Uniqueness results for the critical catenoid

Dong-Hwi Seo

Hanyang University

제18회 고등과학원 기하학 겨울학교

This talk is based on

S., Sufficient symmetry conditions for free boundary minimal annuli to be the critical catenoid, https://arxiv.org/abs/2112.11877

Introduction

- Let $\varphi_t : \Sigma^k \to \mathbb{B}^n \subset \mathbb{R}^n, t \in (-\epsilon, \epsilon)$ be a one-parameter family of immersions with $\varphi_t(\partial\Sigma) \subset \partial\mathbb{B}^n$. Then,

$$\left.\frac{d}{dt}\right|_{t=0} Area(\Sigma_t) = -\int_\Sigma \langle \vec{H}, X \rangle dV + \int_{\partial\Sigma} \langle \eta, X \rangle ds, \tag{1}$$

  where $\vec{H}$ is the mean curvature vector of Σ, η is the outward unit conormal vector along $\partial\Sigma$, and $X := \frac{\partial}{\partial t}\varphi_t$ is the variational vector of ϕ_t.
- If $\phi : \Sigma \to \mathbb{B}^n$ is a critical point of area functional for every variational vector fields, ϕ is said to be free boundary minimal submanifold in $\mathbb{B}^n$.

Free boundary minimal surface in a ball

Definition 1.

An immersed (resp. embedded) submanifold $\varphi : \Sigma^k \to \mathbb{B}^n \subset \mathbb{R}^n$ is said to be an immersed (resp. embedded) free boundary minimal submanifold in $\mathbb{B}^n$ if

- Σ is a minimal submanifold, i.e. mean curvature is zero.
- Σ meets $\partial\mathbb{B}^n$ orthogonally along $\partial\Sigma$.

Note that *embedded* free boundary minimal submanifold in $\mathbb{B}^n$ is proper, i.e. $\varphi(\Sigma) \cap \partial\mathbb{B}^3 = \varphi(\partial\Sigma) \neq \phi$. Equivalently,

Definition 2.

An immersed (resp. embedded) submanifold $\varphi : \Sigma^k \to \mathbb{B}^n \subset \mathbb{R}^n$ is said to be an immersed (resp. embedded) free boundary minimal submanifold in $\mathbb{B}^n$ if the coordinate functions $x_i, i = 1, \ldots, k$ satisfy

- $\Delta_\Sigma x_i = 0$ in Σ.
- $\frac{\partial x_i}{\partial \eta} = x_i$ on $\partial\Sigma$.

Examples of free boundary minimal surfaces

Figure: Equatorial disk

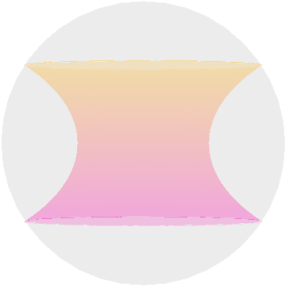


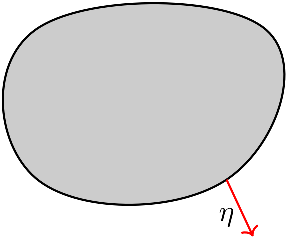
Figure: Critical catenoid

In relation with Steklov eigenvalue problem

Let M^m : Riemannian manifold, $\Omega \subset M$: smooth bounded domain. *Steklov eigenvalue problem* is finding all $\sigma \in \mathbb{R}$ for which there exists $u \in C^\infty(\Omega)$ which satisfies

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \eta} = \sigma u & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where η is the outward unit conormal vector on $\partial\Omega$.



It is known that the eigenvalues σ of this problem are discrete and it form a sequence, $0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots \to \infty$. Note that constant functions are Steklov eigenfunctions with eigenvalue $\sigma_0 = 0$. In addition, for $i = 0, 1, \cdots$, we have

$$\sigma_{i+1}(M) = \inf_{f \in C^\infty(\partial M) \setminus \{0\}} \left\{ \frac{\int_M |\nabla \hat{f}|^2}{\int_{\partial M} f^2} \,\middle|\, \int_{\partial M} f u_k = 0 \text{ for } k = 0, \cdots, i \right\}, \tag{3}$$

where $\hat{f} \in C^\infty(M)$ is the harmonic extension of f and u_k is a Steklov eigenfunction corresponding to σ_k.

