Formal exponential maps and Atiyah class of dg manifolds

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The 18th KIAS Geometry Winter school
Jeongsun, 2023-02-10

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Exponential maps arise naturally in the contexts of linearization problems

1. Lie theory
2. smooth manifolds

PBW isomorphism in Lie theory

- $\mathfrak{g}$: finite dimensional Lie algebra
- $\exp : \mathfrak{g} \to G$
- exp : local diffeomorphism from nbd of 0 to nbd of 1
- Differential operators of $\mathfrak{g}$ evaluated at $0 \in \mathfrak{g}$ is $D'(0) = S\mathfrak{g}$:

$$S\mathfrak{g} = \bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n} / \{x \otimes y = y \otimes x\}$$

  called the symmetric algebra of $\mathfrak{g}$.
- Differential operators of G evaluated at $1 \in G$ is $D'(1) = U\mathfrak{g}$:

$$U\mathfrak{g} = \bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n} / \{x \otimes y - y \otimes x = [x, y]\}$$

  called the universal enveloping algebra of $\mathfrak{g}$.

PBW isomorphism in Lie theory

- exp induces an isomorphism on differential operators evaluated at $0 \in \mathfrak{g}$ and $1 \in G$:

$$\mathrm{pbw} = (\exp)_* : S\mathfrak{g} \xrightarrow{\cong} U\mathfrak{g}$$

$$X_1 \odot \cdots \odot X_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} X_{\sigma(1)} \cdots X_{\sigma(n)}$$

- Both $S\mathfrak{g}$ and $U\mathfrak{g}$ have canonical coalgebra structure.

Fact: Poincaré–Birkhoff–Witt map is an isomorphism of coalgebras.

C is coalgebra $\iff$ its dual $A = C^*$ is an algebra

e.g. $\Delta : C \to C \otimes C$ is a comultiplication
then $\mu : C^* \otimes C^* \to C^*$ is a multiplication defined by

$$\mu(f, g) = (f \otimes g) \circ \Delta : C \to C \otimes C \to \mathbb{R} \otimes \mathbb{R} \cong \mathbb{R}$$

Exponential map on smooth manifolds

- an affine connection ∇ on smooth manifold M
- $\exp^{\nabla} : T_M \to M \times M$ (bundle map)
  defined by $\exp^{\nabla}(X_m) = (m, \gamma(1))$ where γ is the smooth path in M satisfying $\dot{\gamma}(0) = X_m$ and $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$. (i.e. geodesic)
  - $\Gamma(S(T_M))$ seen as space of differential operators on T_M, all derivatives in the direction of the fibers, evaluated along the zero section of T_M
  - $\mathcal{D}(M)$ seen as space of differential operators on $M \times M$, all derivatives in the direction of the fibers, evaluated along the diagonal section $M \to M \times M$
- $\mathrm{pbw}^{\nabla} := \exp^{\nabla}_{*} : \Gamma(S(T_M)) \xrightarrow{\cong} \mathcal{D}(M)$ is an isomorphism of left modules over $C^{\infty}(M)$ called Poincaré–Birkhoff–Witt isomorphism.

pbw^{∇} as infinite jet of exp

The Taylor series of the composition

$$T_m M \xrightarrow{\exp^{\nabla}} \{m\} \times M \xrightarrow{f} \mathbb{R}$$

at the point $0_m \in T_m M$ is

$$\sum_{J \in \mathbb{N}_0^n} \tfrac{1}{J!} \big(\mathrm{pbw}^{\nabla}(\partial_x^J) f \big)(m) \otimes y_J \quad \in \hat{S}(T_m^{\vee} M).$$

- $(x_i)_{i \in \{1,\dots,n\}}$ are local coordinates on M
- $(y_j)_{j \in \{1,\dots,n\}}$ induced local frame of $T_M^{\vee}$ regarded as fiberwise linear functions on T_M

Hence pbw^{∇} is the fiberwise infinite jet of the bundle map $\exp : T_M \to M \times M$ along the zero section of $T_M \to M$.

