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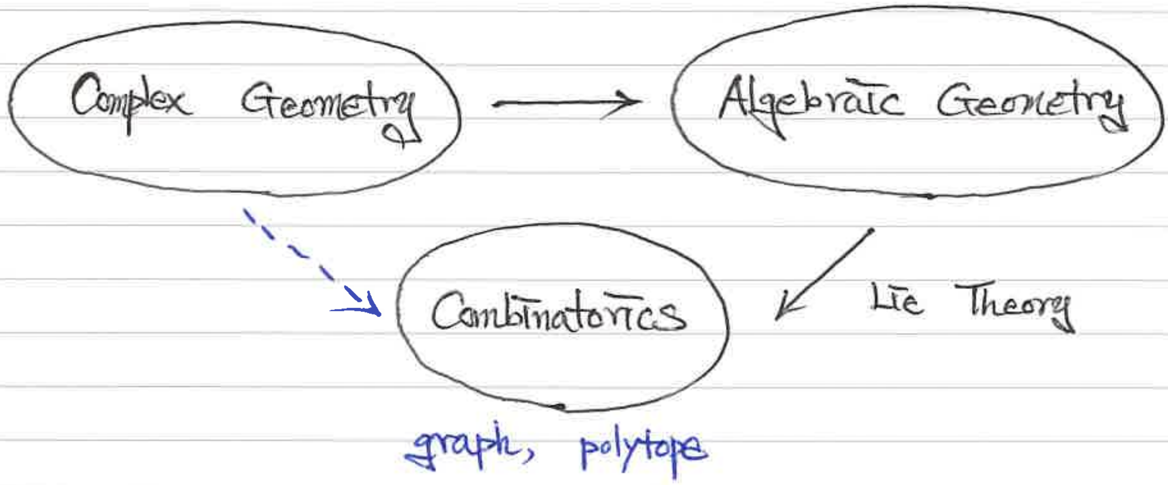
Greatest Ricci lower bounds of projective horospherical manifolds



§1. Existence of Kähler-Einstein metrics and greatest Ricci lower bounds

(X, g) : compact Kähler manifold

Kähler metric $\quad g = \frac{1}{2\pi} \sum_{i,j=1}^{n} g_{i\bar{j}}\, dz_i \otimes d\bar{z}_j$

Kähler form $\quad \omega = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^{n} g_{i\bar{j}}\, dz_i \wedge d\bar{z}_j \quad \left[\begin{array}{c} d\omega = 0 \\ \text{closed } (1,1)\text{-form} \end{array} \right]$

Ricci form $\quad \mathrm{Ric}(\omega) = -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log\left(\det\left(g_{i\bar{j}}\right)\right)$

$$\Longrightarrow \quad [\mathrm{Ric}(\omega)] \in H^{1,1}(X) \cap H^2(X, \mathbb{Z})$$

Existence of a Kähler-Einstein metric

$$\mathrm{Ric}(\omega) = \lambda \cdot \omega \quad (\lambda \in \mathbb{R})$$

$\Updownarrow$

Finding a solution ϕ of complex Monge-Ampère equation

$$(\omega_{ref} + i\,\partial\bar{\partial}\phi)^n = e^{f_{ref} + \phi}\, \omega_{ref}^n$$

: nonlinear elliptic PDE

the first Chern class

① $\lambda < 0$; $c_1(X) < 0$ general type

$\lambda = 0$; $c_1(X) = 0$ Ricci-flat

(Calabi-Yau manifolds)

Aubin and Yau's continuity method (1978) :

For a Kähler form $\omega \in c_1(X) < 0$,

define $I := \left\{ t \in [0,1] \;\middle|\; \begin{array}{l} \exists \text{ a Kähler form } \omega_t \in c_1(X) \\ \text{satisfying } -\mathrm{Ric}(\omega_t) = t\omega_t + (1-t)\,\omega \end{array} \right\}$.

(i) $0 \in I$

(ii) $I \subset [0,1]$ is open. $C^{2,\alpha}$-estimate

(iii) $I \subset [0,1]$ is closed. Hard part (C^0-estimate)

$\Longrightarrow$ $1 \in I$ which gives the KE equation.

