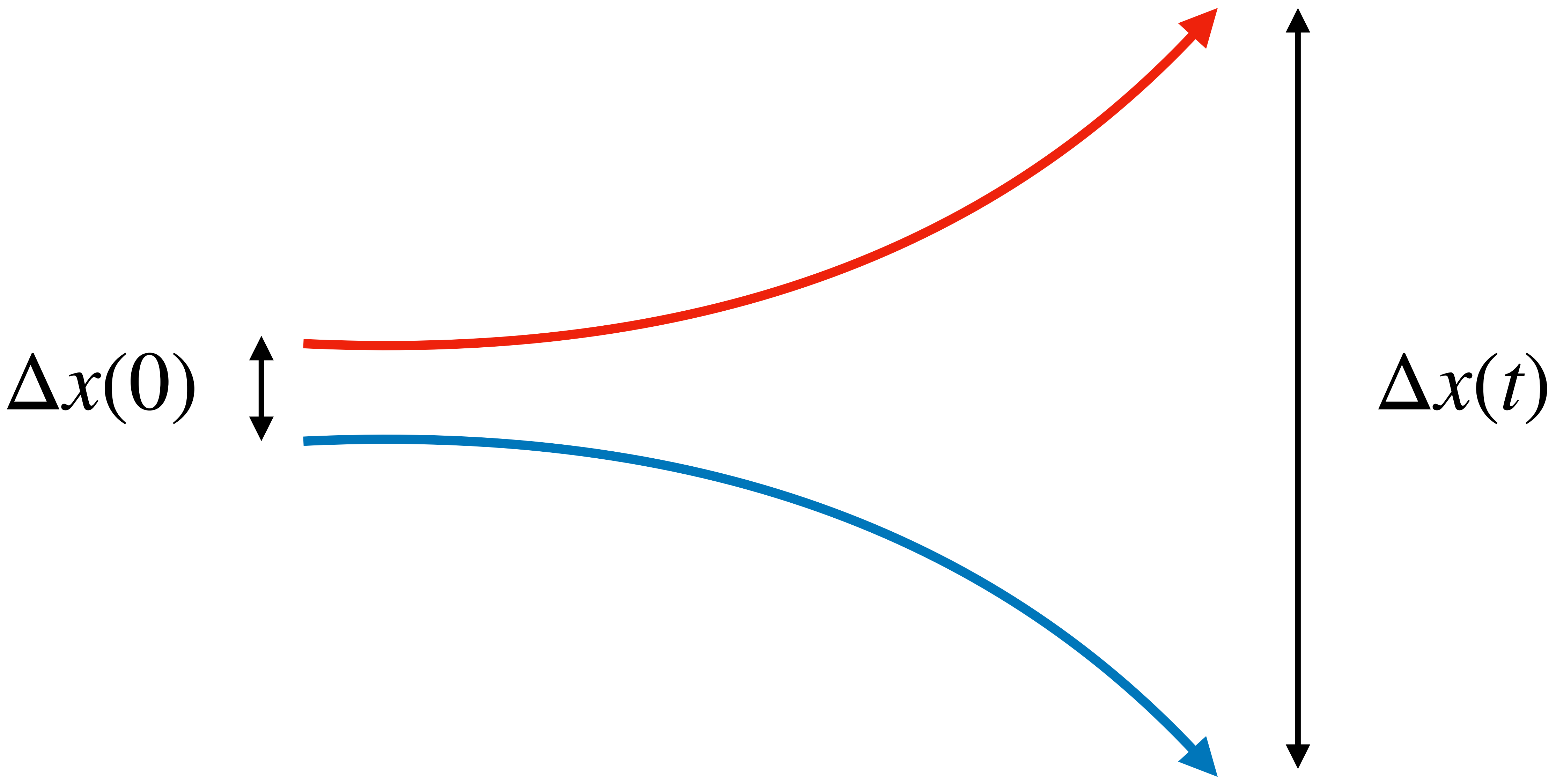
$$C_{ij}(t) = \sum_{\hat{S}} |a_{\hat{S}}(t)|^2 \mathrm{tr}[\hat{S}_j, \hat{Z}_j]^\dagger [\hat{S}_j, \hat{Z}_j] = \sum_{\hat{S}_j = \{\hat{X}_j, \hat{Y}_j\}} |a_{\hat{S}}(t)|^2$$





Semi-Classical Chaos

$$C(t) \sim \langle [\hat{x}(t), \hat{p}]^2 \rangle \longrightarrow \left\langle \{x(t), p\}^2_{\text{P.B.}} \right\rangle_{classical} = \left\langle \left(\frac{dx(t)}{dx(0)} \right)^2 \right\rangle \sim \exp(\lambda_L t)$$



Why "squared" commutator?

$\dfrac{dx(t)}{dx(0)}$ can be positive or negative.

Operator spreading cannot be captured by a two-point correlation function.

$$\langle [A(t), B] \rangle$$

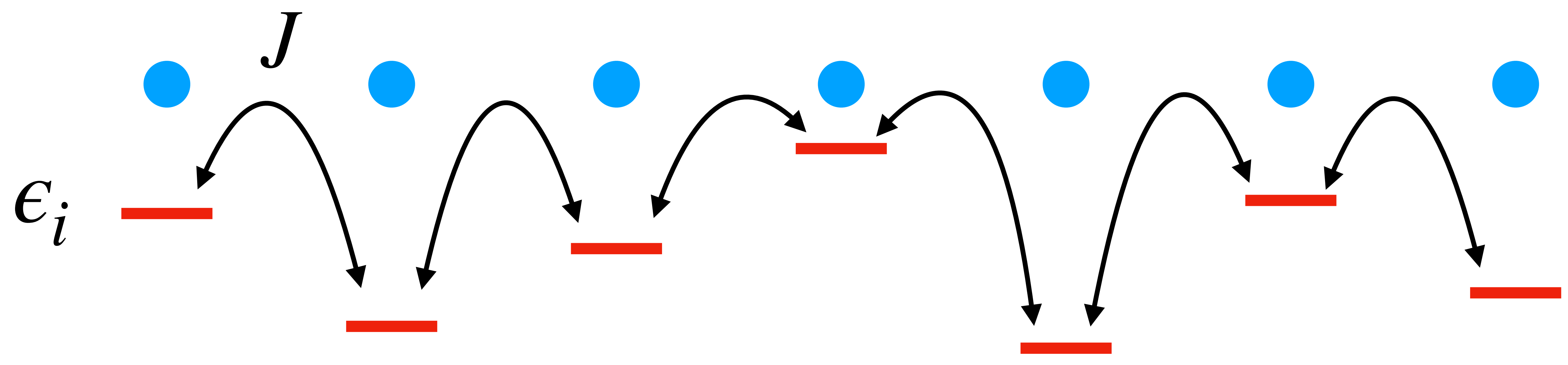
Many-Body Localization

- There are some systems that do not thermalize.
  - Quantum many-body scar, "traditional" integrable systems,
  - Many-body localization (MBL)
- Why is MBL special?
  - It remains "permanently" and "robustly" out of equilibrium.
  - Dephasing without dissipation