Algebraic characterization of pbw^{∇}

Theorem (Laurent-Gengoux, Stiénon, Xu, 2014)

$$\mathrm{pbw}^{\nabla}(f) = f, \quad \forall f \in C^{\infty}(M);$$
$$\mathrm{pbw}^{\nabla}(X) = X, \quad \forall X \in \mathfrak{X}(M);$$
$$\mathrm{pbw}^{\nabla}(X^{n+1}) = X \cdot \mathrm{pbw}^{\nabla}(X^{n}) - \mathrm{pbw}^{\nabla}(\nabla_X X^{n}), \quad \forall n \in \mathbb{N}.$$

Therefore, for all $n \in \mathbb{N}$ and $X_0, \ldots, X_n \in \mathfrak{X}(M)$,

$$\mathrm{pbw}^{\nabla}(X_0 \odot \cdots \odot X_n) = \frac{1}{n+1} \sum_{k=0}^{n} \left\{ X_k \cdot \mathrm{pbw}^{\nabla}(X^{\{k\}}) - \mathrm{pbw}^{\nabla}\left(\nabla_{X_k}(X^{\{k\}})\right) \right\}$$

where $X^{\{k\}} = X_0 \odot \cdots \odot X_{k-1} \odot X_{k+1} \odot \cdots \odot X_n$.

Example

- *G: Lie group,* X_i^L: *Left invariant vector field*
- *Choose a connection* ∇ *such that* $\nabla_{X_i^L} X_j^L = 0$. *Then,*

$$\mathrm{pbw}^\nabla(X_1^L \odot \cdots \odot X_n^L) = \frac{1}{n!} \sum_{\sigma \in S_n} X_{\sigma(1)}^L \cdots X_{\sigma(n)}^L$$

- $(\mathrm{pbw}^\nabla)^{-1} : \mathcal{D}(M) \to \Gamma(S(T_M))$ takes a differential operator to its *complete symbol*
- The algebraic characterization of pbw^∇ does NOT involve any points of M or any geodesic curves of ∇.
- The map pbw^∇ is a sort of formal exponential map defined inductively.

PBW isomorphism in differential geometry

Both $\Gamma(S(T_M))$ and $\mathcal{D}(M)$ are left coalgebras over $R := C^\infty(M)$.

Proposition

$\text{pbw}^\nabla : \Gamma(S(T_M)) \to \mathcal{D}(M)$ *is an isomorphism of coalgebras over* $C^\infty(M)$.

Note: pbw^∇ does NOT preserve multiplication.

Atiyah class of holomorphic vector bundle

- X : complex manifold
- E : holomorphic vector bundle over X
- (smooth) connection $\nabla^{1,0} : \Gamma(E) \to \Omega^{1,0}(E)$ of type $(1,0)$:

$$\nabla^{1,0}(f \cdot s) = \partial(f) \cdot s + f \cdot \nabla^{1,0}(s), \quad s \in \Gamma(E), f \in C^\infty(X)$$

- Choose $\mathcal{R} = \nabla^{1,0}\bar{\partial} + \bar{\partial}\nabla^{1,0} : \Gamma(E) \to \Omega^{1,1}(E)$, then

$$\mathcal{R} \in \Omega^{1,1}(\mathrm{End}(E)).$$

Definition (Atiyah, 1957)

The Atiyah class α_E of E is the cohomology class

$$\alpha_E = [\mathcal{R}] \in H^1\big(X; \Omega^1_X \otimes \mathrm{End}(E)\big)$$

– Atiyah class α_E is an obstruction to the existence of holomorphic connection on E.

Graded vector space

Definition

A graded vector space V is a space $V = \bigoplus_{i \in \mathbb{Z}} V_i$ where V_i consists of degree i vectors.

Example

Let $\Omega^k(M)$ be the space of k-forms. We can declare $w \in \Omega^k(M)$ is of degree k. Then space $\Omega^\bullet(M)$ of all forms on M is a graded vector space (infinite dimensional).

Degrees here have no effect. However, for example, if you consider the multiplication (i.e. graded algebra), then $\Omega^\bullet(M)$ is not commutative, but is graded commutative:

$$\xi \cdot \eta = (-1)^{nm} \eta \cdot \xi$$

for $\xi \in \Omega^m(M), \eta \in \Omega^n(M)$.