Recall that an immersed free boundary minimal surface M in $\mathbb{B}^3$ satisfies

$$\begin{cases} \Delta_M(x_1, x_2, x_3) = (0,0,0) & \text{in } M \\ \frac{\partial}{\partial \eta}(x_1, x_2, x_3) = (x_1, x_2, x_3) & \text{on } \partial M \end{cases} . \tag{4}$$

Lemma 1.

Any coordinate functions, $x_i, i = 1, 2, 3$, are Steklov eigenfunctions of M with eigenvalue 1. It implies $\sigma_1(M) \leq 1$.

Lemma 2 (Orthogonality).

Let u be a first Steklov eigenfunction of M. Then, we have

$$\int_{\partial M} u = 0. \tag{5}$$

If $\sigma_1(M) < 1$, we have

$$\int_{\partial M} u x_i = 0 \text{ for all } i = 1, 2, 3. \tag{6}$$

Some questions posed by Fraser and Li (2012)

Question 1.

Which compact orientable surfaces with boundary can be realized as properly embedded minimal surfaces in the unit ball $\mathbb{B}^3$ with free boundary?

Question 2.

Let M' be a compact embedded free boundary minimal hypersurface in $\mathbb{B}^n$. Is $\sigma_1(M') = 1$?

Question 3.

Is the critical catenoid the only embedded free boundary minimal annulus in $\mathbb{B}^3$ up to congruence?

Closed minimal surfaces in $\mathbb{S}^3$

Those relation and questions are inspired by the relation between closed minimal surface in $\mathbb{S}^3$ and the Laplace eigenvalue problem.
An immersed closed minimal surface N in $\mathbb{S}^3$ satisfies

$$\Delta_N(x_1, x_2, x_3, x_4) + 2(x_1, x_2, x_3, x_4) = (0, 0, 0, 0) \quad \text{in } N \ . \tag{7}$$

Lemma 3.

Any coordinate functions, $x_i, i = 1, 2, 3, 4$, are Laplace eigenfunctions of N with eigenvalue 2. It implies the first (nonzero) eigenvalue $\lambda_1(N) \leq 2$.

Question 1.*

Which closed surfaces can be realized as embedded closed minimal surfaces in the sphere $\mathbb{S}^3$?

Yes, it is proved by Lawson in 1970.

Question 2 (Yau's conjecture (1982)).*

Let N' be a closed embedded minimal hypersurface in $\mathbb{S}^{n+1}$. Is $\lambda_1(N') = n$?

Question 3 (Lawson's conjecture (1970)).*

Is the Clifford torus the embedded closed minimal surface of genus 1 in $\mathbb{S}^3$ up to congruence?

Yes, it is proved by Brendle in 2013.

Rigidity of free boundary minimal surface in a ball

- (Nitsche, 1985) An immersed free boundary minimal disk in $\mathbb{B}^3$ is an equatorial disk.

In the same paper, Nitsche claimed the following statement, which is now well-known by Fraser and Li.

Conjecture.

An embedded free boundary minimal annulus in $\mathbb{B}^3$ is a critical catenoid.

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Conjecture.

An embedded free boundary minimal annulus in $\mathbb{B}^3$ is a critical catenoid.
$\leftrightsquigarrow$ Lawson's conjecture (1970) for $\mathbb{S}^3$, solved by Brendle (2013).

Previous results

- (Fraser-Schoen, 2016) If an immersed free boundary minimal annulus in $\mathbb{B}^n$ has the first Steklov eigenvalue 1, it is congruent to the critical catenoid.

Let Σ^2 be an embedded free boundary minimal annulus in $\mathbb{B}^3$.

- (McGrath, 2018) If Σ is invariant under the reflections of the three coordinate planes, then it is congruent to the critical catenoid.
- (Kusner-McGrath, 2020) If Σ is invariant under the antipodal map, then it is congruent to the critical catenoid.