② $\lambda > 0$; $c_1(X) > 0$ Fano manifold

(positive Ricci curvature)

There are obstructions to the existence of KE metrics.

(trivial unipotent radical)

(i) [Matsushima, 1957] the reductivity of the automorphism group $\mathrm{Aut}(X) = \{ \varphi : X \longrightarrow X \}$

biholomorphism

(ii) vanishing of the $\underbrace{\text{Futaki invariant}}_{}$ (1983)

a functional on the Lie algebra of $\mathrm{Aut}(X)$

Example the automorphism group of the complex projective plane.

$\mathrm{Aut}(\mathbb{P}^2) = PGL_3(\mathbb{C})$: reductive Lie group

$$\mathrm{Aut}\left(Bl_p(\mathbb{P}^2)\right) = \left(\mathbb{C}^* \times PGL_2(\mathbb{C})\right) \ltimes \mathbb{C}^2$$

$$= \left\{ \left(\begin{array}{c|cc} * & * & * \\ \hline 0 & * & * \\ 0 & * & * \end{array} \right) \right\}$$

: non-reductive

unipotent radical

Rad_u

By Matsushima's theorem,

the blowup $Bl_p(\mathbb{P}^2)$ of $\mathbb{P}^2$ at a point $p \in \mathbb{P}^2$ cannot admit a Kähler-Einstein metric.

Moment polytopes of toric varieties :

$\mathbb{P}^2$	$Bl_p(\mathbb{P}^2)$	$Bl_{p,q}(\mathbb{P}^2)$	$Bl_{p,q,r}(\mathbb{P}^2)$
KE	Not KE	Not KE	KE

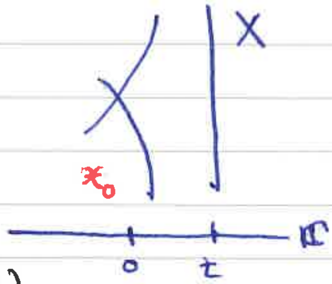
Yau – Tian – Donaldson conjecture

proven by Chen – Donaldson – Sun and Tian (2015)

$\boxed{\text{Theorem}}$ A Fano manifold admits a KE metric.

$\Longleftrightarrow$ it is K-polystable.

- test configuration $(\mathcal{X}, \mathcal{L})$
- the Donaldson – Futaki invariant $DF(\mathcal{X}, \mathcal{L})$



$\boxed{\text{Definition}}$ The "greatest Ricci lower bound" $R(X)$ of a Fano manifold X is defined as

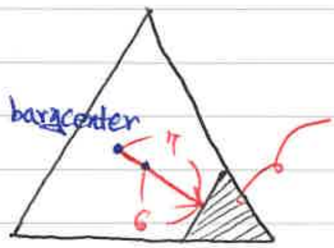
$$R(X) := \sup \left\{ 0 \leq t \leq 1 \;\middle|\; \begin{array}{l} \exists \text{ a Kähler form } \omega \in c_1(X) \\ \text{with } Ric(\omega) \geq t\omega \end{array} \right\}.$$

: the maximum existence time $t \in [0,1]$ of Aubin – Yau's continuity path

Remark X admit KE metric $\overset{\circ}{\Longrightarrow}$, $\overset{\times}{\Longleftarrow}$ $R(X) = 1$

Examples [Székelyhidi, 2011]

$$R(Bl_p(\mathbb{P}^2)) = \frac{6}{7}$$



[Chi Li, 2011]

For different points $p, q \in \mathbb{P}^2$,

$$R(Bl_{p,q}(\mathbb{P}^2)) = \frac{21}{25}.$$

§2. Rational homogeneous varieties and horospherical varieties

G : connected reductive complex Lie group

$\mathfrak{g} = T_e G$: corresponding Lie algebra

Fix a maximal torus $T \subset G$, $\mathfrak{t} = \mathrm{Lie}(T)$

T is $(\mathbb{C}^*)^n$, $\mathfrak{t}$ Cartan subalgebra

<u>Example</u> $G = GL_n(\mathbb{C})$, $T = \left\{ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} : a_i \in \mathbb{C}^* = \mathbb{C} \setminus \{0\} \right\}$