Duality between space and algebra

There is a certain (heuristic) duality between space and commutative algebras.

- X: space $\Rightarrow A = \{f : X \to \mathbb{R}\}$: algebra of functions on X
- A: commutative algebra $\Rightarrow X = \{\text{maximal ideals in } A\}$: space

Idea:

use "algebra of smooth functions" to define a smooth manifold!

We will define graded manifold by a graded commutative algebras of special type.

Differential graded manifolds

M: smooth manifold, $\mathcal{O}_M$: (sheaf of) ring of smooth functions.

Definition

A $\mathbb{Z}$-graded manifold $\mathcal{M}$ with base manifold M is a sheaf $\mathcal{A}$ of $\mathbb{Z}$-graded commutative $\mathcal{O}_M$-algebras such that

$$\mathcal{A}(U) \cong \mathcal{O}_M(U)\hat{\otimes}\hat{S}(V^\vee)$$

for sufficiently small open subsets $U \subset M$ and some $\mathbb{Z}$-graded vector space V. In other words, smooth functions on $\mathcal{M}$ are locally formal power series in V with coefficients in $\mathcal{O}_M$.

$C^\infty(\mathcal{M}) := \mathcal{A}(\mathcal{M})$

Definition

A dg manifold is a $\mathbb{Z}$-graded manifold $\mathcal{M}$ endowed with a (homological) vector field $Q \in \mathfrak{X}(\mathcal{M}) = \mathrm{Der}_{\mathbb{K}}(C^\infty(\mathcal{M}))$ of degree $+1$ such that $[Q, Q] = 2\, Q \circ Q = 0$.

examples

Example

Given a Lie algebra $\mathfrak{g}$, $(\mathfrak{g}[1], Q = d_{\mathrm{CE}})$ *is a dg manifold.*

- $C^\infty(\mathfrak{g}[1]) = \Lambda^\bullet \mathfrak{g}^\vee$
- $Q = d_{\mathrm{CE}} : \Lambda^\bullet \mathfrak{g}^\vee \to \Lambda^{\bullet+1} \mathfrak{g}^\vee$ — *Chevalley–Eilenberg differential*

Example

Given a smooth manifold M, $(T_M[1], d_{dR})$ *is a dg manifold;*

- $C^\infty(T_M[1]) = \Omega^\bullet(M)$ — *Space of differential forms on* M
- $Q = d_{dR} : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ — *de Rham differential*

Example

Given a complex manifold X, $(T_X^{0,1}[1], \bar\partial)$ *is a dg manifold;*

- $C^\infty(T_X^{0,1}[1]) = \Omega^{0,\bullet}(X)$ — *Space of anti-holomorphic forms*
- $Q = \bar\partial : \Omega^{0,\bullet}(X) \to \Omega^{0,\bullet+1}(X)$ — *Dolbeault operator*

Example

Given a regular foliation F of M and the associated (regular) integrable distribution $T_F \subset T_M$, $(T_F[1], d_{\mathrm{dR}})$ is a dg manifold;

- $C^\infty(T_F[1]) = \Omega_F^\bullet$ — *Space of leafwise differential forms*
- $Q = d_{dR} : \Omega_F^\bullet \to \Omega_F^{\bullet+1}$ — *de Rham differential*

Example

Given a vector bundle $E \to M$ and smooth section s, $(E[-1], i_s)$ is a dg manifold, called derived intersection of s with the zero section.

- $C^\infty(E[-1]) = \Gamma(\Lambda^{-\bullet} E^\vee)$
- $Q = i_s : \Gamma(\Lambda^{-\bullet} E^\vee) \to \Gamma(\Lambda^{-\bullet+1} E^\vee)$ — *interior product with s*

For instance, if $f \in C^\infty(M)$, then $(T_M^\vee[-1], i_{df})$ is a dg-manifold called derived critical locus of f.

Connection on a graded manifold

Definition

An affine connection on a graded manifold $\mathcal{M}$ is a $\mathbb{k}$-linear map

$$\nabla : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$$

of degree 0 satisfying

$$\nabla_{fX} Y = f \nabla_X Y,$$
$$\nabla_X (fY) = X(f) Y + (-1)^{|X||f|} f \nabla_X Y,$$

for all $f \in C^\infty(\mathcal{M})$ and all homogeneous $X, Y \in \mathfrak{X}(\mathcal{M})$.