Many-Body Localization

Anderson Localization

- Non-interacting spinless fermions in a 1D chain

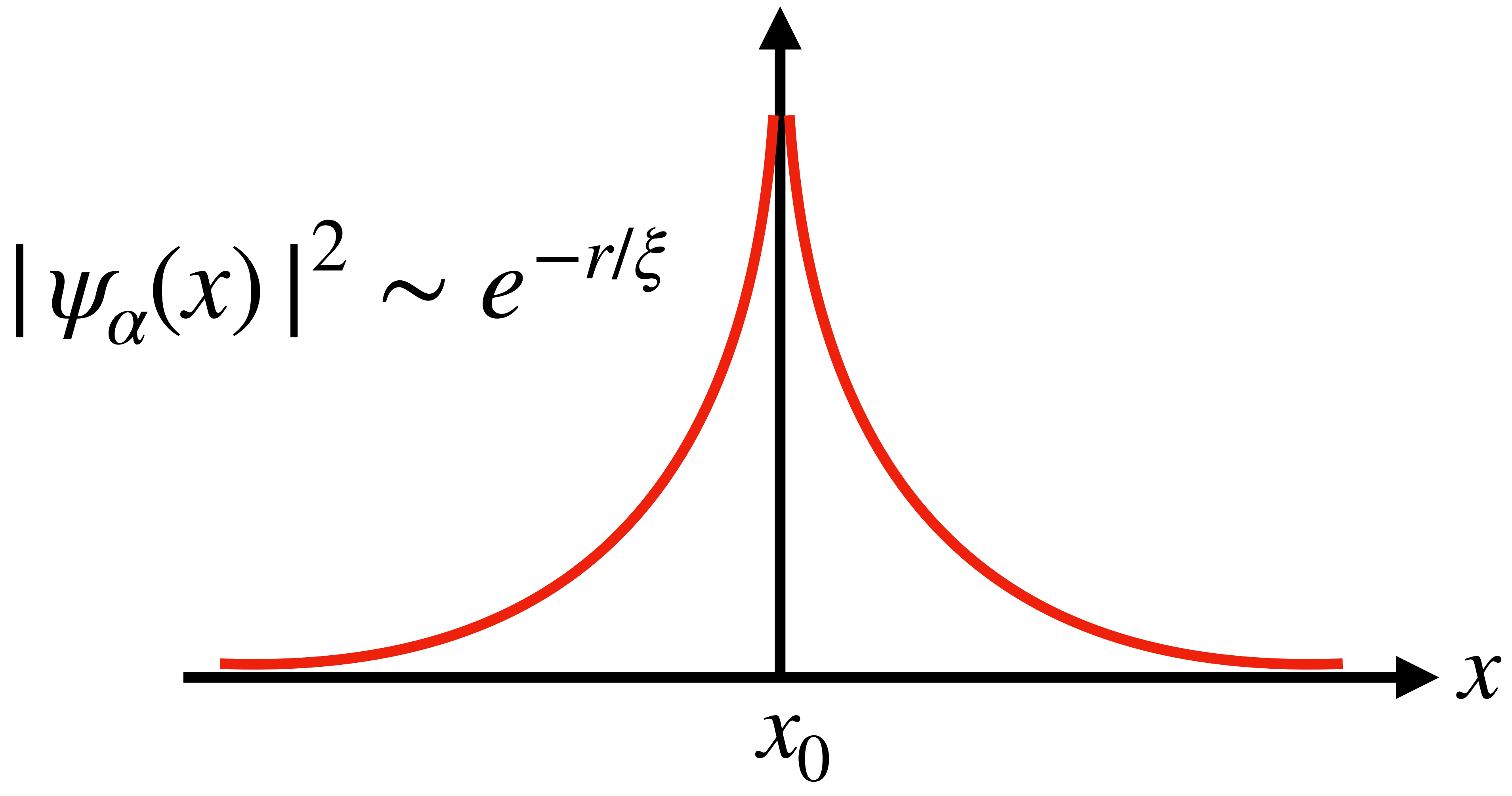


$$\hat{H}_0 = \sum_i \epsilon_i \hat{n}_i + J \sum_{\langle i,j \rangle_{n.n.}} \left(\hat{c}_i^\dagger \hat{c}_j + \hat{c}_j^\dagger \hat{c}_i \right)$$

"Strong" localization limit; no degeneracy

$$J \ll |\epsilon_i - \epsilon_{i+1}|$$

All single-particle eigenstates are localized.



AL as an MBL

- All single-particle eigenstate are localized.

➡ Complete set of localized conserved operators.

$$\hat{H}_0 = \sum_i \epsilon_i \hat{n}_i + J \sum_{\langle i,j \rangle_{n.n.}} \left(\hat{c}_i^\dagger \hat{c}_j + \hat{c}_j^\dagger \hat{c}_i \right) = \sum_\alpha E_\alpha \hat{c}_\alpha^\dagger \hat{c}_\alpha$$

$$[\hat{H}_0, \hat{n}_\alpha] = 0 = [\hat{n}_\alpha, \hat{n}_\beta]$$

"A basic structure of MBL"

"Localization in Fock space"

MBL with an interaction

- Consider a spin-1/2 chain.

"complete set of conserved operators"

$$\hat{H}_0 = \sum_i h_i \hat{Z}_i \qquad [\hat{H}_0, \hat{Z}_i] = 0 = [\hat{Z}_i, \hat{Z}_j]$$

- Add a nearest-neighbor interaction.

$$\hat{H} = \sum_i h_i \hat{Z}_i + g \sum_{i,j} \hat{J}_i(\hat{\boldsymbol{\sigma}}_i, \hat{\boldsymbol{\sigma}}_{i+1})$$

- MBL is stable at small g!

Basko, Aleiner, Altshuler 2006

(perturbatively; any dimensions)

Imbrie 2016

(non-perturbatively; MBL unstable at D>1)

Phenomenology of MBL

- Local observable: relaxation, but no thermalization
- Single-particle transport: completely frozen like AL
- "Area-law" of eigenstate entanglement. (cf. volume-law in ETH)

$$S_A \sim \mathrm{vol}(\partial A) \qquad \textbf{(cf. ETH: } S_A \sim V_A \textbf{)}$$

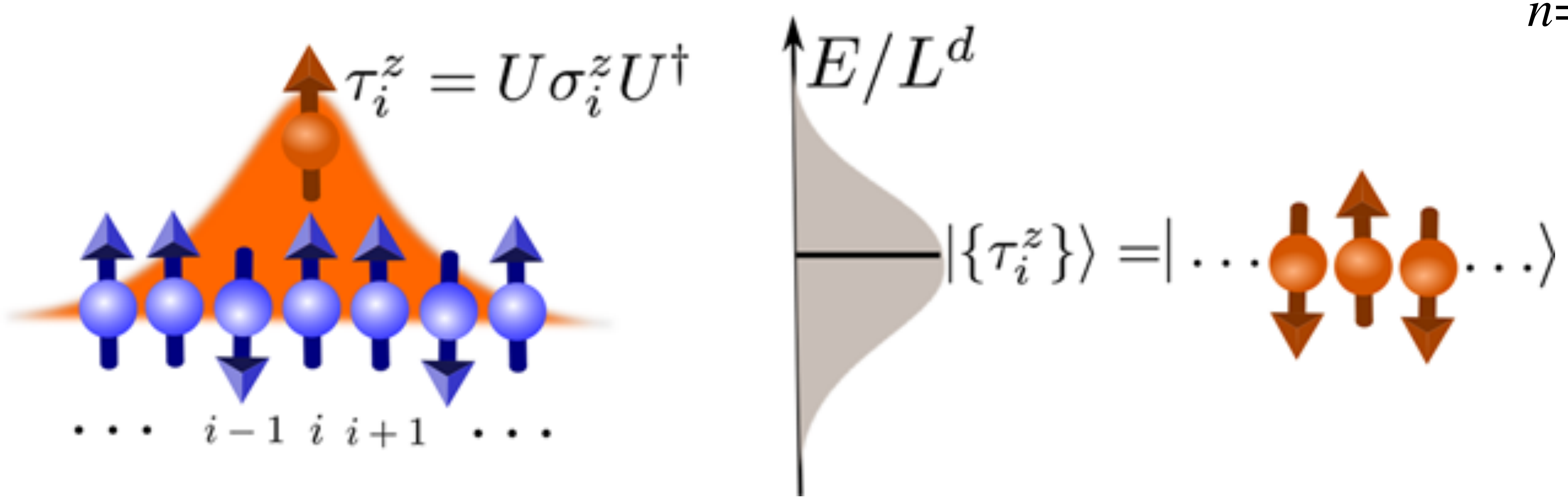
- "Logarithmic" spreading of quantum information

$$S_A \propto \ln t \qquad \mathbb{E}_\mu \| [O_A(t), O_B] \| \leq ct\, |\partial A|\, e^{-\frac{x}{2\xi}}$$

