Project 1: Hubbard-Stratonovich transformation

In this project we consider the Hamiltonian for M Ising spins

$$\mathcal{H} = -\frac{1}{2}\sum_{i\neq j} J_{ij} S_i S_j - \sum_i H_i S_i.$$

1. Prove the identity

$$\int_{-\infty}^{\infty} \prod_i \left(\frac{dx_i}{\sqrt{2\pi}}\right) \exp\left[-\frac{1}{2}\sum_{i,j} x_i A_{ij} x_j + \sum_i x_i B_i\right] = \frac{1}{\sqrt{\det \boldsymbol{A}}} \exp\left[\frac{1}{2}\sum_{i,j} B_i (\boldsymbol{A}^{-1})_{ij} B_j\right]$$

where $\boldsymbol{A}$ is a real symmetric positive matrix and $\boldsymbol{B}$ is an arbitrary vector.

2. Apply the above identity, making the identification $(\boldsymbol{A}^{-1})_{ij} = J_{ij}$ and $B_i = S_i$ to write the partition function Z in the form

$$Z = \int_{-\infty}^{\infty} \prod_{i=1}^{M} d\psi_i e^{-\beta S(\{\psi_i\},\{H_i\},\{J_{ij}\})}.$$

Find the function $S(\{\psi_i\},\{H_i\},\{J_{ij}\})$.

3. Assume that we can approximate Z by the maximum term in the functional integral: $Z \approx \exp[-\beta S(\bar{\psi}_i)]$ where $\bar{\psi}_i$ is the value of ψ_i which minimizes S. Find the equation satisfied by $\bar{\psi}_i$, and show that the magnetization at site i

$$m_i \equiv \langle S_i \rangle = -\frac{\partial F}{\partial H_i} \approx -\frac{\partial S}{\partial H_i}.$$

Combine the equations to find the self-consistent equations for $\{m_i\}$.

4. Use the self-consistent equation to find critical properties for two-dimensional nearest-neighbor Ising model.