- ∇ is torsion-free if

$$\nabla_X Y - (-1)^{|X||Y|} \nabla_Y X = [X, Y].$$

Atiyah class of a differential graded manifold

- Choose a torsion-free connection ∇ on a dg manifold $(\mathcal{M}, Q)$.
- Define a degree +1 section $\mathrm{At}^{\nabla} \in \Gamma(\mathrm{Hom}(S^2(T_{\mathcal{M}}), T_{\mathcal{M}}))$

$$\mathrm{At}^{\nabla}(X, Y) = L_Q(\nabla_X Y) - \nabla_{L_Q X} Y - (-1)^{|X|} \nabla_X (L_Q Y)$$

for homogeneous vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$.

Note: $\mathrm{At}^{\nabla} := L_Q \nabla$ but $\nabla \notin \Gamma(\mathrm{Hom}(S^2(T_{\mathcal{M}}), T_{\mathcal{M}}))$

Lemma

- $L_Q \circ L_Q = 0$ *and* $L_Q \mathrm{At}^{\nabla} = 0$
- $\left[\mathrm{At}^{\nabla}\right] \in H^1\left(\Gamma(\mathrm{Hom}(S^2(T_{\mathcal{M}}), T_{\mathcal{M}})), L_Q\right)$ *is independent of the connection* ∇.

Definition

1. $\mathrm{At}^{\nabla} \in \Gamma(\mathrm{Hom}(S^2(T_{\mathcal{M}}), T_{\mathcal{M}}))$ is an Atiyah cocycle of ∇.
2. The Atiyah class of the dg manifold $(\mathcal{M}, Q)$

$$\alpha_{\mathcal{M}} := \left[\mathrm{At}^{\nabla}\right] \in H^1\big(\Gamma(\mathrm{Hom}(S^2(T_{\mathcal{M}}), T_{\mathcal{M}})), L_Q\big)$$

is the obstruction to existence of a connection on $\mathcal{M}$ compatible with the homological vector field Q.

A connection ∇ on a dg manifold $(\mathcal{M}, Q)$ is said to be *compatible* with the homological vector field if

$$L_Q(\nabla_X Y) = \nabla_{L_Q X} Y + (-1)^{|X|} \nabla_X (L_Q Y) \quad \text{for all } X, Y \in \mathfrak{X}(\mathcal{M}).$$

Example

Let $\mathfrak{g}$ *be a finite-dimensional Lie algebra*

- $(\mathcal{M}, Q) = (\mathfrak{g}[1], d_{\mathsf{CE}})$ *is the corresponding dg manifold*
- $T_{\mathcal{M}} \cong \mathfrak{g}[1] \times \mathfrak{g}[1]$ *implies*

$$H^1\big(\Gamma(S^2(T^\vee_{\mathcal{M}}) \otimes T_{\mathcal{M}}), Q\big) \cong H^0_{\mathsf{CE}}(\mathfrak{g}; \Lambda^2\mathfrak{g}^\vee \otimes \mathfrak{g}) \cong (\Lambda^2\mathfrak{g}^\vee \otimes \mathfrak{g})^{\mathfrak{g}}$$

- $\alpha_{\mathfrak{g}[1]} \leftrightarrow$ *the Lie bracket of* $\mathfrak{g}$

Example

Let X be a complex manifold.

- $(\mathcal{M}, Q) = (T_X^{0,1}[1], \bar{\partial})$ *is a corresponding dg manifold;*
- *There exists a "good" quasi-isomorphism*

$$(\Gamma(T_{\mathcal{M}}), L_Q) \xrightarrow{q.i.} (\Omega^{0,\bullet}(T_X^{1,0}), \bar{\partial})$$

- *There is an isomorphism*

$$H^1(\Gamma(\mathrm{Hom}(S^2(T_{\mathcal{M}}), T_{\mathcal{M}})), L_Q) \cong H^1(\Omega^{0,\bullet}(\mathrm{Hom}(S^2(T_X^{1,0}), T_X^{1,0}),), \bar{\partial}) \subset H^1(X, \Omega_X^1 \otimes \mathrm{End}(T_X))$$

- $\alpha_{T_X^{0,1}[1]} \leftrightarrow$ *the classical Atiyah class of X*

$T_F \leftrightarrow T_X^{0,1}, \quad \mathcal{N}_F \leftrightarrow T_X^{1,0}, \quad T_M \leftrightarrow T_X^{\mathbb{C}}$

Example

Let T_F be a regular integrable distribution of T_M and $\mathcal{N}_F := T_M/T_F$ be its normal bundle.