<u>Root space decomposition</u> $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$

$\mathfrak{g}_\alpha := \{ x \in \mathfrak{g} \mid [t, x] = \alpha(t)\, x \quad \forall\, t \in \mathfrak{t} \}$

root space

$\Phi = \Phi(\mathfrak{g}, \mathfrak{t}) = \{ \alpha \in \mathfrak{t}^* \setminus \{0\} : \mathfrak{g}_\alpha \neq \{0\} \}$

root system; $\mathfrak{t}^* = \mathrm{Hom}(\mathfrak{t}, \mathbb{C})$

<u>Example</u> $G = SL_3(\mathbb{C})$

$\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C}) = \{ A \in \mathrm{Mat}_{3\times3}(\mathbb{C}) \mid \mathrm{tr}(A) = 0 \}$

$(\because)$ $\det(e^A) = e^{\mathrm{tr}(A)} = 1$

$\cup$

Cartan subalgebra $\mathfrak{t} = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} : a_1 + a_2 + a_3 = 0 \right\}$

linear functional $L_i : \mathrm{diag}(a_1, a_2, a_3) \longmapsto a_i$

root system of $\mathfrak{sl}_3(\mathbb{C})$: $\Phi = \{ L_i - L_j \mid 1 \leq i \neq j \leq 3 \}$

$\Delta = \{ \alpha_1, \cdots, \alpha_n \}$: a choice of simple roots of G

- coroot $\alpha_i^\vee = \dfrac{2\alpha_i}{(\alpha_i, \alpha_i)}$
- fundamental weights ϖ_i defined by $(\varpi_i, \alpha_j^\vee) = \delta_{ij}$

$\boxed{\text{Fundamental Theorem in Representation Theory}}$

For a dominant integral weight $\lambda = \sum_{i=1} a_i \varpi_i \in \mathfrak{t}^*$ $(a_i \in \mathbb{Z}_{\geq 0})$,

$\exists\,!$ irreducible G-representation space $V_G(\lambda)$ with highest weight λ.

Example ① $G = SL_2(\mathbb{C})$; $V_G(k\varpi_1) = \mathrm{Sym}^k(\mathbb{C}^2)$

② $G = SL_4(\mathbb{C})$; $V_G(\varpi_1) = \mathbb{C}^4$ (standard representation)

$V_G(\varpi_2) = \Lambda^2 \mathbb{C}^4$, $V_G(2\varpi_1) = \mathrm{Sym}^2(\mathbb{C}^4)$

Rational homogeneous varieties (generalized flag varieties)

Let v_i be a nonzero highest weight vector of $V_G(\varpi_i)$.

$$\Rightarrow \quad G/P_i = G.[v_i] \subset \mathbb{P}(V_G(\varpi_i))$$

$\uparrow$: rational homogeneous variety of Picard number 1

P_i : the i-th maximal parabolic subgroup of G

Example $G = SL_4(\mathbb{C})$, $V_{SL_4}(\varpi_2) = \Lambda^2 \mathbb{C}^4$, $\mathbb{C}^4 = \langle e_1, e_2, e_3, e_4 \rangle$

$e_1 \wedge e_2$: highest weight vector

$G/P_2 = SL_4 . [e_1 \wedge e_2] \subset \mathbb{P}(\Lambda^2 \mathbb{C}^4)$

$= \{$ 2-dimensional subspaces in $\mathbb{C}^4 \}$

$= Gr(2, 4)$ the Grassmannian

(A_3, α_2)

Let v_i and v_j be highest weight vectors of $V_G(\varpi_i)$ and $V_G(\varpi_j)$ $(i \neq j)$.

Horospherical variety

$$G.[v_i + v_j] \subset \mathbb{P}(V_G(\varpi_i) \oplus V_G(\varpi_j))$$

$\|$

(G, ϖ_i, ϖ_j)

$G/H = G.[v_i + v_j]$: rank 1 horospherical homogeneous space

$\downarrow \mathbb{C}^*$

$G/(P_i \cap P_j)$

Theorem [Pasquier, 2009] classified smooth projective horospherical varieties X of Picard number one :

if X is nonhomogeneous, then X is one of the following

(i) $(SO_{2n+1}, \varpi_{n-1}, \varpi_n)$, $n \geq 3$

(ii) $(SO_7, \varpi_1, \varpi_3)$

(iii) $(Sp_{2n}, \varpi_k, \varpi_{k-1})$, $n \geq k \geq 2$

(iv) $(F_4, \varpi_2, \varpi_3)$

(v) $(G_2, \varpi_2, \varpi_1)$,

and their automorphism groups are non-reductive.