Unified picture of MBL

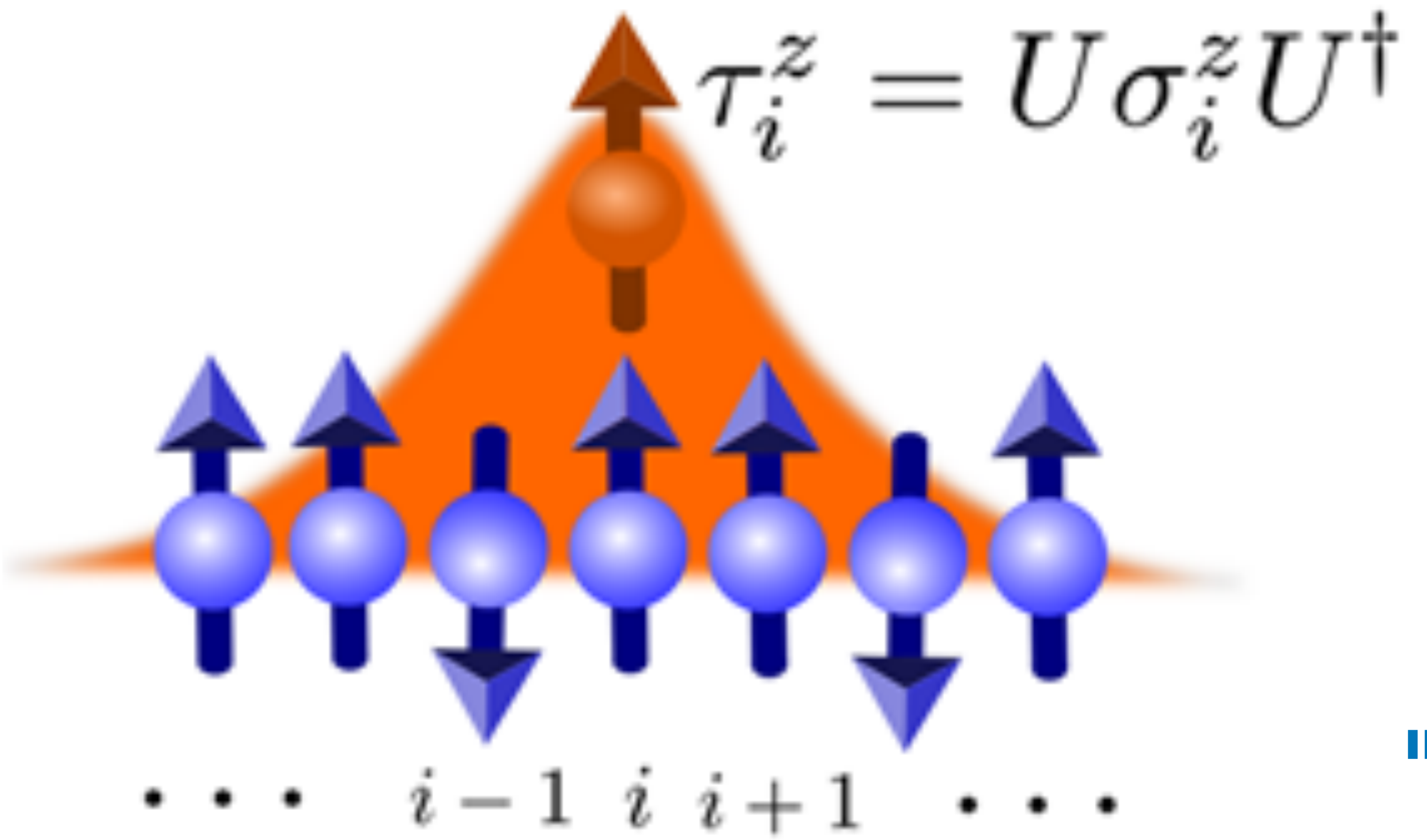
- Serbyn, Papic, Abanin 2013; Huse, Nandkishore, Oganesyan 2014
- (quasi-) Local Integral of Motion (LIOM) or "l-bit"
  : <u>complete set of conserved operators</u>

$$\hat{\tau}_i^z = Z\hat{\sigma}_i^z + \sum_{n=1}^{\infty} V_i^{(n)} \hat{O}_i^{(0)}$$