- $(\mathcal{M}, Q) = (T_F[1], d_{dR})$ *is a corresponding dg manifold;*
- *There exists a "good" quasi-isomorphism*

$$(\Gamma(T_{\mathcal{M}}), L_Q) \xrightarrow{q.i.} (\Omega_F^\bullet(\mathcal{N}_F), d_{dR})$$

- *There is an isomorphism*

$$H^1(\Gamma(\mathrm{Hom}(S^2(T_{\mathcal{M}}), T_{\mathcal{M}})), L_Q) \cong H^1(\Omega_F^\bullet(\mathrm{Hom}(S^2(\mathcal{N}_F), \mathcal{N}_F)), d_{dR})$$

- $\alpha_{T_F[1]} \leftrightarrow$ *the Atiyah-Molino class of the regular foliation F*

Exponential map on graded manifolds

$\mathcal{M}$: graded manifold

Theorem (Liao, Stiénon, 2015)

1. *The formal exponential map associated to an affine connection ∇ on $\mathcal{M}$ is the morphism of left $C^\infty(\mathcal{M})$-modules*

$$\mathrm{pbw}^\nabla : \Gamma(S(T_\mathcal{M})) \to \mathcal{D}(\mathcal{M})$$

inductively defined by an algebraic formula.

2. *Moreover,*

$$\mathrm{pbw}^\nabla : \Gamma(S(T_\mathcal{M})) \to \mathcal{D}(\mathcal{M})$$

is an isomorphism of graded coalgebras over $C^\infty(\mathcal{M})$.

$$\mathrm{pbw}^\nabla(f) = f, \qquad \mathrm{pbw}^\nabla(X) = X$$

$$\mathrm{pbw}^\nabla(X_0 \odot \cdots \odot X_n) = \frac{1}{n+1} \sum_{k=0}^{n} \pm \left\{ X_k \cdot \mathrm{pbw}^\nabla(X^{\{k\}}) - \mathrm{pbw}^\nabla\left(\nabla_{X_k}(X^{\{k\}})\right) \right\}$$

Exponential map on differential graded manifolds

Given a dg manifold $(\mathcal{M}, Q)$, there exists two differential graded coalgebras:

1. $(\Gamma(S(T_{\mathcal{M}})), L_Q)$
2. $(\mathcal{D}(\mathcal{M}), \mathcal{L}_Q := [Q, -])$

Question

When is

$$\mathrm{pbw}^{\nabla} : (\Gamma(S(T_{\mathcal{M}})), L_Q) \to (\mathcal{D}(\mathcal{M}), \mathcal{L}_Q)$$

an isomorphism of differential graded coalgebras?

– If pbw^{∇} is an isomorphism of differential graded coalgebras, then we can consider it as "a formal exponential map" of $(\mathcal{M}, Q)$.

Existence of formal exponential map of dg mfd

Theorem (S, Stiénon, Xu, 2022)

The Atiyah class $\alpha_{\mathcal{M}}$ vanishes if and only if there exists a torsion-free connection ∇ such that

$$\mathrm{pbw}^{\nabla} : \Gamma(S(T_{\mathcal{M}})) \to \mathcal{D}(\mathcal{M})$$

is an isomorphism of differential graded coalgebras over $C^{\infty}(\mathcal{M})$.

In general, the failure of pbw^{∇} to preserve dg structure is measured by

$$(\mathrm{pbw}^{\nabla})^{-1} \circ \mathcal{L}_Q \circ \mathrm{pbw}^{\nabla} - L_Q = \sum_{k=0}^{\infty} R_k$$

where $R_k \in \Gamma(\mathrm{Hom}(S^k(T_{\mathcal{M}}), T_{\mathcal{M}}))$ are sections of degree $+1$.