$\Downarrow$

They cannot admit Kähler–Einstein metrics.

§ 3. Greatest Ricci lower bounds of projective horospherical manifolds of Picard number one

Ingredients :

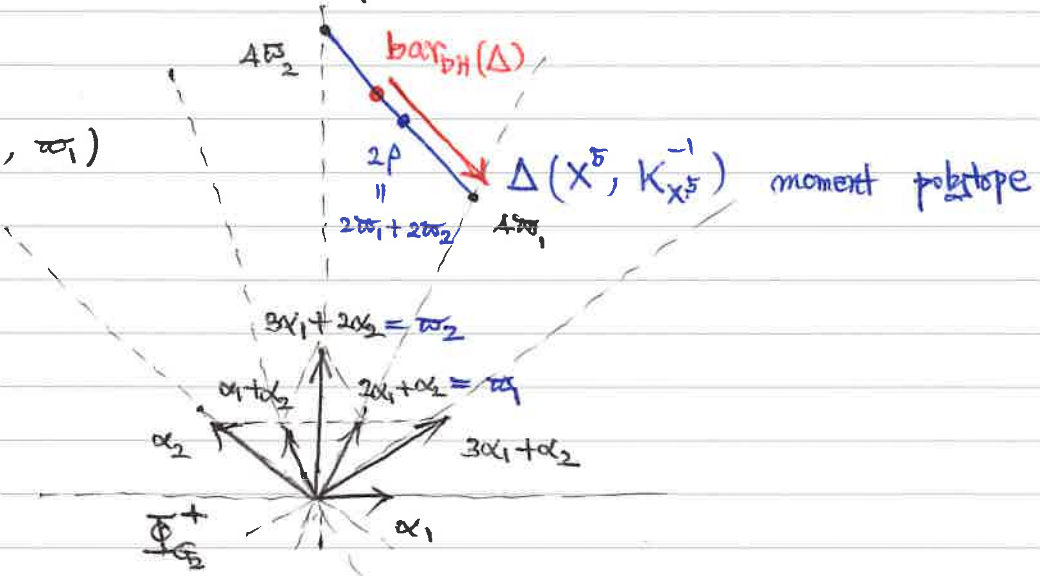
- Algebraic moment polytope $\Delta(X, K_X^{-1})$
- Duistermaat – Heckman measure

$\Longrightarrow$ compute the barycenter of the moment polytope with respect to the Duistermaat – Heckman measure $\mathrm{bar}_{DH}(\Delta)$

<u>Example</u> The greatest Ricci lower bound of the horospherical manifold $(G_2, \varpi_2, \varpi_1)$ is

$$R(G_2, \varpi_2, \varpi_1) = \frac{56}{67} \approx 0.8358 .$$

$X^5 = (G_2, \varpi_2, \varpi_1)$



Duistermaat – Heckman measure

$$P_{DH}(x, y)\, dx dy = \prod_{\alpha \in \Phi^+} \kappa(\alpha, p)\, dp$$
$$= x\left(-\frac{3}{2}x + \frac{\sqrt{3}}{2}y\right)\left(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y\right)\left(\frac{1}{2}x + \frac{\sqrt{3}}{2}y\right)\left(\frac{3}{2}x + \frac{\sqrt{3}}{2}y\right)\sqrt{3}y\, dx dy$$

barycenter $$\mathrm{bar}_{DH}(\Delta) = \frac{1}{\mathrm{Vol}_{DH}(\Delta)} \int_\Delta p \cdot P_{DH}(p)\, dp$$
$$= \left(\frac{45}{56}, \frac{179\sqrt{3}}{56}\right)$$

$$\therefore \; R(X^5) = \frac{2}{2 + \frac{11}{28}} = \frac{56}{67} .$$