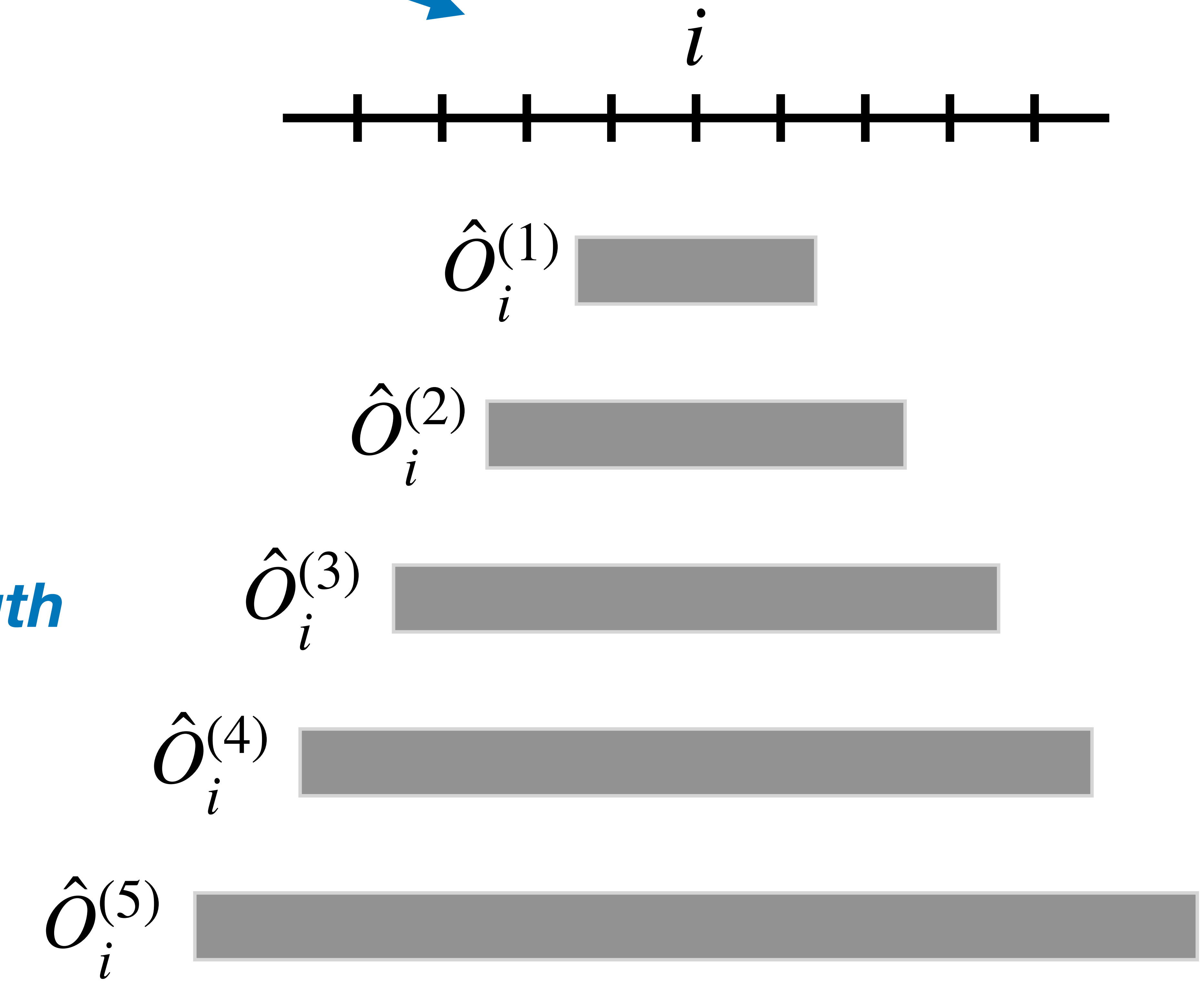
Construction of LIOM

$$\hat{\tau}_i^z = Z\hat{\sigma}_i^z + \sum_{n=1}^{\infty} V_i^{(n)} \hat{O}_i^{(n)}$$



$$V_i^{(n)} \sim \exp(-n/\xi_{\mathrm{op}})$$

"operator" localization length



Effective "l-bit" Hamiltonian

- "fully" many-body localization

$J_{ij} \sim J_0 e^{-|i-j|/\kappa}$

$J_{ijk} \sim J_0 e^{-|i-k|/\kappa}$

$$\hat{H} = \sum_i h_i \hat{\tau}_i^z + \sum_{i>j} J_{ij} \hat{\tau}_i^z \hat{\tau}_j^z + \sum_{i>j>k} J_{ijk} \hat{\tau}_i^z \hat{\tau}_j^z \hat{\tau}_k^z + \cdots$$

J^{eff} ***decay length***

- Important: effective interaction between two remote l-bits

$$\hat{J}_{ab}^{\mathrm{eff}} = J_{ab} + \sum_k J_{akb} \hat{\tau}_k^z + \sum_{k<l} J_{aklb} \hat{\tau}_k^z \hat{\tau}_l^z + \cdots \sim J_0 \exp(-x/\xi_{\mathrm{eff}})$$

Note: length scales in MBL

- LIOM localization length ξ_{op} in $V_i^{(n)} \sim \exp(-n/\xi_{\mathrm{op}})$
- Interaction decay length κ in $J_{ij} \sim J_0 e^{-|i-j|/\kappa}$
- Effective interaction decay length ξ_{eff} in $J^{\mathrm{eff}} \sim J_0 \exp(-x/\xi_{\mathrm{eff}})$
- They are all different, and their relationship is nontrivial.
- Abanin, Altman, Bloch, Serbyn, RMP 2019

$$\kappa^{-1} \geq (\xi_{\mathrm{op}}^{-1} + \ln 2)/2 \qquad \xi_{\mathrm{eff}}^{-1} \leq \kappa^{-1} - (\ln 2)/2 \qquad \xi_{\mathrm{eff}} \leq 2\xi_{\mathrm{op}}$$

"dephasing" dynamics in MBL

- Key feature: exponential decay of $J^{\mathrm{eff}}(x) \sim J_0 \exp(-x/\xi_{\mathrm{eff}})$

- Logarithmic growth of entanglement entropy, when quenched from nonent.

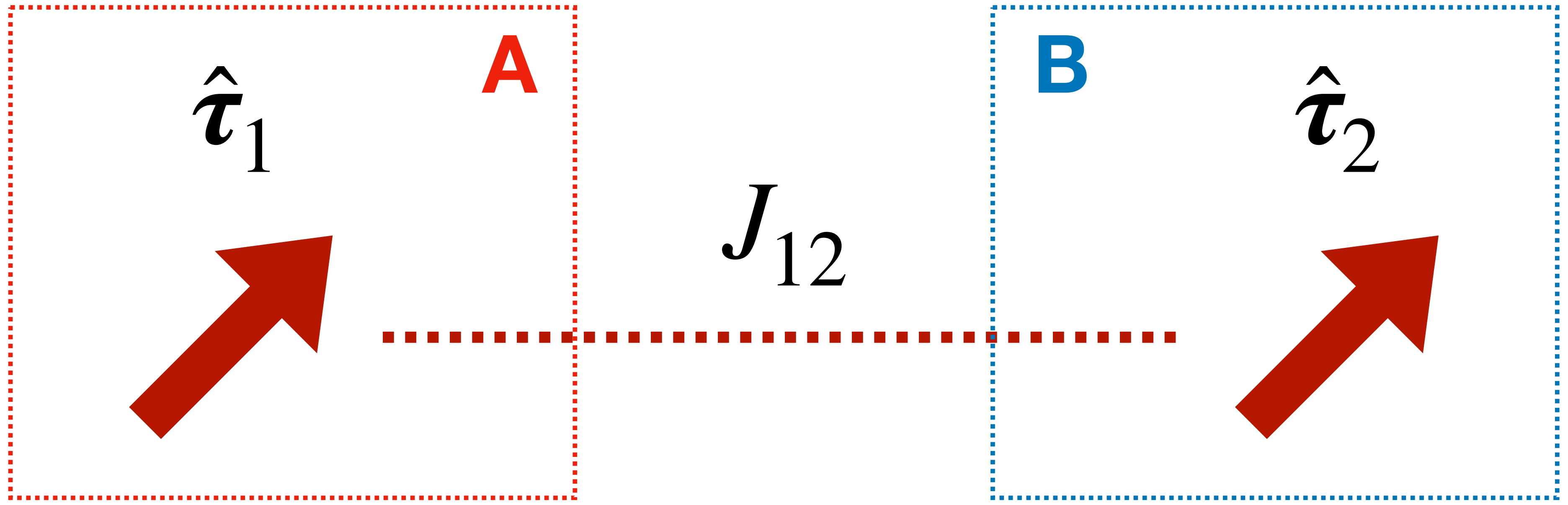
$$x_{\mathrm{ent}}(t) \sim \xi_{\mathrm{eff}} \ln(J_0 t) \qquad\qquad S(t) \sim \xi_{\mathrm{eff}} \ln(J_0 t)$$

- Equilibration of an local observable

$$|\langle \hat{\tau}_i^x \rangle| \sim \frac{1}{(J_0 t)^a}$$

- Logarithmic light cone measured by OTOC

Entanglement spreading



Time-evolution subject to

$$\hat{H} = J_{12}\hat{\tau}_1^z\hat{\tau}_2^z$$

$$\frac{1}{\sqrt{2}}\left(|\Uparrow_1\rangle + |\Downarrow_1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|\Uparrow_2\rangle + |\Downarrow_2\rangle\right) = |\Psi(t=0)\rangle$$

