Cosmology with Gravitational Waves

Sunghoon Jung

Note to students: Problems are aimed at helping you play with 'chirping' and 'primordial' GWs by yourself and taste cosmology with them: dark energy, dark matter, and inflationary Hawking radiation. Each problem can be solved independently, so you can select any subset of them. But since all are under active research, motivated students are recommended to challenge themselves as much as possible.

1 Chirping GWs from Black-Hole Mergers

Consider the 'chirping' GW, produced by a binary black-hole merger.

(1) Chirping waveforms. Derive the chirping waveform $h(t)$ as a function of time, and the chirping evolution of the frequency f, by completing the steps outlined in the lecture

$$h(t) \sim \frac{1}{d_L}(G\mathcal{M})^{5/3}(\pi f)^{2/3}, \qquad \frac{df}{dt} = \frac{96}{8}\pi^{8/3}(G\mathcal{M})^{5/3}f^{11/3}. \tag{1}$$

Solve the latter for $t(f)$. What does this time mean, in regard of f, with respect to what?

Finally our main goal. Plot the waveform $h(t)$ of the first-ever LIGO observation – GW150914. Draw for $f = 10 - 1000$ Hz, using the following parameters for the merger: $M_1 = M_2 = 30\,\mathrm{M}_\odot$ and $d_L = 400$ Mpc. [Refer to textbooks, e.g., [1, 2].]

(Aside: Now, you have reproduced the Nobel Prize-winning observation! Further, why is chirping so useful for detection; why does $h(t)$ grow with t? Derive $\widetilde{h}(f) \sim \frac{(G\mathcal{M})^{5/6}f^{-7/6}}{d_L}$.)

(2) Dark energy mock-data challenge. I have generated the following mock data, assuming some values of $\Omega_{m,\Lambda}, H_0$ with $\Omega_m + \Omega_\Lambda = 1$. Use standard siren physics given in the lecture to find out my dark energy density, Hubble constant, and their uncertainties.

$$\{d_L(\mathrm{Mpc}), z\} = \{450, 0.1\}, \{4200, 0.7\}, \{8000, 1.2\}. \tag{2}$$

Assume 2% errors on d_L measurements. Check your answers on 12/29!

(Aside: Note that this standard siren physics requires the measurement of z, which is impossible with just chirping GWs. How is z measured separately in Ref. [3]?)

(3) Seeing dark matter. Suppose the chirping GW passes by a compact dark matter, or simply another black hole. The path of the GW will be deflected by the gravitational potential of the black hole. First, review the usual deflection angle

$$\alpha = \frac{4GM}{b}, \tag{3}$$

where M is the mass of the lens black hole and b is the impact parameter. Note that this does not depend on the GW frequency f.

But α must vanish in the limit $f \to 0$. Why is that? At what frequencies does α transition from Eq. (3) to 0? In Ref. [4], how is such frequency-dependence of lensing utilized to "see" compact dark matter in the universe? Why is chirping so powerful for this mission? [Hint: What are the available length scales in this simple system of a black hole with M and a passing GW with f?]

2 Primordial GWs from Inflation

Consider 'primordial' GWs, produced by quantum fluctuations during inflaton – inflationary Hawking radiation.

(4) Quantum nature. In the lecture, we have derived the statistical variance of the primordial GW amplitude (per log interval of comoving frequency k)

$$\frac{d\langle h^2(x)\rangle}{d\log k} \simeq \left(\frac{H_{inf}}{2\pi}\right)^2 \frac{1}{M_{\rm Pl}^2}. \tag{4}$$

The variance is inevitable by its quantum origin. What does this variance mean if we measure primordial GWs coming from various directions?

To answer this, consider an ensemble of s. The value of s follows a Gaussian probability distribution

$$P(s) \propto e^{-s^2/2\sigma^2}. \tag{5}$$

Plot $P(s)$, calculate $\langle s^2\rangle$, and identify what this corresponds to in the plot.

(5) Inflationary origin. The constancy of the right-hand side of Eq. (4) dictates the inflationary origin of primordial GWs. Let's study how to measure/confirm this. (Aside: think about why so.)

Calculate the energy density of the primordial GW observed today. Start with

$$\frac{d\rho_{\rm GW}}{d\log k} \equiv \frac{1}{32\pi G}\frac{d\langle \dot{h}(x)^2\rangle}{d\log k} \simeq \frac{1}{32\pi G}k^2\frac{H_{inf}^2}{4\pi^2 M_{\rm Pl}^2}. \tag{6}$$

This is not final yet because radiation energy must be redshifted by universe expansion since the k mode has re-entered the horizon (at t_k where $k = a(t_k)H(t_k)$ is satisfied) until today. Using $a(t) \propto t^{1/2}$, find the energy redshift $a(t_k)/a(t_0) = a(t_k)$ for given k. Show that the correct energy density Eq. (6) is k independent.

So, what would be the observational evidence of inflation? What others about cosmology and particle physics can we learn, e.g., from more complete result in Ref. [5]?

References

[1] S. Carroll, "An Introduction to General Relativity", Ch.7

[2] M. Maggiore, "Gravitational Waves", vol.1 Ch.4.1

[3] S. Nissanke, D. E. Holz, S. A. Hughes, N. Dalal and J. L. Sievers, "Exploring short gamma-ray bursts as gravitational-wave standard sirens," Astrophys. J. **725**, 496-514 (2010) doi:10.1088/0004-637X/725/1/496 [arXiv:0904.1017 [astro-ph.CO]].

[4] S. Jung and C. S. Shin, "Gravitational-Wave Fringes at LIGO: Detecting Compact Dark Matter by Gravitational Lensing," Phys. Rev. Lett. **122**, no.4, 041103 (2019) doi:10.1103/PhysRevLett.122.041103 [arXiv:1712.01396 [astro-ph.CO]].

[5] L. A. Boyle and P. J. Steinhardt, "Probing the early universe with inflationary gravitational waves," Phys. Rev. D **77**, 063504 (2008) doi:10.1103/PhysRevD.77.063504 [arXiv:astro-ph/0512014 [astro-ph]].