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Notation

- Σ : embedded free boundary minimal annulus in $\mathbb{B}^3$.
- M_0 : embedded free boundary minimal surface of genus zero in $\mathbb{B}^3$ with $b \geq 2$ boundary components,

$$(\partial M_0)_1, \ldots, (\partial M_0)_b.$$

- M will be defined in various situations.

Uniqueness results

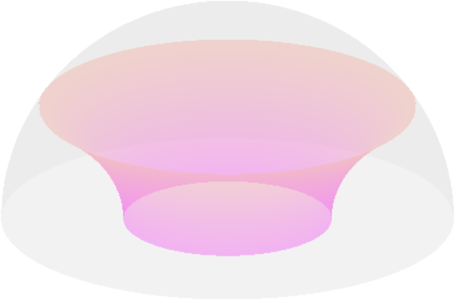
Theorem 1 (S.).

If Σ has one of the following symmetry conditions, then Σ is congruent to the critical catenoid.

- Σ *is invariant under the reflections through two distinct planes.*
- Σ *is invariant under the reflection through a plane that does not meet* $\partial\Sigma$.

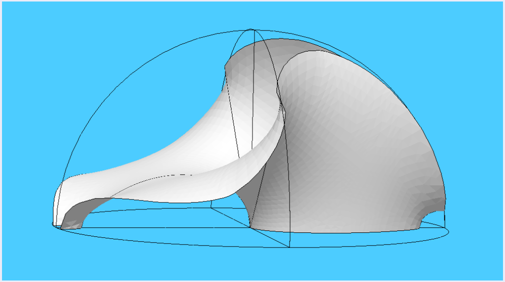
Corollary 1.

Let Σ' be an embedded free boundary minimal annulus in $\mathbb{B}^3 \cap \{x_3 \geq 0\}$. If one boundary component is contained in the open hemisphere and the other boundary component is contained in the equatorial disk, then Σ' is congruent to the half of the critical catenoid, $\Sigma_c \cap \{x_3 \geq 0\}$.



Remark.

The boundary condition in Corollary 1 would be necessary. Carlotto, Franz, and Schulz constructed a compact embedded free boundary minimal surface $\Sigma_{1,1}$ of genus 1 in $\mathbb{B}^3$ with only one boundary component. Note that $\Sigma_{1,1}$ has the dihedral symmetry D_2. The author expect to have additional reflection symmetry and gives the following expected example.



An Idea of the proof of Theorem 1

The strategy of the proof is the method of proof by contradiction by assuming $\sigma_1(\Sigma) < 1$. Then, by the work of Fraser and Schoen, Σ is congruent to the critical catenoid.
We briefly review the previous approach of the works of McGrath and Kusner-McGrath.

Definition 3 (Nodal domain).

Let u be a Steklov eigenfunction of Σ. Then the nodal set of u is $\mathcal{N} = \{p \in \Sigma \mid u(p) = 0\}$. A nodal domain of u is a component of $\Sigma \setminus \mathcal{N}$.

Lemma 4 (Courant nodal domain theorem).

If u is a first Steklov eigenfunction, then u has exactly two nodal domains.

Using the Courant nodal domain theorem, McGrath obtained

Lemma 5 (Symmetry of a first eigenfunction).

If Σ is invariant under the reflection through a plane and $\sigma_1(\Sigma) < 1$, then a first Steklov eigenfunction is invariant under the reflection.

In addition, using the two-piece property by Lima-Menezes (2021), Kusner and McGrath obtained the following convexity result.

Lemma 6 (Convexity of the boundary components of M_0).

Each boundary component of M_0 is strictly convex on $\mathbb{S}^2$. In other words, there are at most two intersection points of a boundary component and a great circle.

From the convexity we can further observe that the existence of a coordinate plane Π that satisfies

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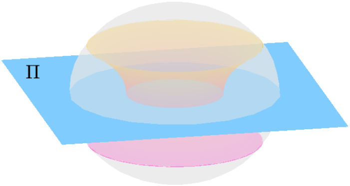
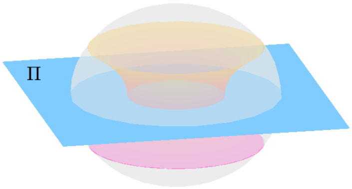
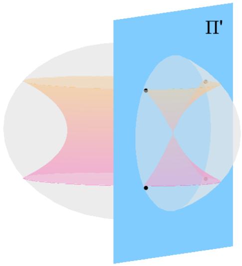


From the convexity we can further observe that the existence of a coordinate plane Π that satisfies

Observation.