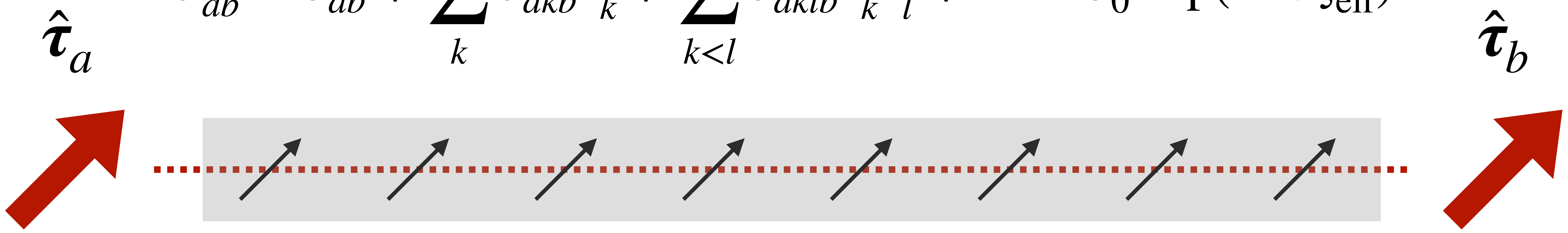
Non-entangled initial state

$$\longrightarrow \quad |\Psi(t)\rangle = \frac{1}{2}e^{-iJ_{12}t}\left(|\Uparrow\Uparrow\rangle + |\Downarrow\Downarrow\rangle\right) + \frac{1}{2}e^{iJ_{12}t}\left(|\Uparrow\Downarrow\rangle + |\Downarrow\Uparrow\rangle\right)$$

$$\hat{\rho}_A = \mathrm{tr}_B\,\hat{\rho} = \frac{1}{2}\begin{pmatrix} 1 & \cos(2J_{12}t) \\ \cos(2J_{12}t) & 1 \end{pmatrix} \longrightarrow \boxed{S_A = \ln 2 \quad \text{when} \quad J_{12}t = \pi/4}$$

Entanglement spreading

$$\hat{J}^{\mathrm{eff}}_{ab} = J_{ab} + \sum_k J_{akb}\hat{\tau}^z_k + \sum_{k<l} J_{aklb}\hat{\tau}^z_k\hat{\tau}^z_l + \cdots \sim J_0 \exp(-x/\xi_{\mathrm{eff}})$$



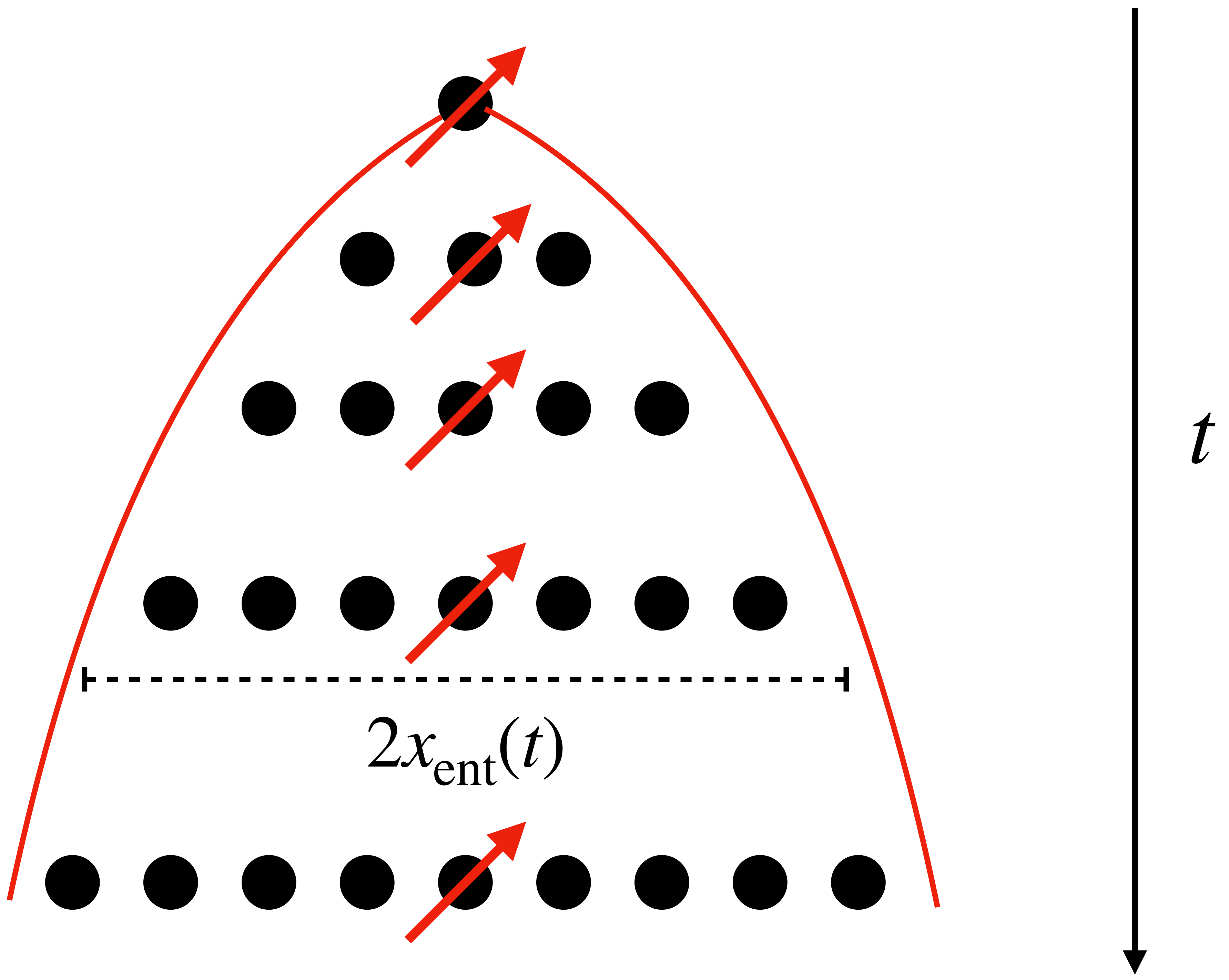
The effective interaction includes all the intervening spins in its structure.
It is still the same two-spin problem even if they are separated over the distance.

These two spins at both ends get entangled when $J_{\mathrm{eff}}\, t \sim 1$.

During the time period t, entanglement spreads over the distance $x_{\mathrm{ent}}(t) \sim \xi_{\mathrm{eff}} \ln(J_0 t)$.