$R_0 = R_1 = 0, \quad R_2 = -\mathrm{At}^{\nabla}$

Kapranov $L_\infty[1]$ algebra on dg manifolds

Theorem (S, Stiénon, Xu, 2022)

The $R_k \in \Gamma(\operatorname{Hom}(S^k(T_\mathcal{M}), T_\mathcal{M}[1]))$ for $k \geq 2$ are completely determined by Atiyah cocycle At^∇, the curvature R^∇, and their exterior derivatives.

In particular, if the curvature vanishes (i.e. $R^\nabla = 0$), then

$$R_2 = -\operatorname{At}^\nabla, \quad R_{n+1} = \frac{1}{n+1} d^\nabla R_n \quad \text{for } n \geq 2$$

Why do we care about R_k?

Theorem (S, Stiénon, Xu, 2022)

The $R_k \in \operatorname{Hom}(S^k T_\mathcal{M}, T_\mathcal{M}[1])$ for $k = 2, 3, \ldots$, together with L_Q induce an $L_\infty[1]$ algebra on the space of vector fields $\mathfrak{X}(\mathcal{M})$.

Example

- $\mathfrak{g}$: *finite-dimensional Lie algebra*
- $(\mathcal{M}, Q) = (\mathfrak{g}[1], d_{\mathrm{CE}})$ *is a dg manifold*
- $\mathfrak{X}(\mathcal{M})[-1] = \Lambda\mathfrak{g}^\vee \otimes \mathfrak{g}$ *is an* L_∞ *algebra equipped with*

$$L_Q = d_{CE}^{\mathfrak{g}}, \quad R_2 = 1 \otimes [\ ,\ ]_{\mathfrak{g}}, \quad R_{\geqslant 3} = 0$$

– Chevalley-Eilenberg cohomology $H_{\mathrm{CE}}(\mathfrak{g}, \mathfrak{g})$ is a Lie algebra.

Here, $\mathfrak{g}$ action on $\mathfrak{g}$ is the adjoint action

Theorem (Kapranov, 1999)

Let X be a Kähler manifold. The Dolbeault complex $\Omega^{0,\bullet}(T_X^{1,0})$ admits an $L_\infty[1]$ algebra structure $(\lambda_k)_{k\geq 1}$ where λ_k is the wedge product

$$\Omega^{0,j_1}(T_X^{1,0}) \odot \cdots \odot \Omega^{0,j_k}(T_X^{1,0}) \to \Omega^{0,j_1+\cdots+j_k}(S^k(T_X^{1,0}))$$

composed with

$$R_k : \Omega^{0,\bullet}(S^k(T_X^{1,0})) \to \Omega^{0,\bullet+1}(T_X^{1,0})$$

where $\odot$ is graded symmetric tensor (w.r.t to $j_1, j_2, \cdots$) and

- $R_1 = \bar{\partial}$,
- $R_2 = \mathcal{R} = \mathcal{R}^{\nabla^{1,0}}$ *is an Atiyah cocycle,*
- $R_{n+1} = d^{\nabla^{1,0}}(R_n)$ *for* $n \geq 2$

Theorem (S. Stiénon, Xu, 2022)

- *X: Kähler manifold*
- $(\mathcal{M}, Q) = (T_X^{0,1}[1], \bar{\partial})$ *is a dg manifold*
- $\mathfrak{X}(\mathcal{M}) = \mathfrak{X}(T_X^{0,1}[1])$ *admits an $L_\infty[1]$ algebra structure*
- *There is an $L_\infty[1]$ quasi-isomorphism*

$$(\mathfrak{X}(T_X^{0,1}[1]), \{R_i\}) \xrightarrow{L_\infty[1]\ q.i.} (\Omega^{0,\bullet}(T_X^{1,0}), \{\lambda_i\})$$

Moreover, our $L_\infty[1]$ algebra structure on $\mathfrak{X}(T_X^{0,1}[1])$ can be transferred to Kapranov's $L_\infty[1]$ algebra structure on $\Omega^{0,\bullet}(T_X^{1,0})$.

Thank you!