1. Π does not meet $\partial\Sigma$.
2. For a plane Π' that is perpendicular to Π, it intersects $\partial\Sigma$ in at most four points.





One main ingredient of the observation is as follows.

Claim.

A strictly convex closed curve $\mathcal{C}$ on $\mathbb{S}^2$ does not meet a coordinate plane perpendicular to $F(\mathcal{C}) := \int_{\mathcal{C}} x$, where $x = (x_1, x_2, x_3)$ is the position vector in $\mathbb{R}^3$.

Using this claim, we can show that $\Pi \cap \partial\Sigma = \phi$. By minimality of Σ,

$$0 = \int_{\Sigma} \Delta_{\Sigma} x = \int_{\partial\Sigma} \frac{\partial x}{\partial \nu} = \int_{\partial\Sigma} x = \int_{(\partial\Sigma)_1} x + \int_{(\partial\Sigma)_2} x. \tag{8}$$

Thus, if we find Π by the claim with $(\partial\Sigma)_1$, then $\Pi \cap \partial\Sigma = \emptyset$.

Using the previous observation, we can show that the following :

Lemma 7.

If $\sigma_1(\Sigma) < 1$, then a first Steklov eigenfunction u is sign-changing in one of boundary components of Σ. Furthermore, the nodal set of u in this component of $\partial\Sigma$ is exactly two points.

(Sketch of proofs of Theorem 1) Let

$$f_i := \frac{\int_{(\partial\Sigma)_i} x}{\left|\int_{(\partial\Sigma)_i} x\right|} \in \mathbb{S}^2, i = 1, 2. \tag{9}$$

By assumption, we say Σ is invariant under the reflection R_Π through a plane Π. Then, $R_\Pi(f_1) = f_1$ or $R_\Pi(f_1) = f_2$. For simplicity, let $f_1 = (0, 0, 1)$. The two cases are equivalent to
Case 1. The Reflection planes are $\{x_1 = 0\}$ and $\{x_2 = 0\}$.
Case 2. One of the reflection planes is $\{x_3 = 0\}$.

Let $\mathcal{N}$ be the nodal set of the first Steklov eigenfunction u whose eigenvalue < 1. Using the previous lemma, we may assume that u is sign-changing in $(\partial\Sigma)_1$ and let $p_1, p_2 \in \mathcal{N} \cap (\partial\Sigma)_1$.

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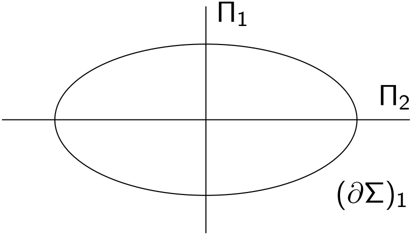

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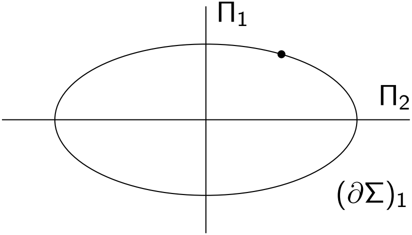

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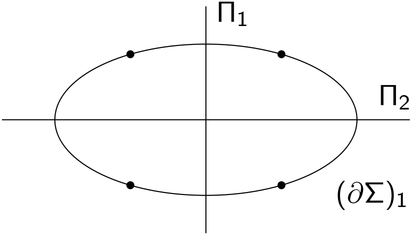

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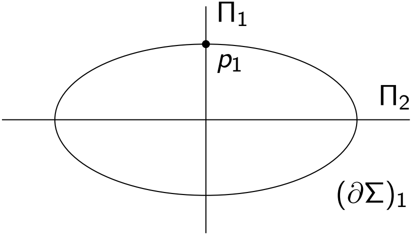