Entanglement spreading

$x_{\mathrm{ent}}(t) \sim \xi_{\mathrm{eff}} \ln(J_0 t)$



Logarithmic light cone

entanglement entropy

Entangled region of length $x_{\mathrm{ent}}(t)$
+ non-entangled area outside

$$\boxed{S(t) \propto x_{\mathrm{ent}} = \xi_{\mathrm{eff}} \ln(J_0 t)}$$

Logarithmic growth of EE

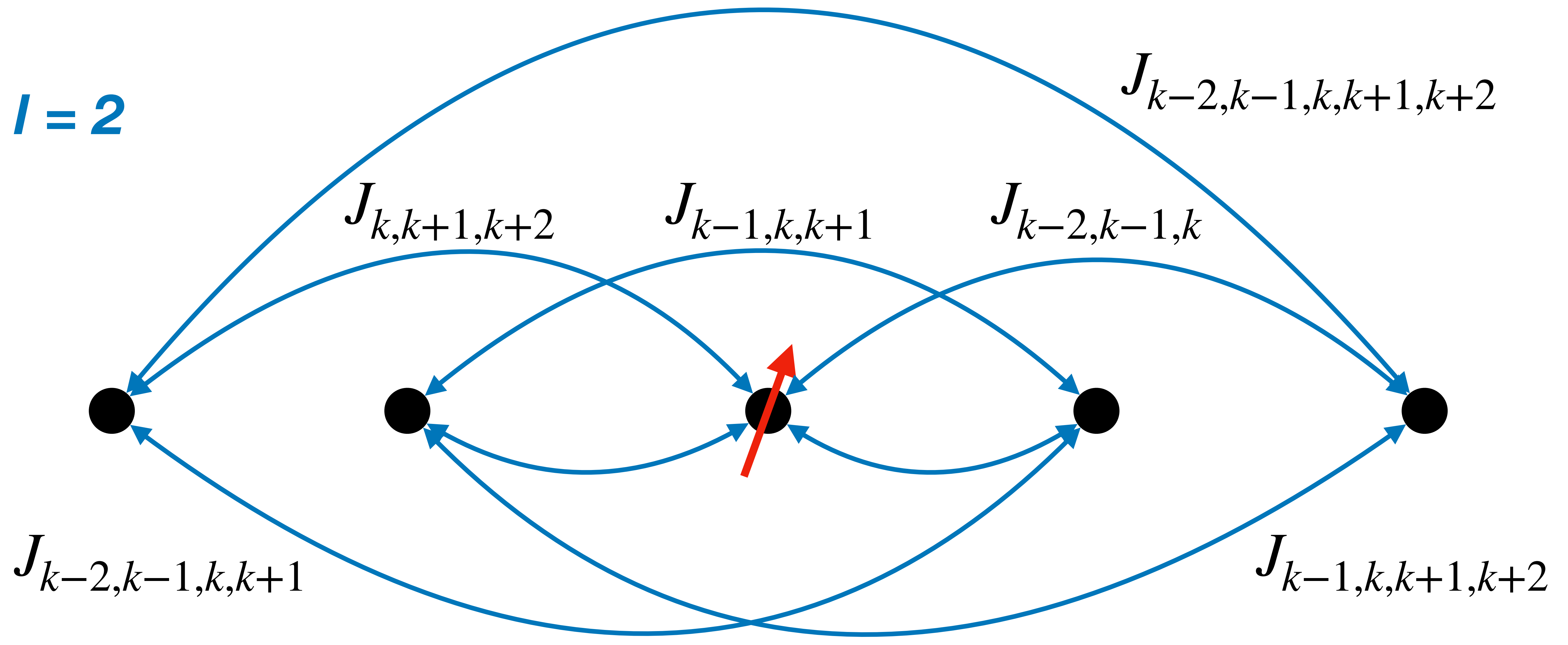
Equilibration of a local spin

- The effective interaction is a "dephasing" interaction.

- Effective "magnetic field operator" acting on an l-bit at site *k*

$$\hat{h}_k^{(\mathrm{eff})}(l) \equiv \hat{h}_k(\tau^z_{k-l}, \tau^z_{k-l+1}, \ldots, \tau^z_{k-1}, \tau^z_{k+1}, \ldots, \tau^z_{k+l-1}\tau^z_{k+l})$$

(example) l = 2



Equilibration of a local spin

Initial state (non-entangled)

$$|\Psi(t=0)\rangle = \otimes_i \frac{1}{\sqrt{2}} \left(|\Uparrow_i\rangle + |\Downarrow_i\rangle \right)$$

Reduced density matrix of l-bit at k (off-diagonal component)

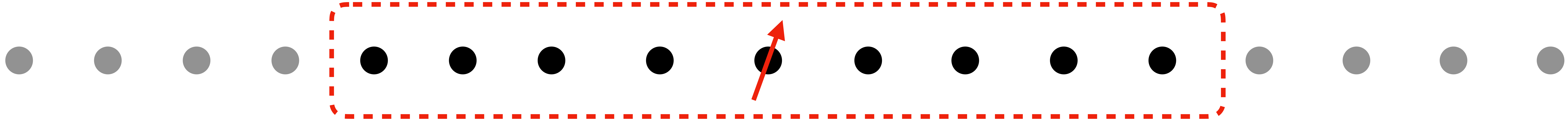
$$\rho_{\Uparrow\Downarrow} \sim \frac{1}{\sqrt{2^{2l}}}$$

$$\rho_{\Uparrow\Downarrow}(t) = \frac{1}{2^{2l}} \sum_{\{\tau'\}} \exp\left[2ih_k^{\mathrm{eff}}(\{\tau'\})t\right]$$

Q. What is the relevant value of l ?

$$\{\tau'\} \equiv \{(\tau^z_{k-l}, \tau^z_{k-l+1}, \ldots, \tau^z_{k-1}, \tau^z_{k+1}, \ldots, \tau^z_{k+l-1}\tau^z_{k+l})\}$$

Equilibration of a local spin



Entangled area = $2x_{\mathrm{ent}}(t) \sim 2\xi_{\mathrm{eff}} \ln(J_0 t)$

$\longrightarrow$ $l \approx x_{\mathrm{ent}}(t)$

Reduced density matrix

$$\rho_{\Uparrow\Downarrow}(t) = \frac{1}{2^{2l}} \sum_{\{\tau'\}} \exp\left[2ih_k^{\mathrm{eff}}(\{\tau'\})t\right] \xrightarrow{\text{CLT}} |\rho_{\Uparrow\Downarrow}| \sim \frac{1}{\sqrt{2^{2l}}} = \frac{1}{(J_0 t)^a} \quad (a = \xi_{\mathrm{eff}} \ln 2)$$

$$\{\tau'\} \equiv \{(\tau^z_{k-l}, \tau^z_{k-l+1}, \ldots, \tau^z_{k-1}, \tau^z_{k+1}, \ldots, \tau^z_{k+l-1}\tau^z_{k+l})\}$$

$\longrightarrow$ $\boxed{|\langle \hat{\tau}^x_k \rangle| \sim \frac{1}{t^a}}$

Out-of-time-ordered correlator

OTOC (effective l-bit model; $\beta = 0$)

$$F(t) = \mathrm{tr}[\hat{\tau}_i^x(t)\hat{\tau}_j^x\hat{\tau}_i^x(t)\hat{\tau}_j^x]$$

$$F(t) = \cos(4tJ_{ij}^{\mathrm{eff}})$$

disorder average

$J_0 \in [-J, J]$

$$\overline{F}(t) = \frac{\sin[4tJ\exp(-|i-j|\xi)]}{4tJ\exp(-|i-j|\xi)}$$

exponentially decaying effective interaction

$$\boxed{J_{ij}^{\mathrm{eff}} \sim J_0 \exp(-|i-j|/\xi)}$$

Logarithmic light cone

$$\boxed{t_0 = \frac{\pi}{4J} e^{-|i-j|/\xi}}$$

R. Fan et al., Sci. Bull. 62, 707 (2017)

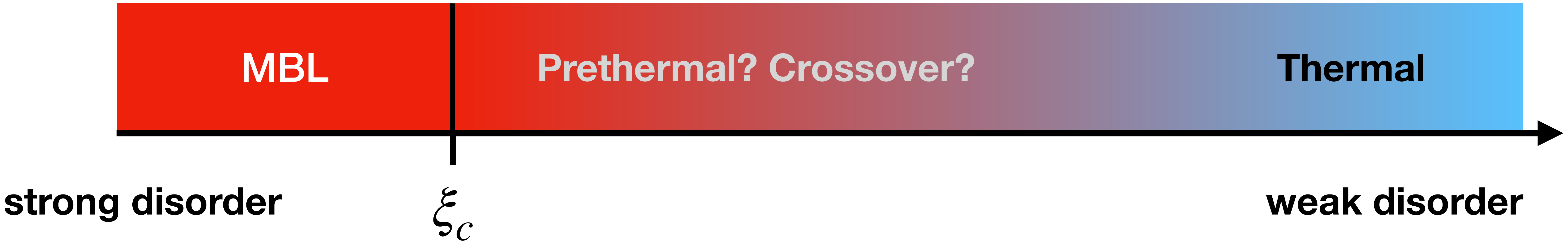
X. Chen et al., Ann. Phys. 1600332 (2016)

B. Swingle and D. Chowdhury, PRB 95, 060201(R) (2017)

Phase diagram

- One dimension

Stability of MBL is well accepted, but true transition point is hard to see.



- Higher dimensions

No MBL transition in d > 1, but there are MBL signatures at strong disorders.

Avalanche scenario

- De Roeck and Huveneers, PRB 95, 155129 (2017)
- Start with a thermal block in a chain, trying to expand by absorbing l-bits.

Condition to stop avalanche:

level spacing >> off-diagonal matrix element

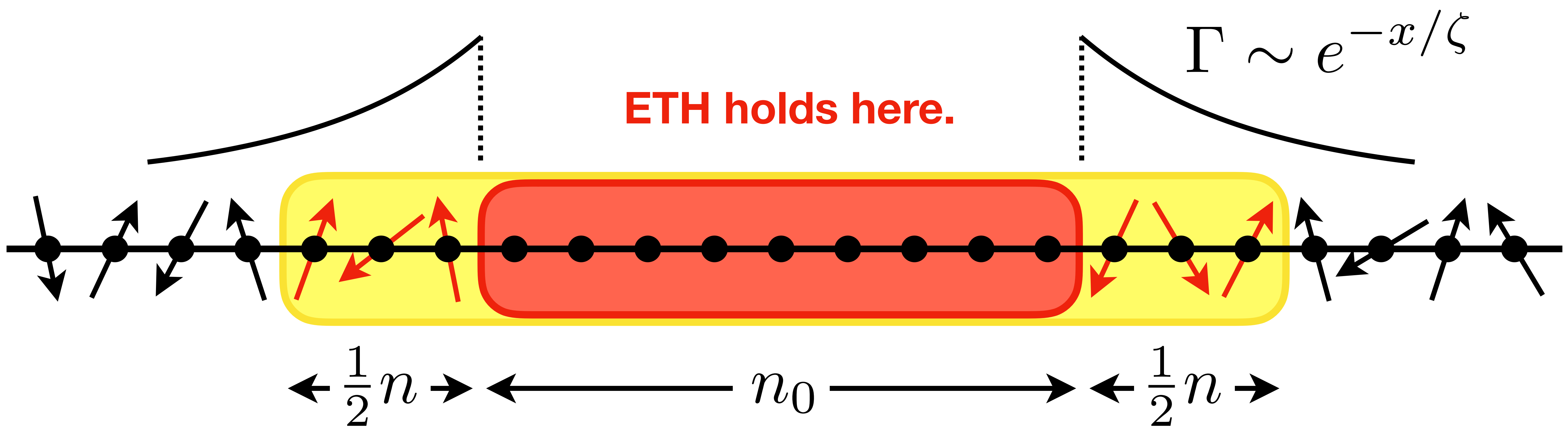
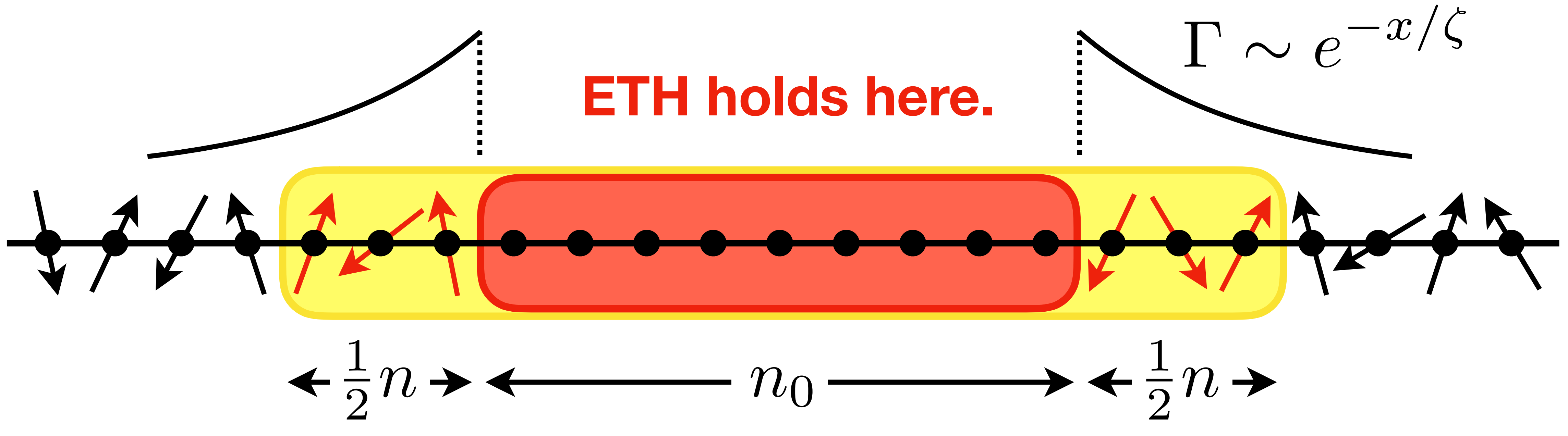


Figure from Dumitrescu et al., PRB 99, 094205 (2019)

Avalanche scenario (1D)

Off-diagonal matrix element: $\Gamma = \dfrac{e^{-n/2\zeta}}{\sqrt{2^{n+n_0}}}$

(ETH: $A_{\alpha\beta} = e^{-S(\bar{E})/2} f(\bar{E},\omega)\, R_{\alpha,\beta}$**)**