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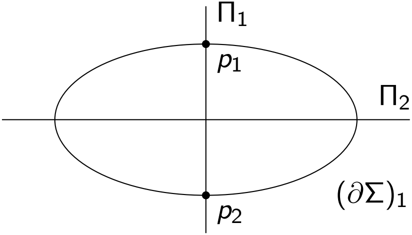

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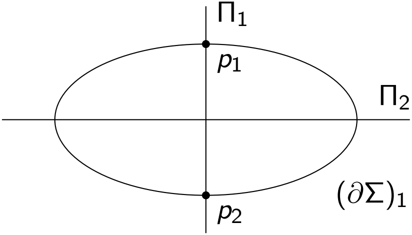

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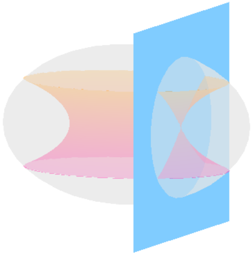
Case 2. Let $\Pi_1 := \{x_3 = 0\}$ be one of the reflection planes. by the symmetry of u, we have $p_3, p_4 \in \mathcal{N} \cap (\partial\Sigma)_2$ and $R_{\Pi_1}(\{p_1, p_2\}) = \{p_3, p_4\}$. Then, we have a plane Π_2 passing through p_1, p_2, p_3, p_4 which is perpendicular to Π_1.

Let $\Pi_2 = \{x_2 = c\}$. By the Observation, $\Pi_2 \cap \partial\Sigma = \{p_1, p_2, p_3, p_4\}$. by the symmetry of u, the sign of u does not change at each $((\partial\Sigma)_1 \cap \{x_2 > c\}) \cup ((\partial\Sigma)_2 \cap \{x_2 > c\})$ and $((\partial\Sigma)_1 \cap \{x_2 < c\}) \cup ((\partial\Sigma)_2 \cap \{x_2 < c\})$.



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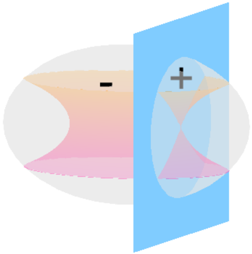

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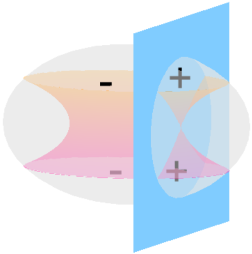
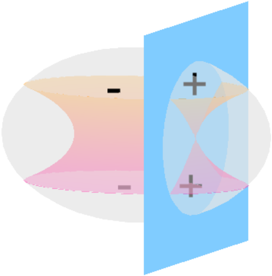
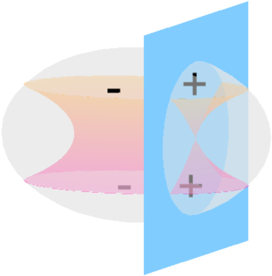




Thus, we have

$$\int_{\partial\Sigma} u(x_2 - c) \neq 0. \qquad (10)$$



Thus, we have

$$\int_{\partial\Sigma} u(x_2 - c) \neq 0. \qquad (10)$$

On the other hand, $\sigma_1(\Sigma) < 1$ implies that $\int_{\partial\Sigma} u = \int_{\partial\Sigma} ux_2 = 0$ (see Lemma 2 (Orthogonality)), which leads a contradiction with the previous identity. □

Conclusion

Using our method, we have the following sufficient conditions for Σ to be the critical catenoid.

A Condition on a component of $\partial\Sigma$

- The reflection symmetries through two distinct planes.

Conditions on $\partial\Sigma$

- The reflection symmetries through two distinct planes.
- The reflection symmetry through a plane with additional conditions.
  - The reflection plane Π does not meet $\partial\Sigma$.
  - The reflection plane Π intersects $\partial\Sigma$ and the two components of $\partial\Sigma$ are congruent.

- The rotoreflection symmetry by $A := B \circ R \in O(3)$, where R is the reflection through a plane Π with $R(f_1) = f_2$, and B is a rotation about the axis perpendicular to Π.
  - B is an irrational rotation.
  - B is a rotation by an angle $\theta := \frac{b}{a} \cdot \pi$, where a is an odd number and they are relatively primes.

Thank you!