Level spacing: $\delta = \dfrac{W}{2^{n+n_0}}$

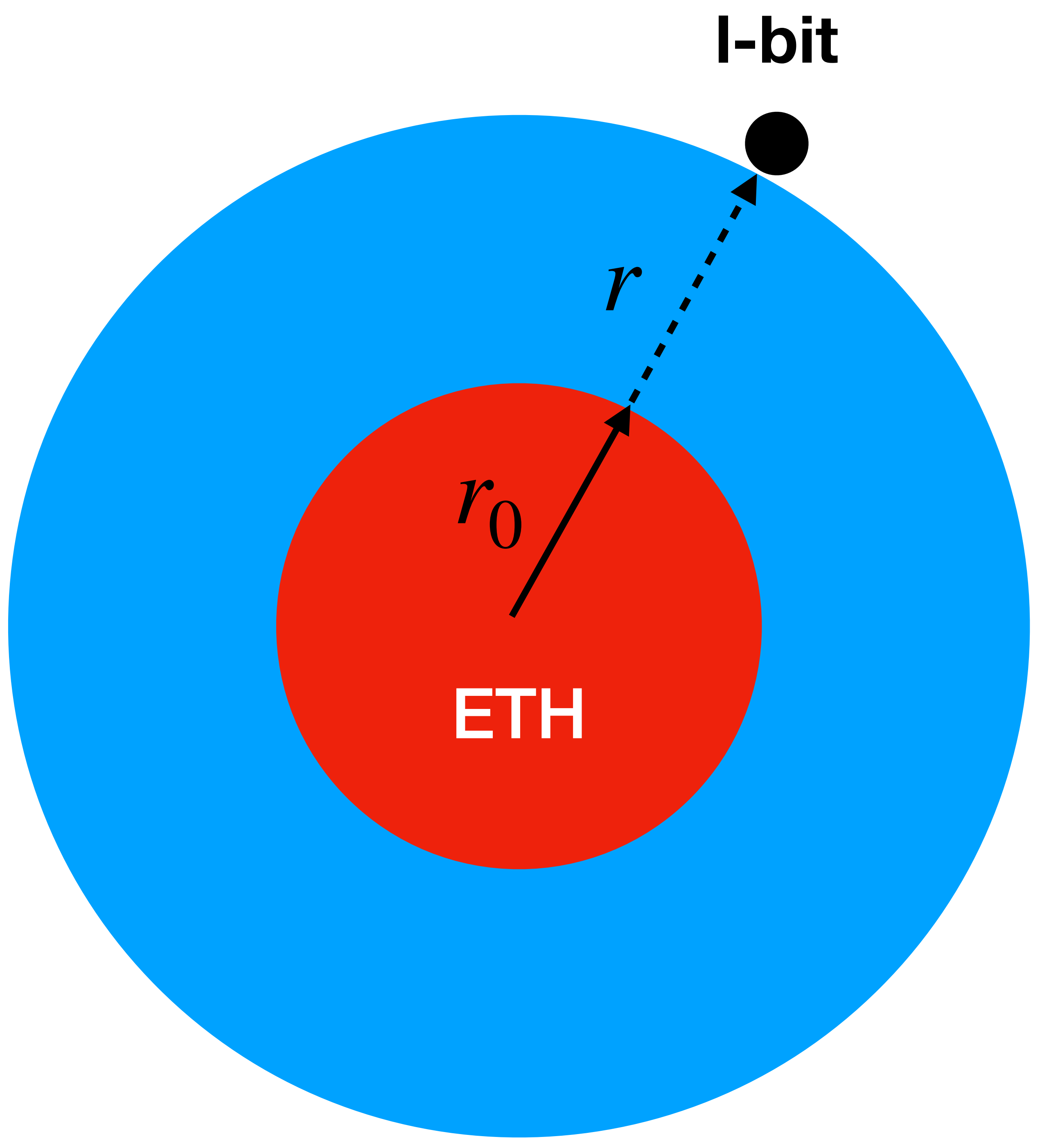
Avalanche stopping criterion

$$\boxed{g \equiv \frac{\Gamma}{\delta} = \exp\left(-\frac{n}{2\zeta} + \frac{\ln 2}{2}(n+n_0)\right) \ll 1}$$

There is discontinuity in ζ !

MBL is stable when $\dfrac{1}{\zeta} > \ln 2.$

Avalanche scenario in d>1



Off-diagonal matrix element: $\Gamma = \dfrac{e^{-r/\zeta}}{\sqrt{2^{N_{r+r_0}}}}$

Level spacing: $\delta = \dfrac{W}{2^{N_{r+r_0}}}$

Number of spins : $N_{r+r_0} \sim (r + r_0)^d$

Avalanche never stops for sufficiently large r_0!

$$\boxed{g \equiv \frac{\Gamma}{\delta} = \exp\left(-\frac{r}{\zeta} + a(r + r_0)^d\right) \gg 1}$$

RG: KT-like transition

Dumitrescu et al., PRB 99, 094205 (2019)
Goremykina et al., PRL 122, 040601 (2019)

$\rho(l)$: density of thermal block with size > l

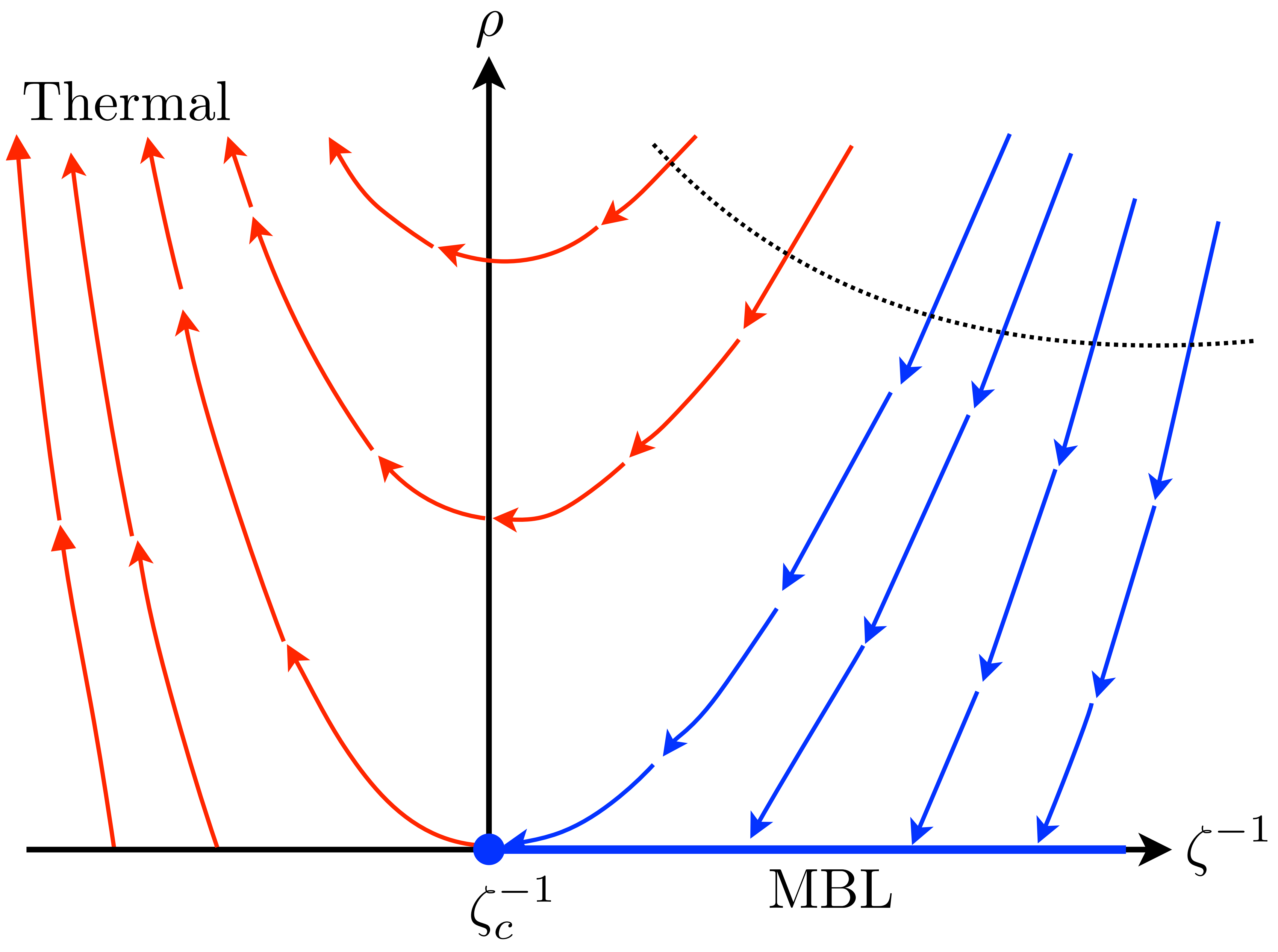
Fixed points

$$\rho_* = 0,\ \zeta_*^{-1} > \zeta_c^{-1} \qquad \textbf{(MBL)}$$

$$\rho_* = 1,\ \zeta_*^{-1} = 0 \qquad \textbf{(Thermal)}$$

Flow equation (phenomenological)

$$\frac{d\rho}{dl} \approx b\rho(\zeta - \zeta_c) \qquad \frac{d\zeta^{-1}}{dl} \approx -c\rho\zeta^{-1}$$



KT-like transition

KT	MBL
low-T critical phase	MBL
vortex fugacity	density of thermal blocks
stiffness	$1/\zeta$

BUT, dynamical scaling exponent

$$z \to \infty$$

Vosk, Huse, Altman, PRX 2015

$$\tau_{\mathrm{th}} \sim L^z$$

(thermalization time)

References

- Review papers
  - Abanin, Altman, Bloch & Serbyn, Reviews of Modern Physics 91, 021001 (2019)
  - Alet & Laflorencie, Comptes Rendus Physique 19, 498 (2018)
  - Nandkishore & Huse, Annual Review of Condensed Matter Physics 6, 16 (2015)
- Lectures by D. Huse, E. Altman, M. Serbyn (available on Youtube)