A few approaches to the critical catenoid conjecture

I. <u>Review</u> (1985, Nitsche)

The only immersed capillary minimal disk in B^3 → equatorial disk

<u>Rmk</u> ① Hopf's method was applied

Hopf differential $\Phi(z)dz^2 = (L - iM)\,dz^2$: holomorphic differential

where $\mathrm{II} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$ and z is a complex coordinate on Σ $(D \xrightarrow{X} \Sigma)$

$\Phi(z)\, dz^2\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right) = z^2\Phi(z) =: \alpha + i\beta$

Johimsthal's thm : $X(\partial D)$ is line of curv. → $\beta = 0$ on ∂D → $\begin{cases} \beta = 0 \\ \alpha = 0 \end{cases}$

"2nd fundamental form gives a strong rigidity"

② Nitsche proposed

"Immersed free bdry minimal annulus in B^3 → catenoid?"

↳ Negative

Fraser-Li

"Immsersed → Embedded"

↳ unknown

③ Nitsche' proof applied to minimal annulus

∃ a conformal equivalence $X : A \to \Sigma$

$z^2\Phi(z) = \alpha + i\beta$ → $\beta = 0$, α : nonzero constant.

foliated by lines of curv. (no umbilic pt)

↳ $\frac{\alpha}{r^2}$: geodesic curv. of bdry curve.

II. Embeddedness condition

① Steklov eigenvalue method

Mcgrath (2017) – FBMA symmetric to three coordinate planes → critical catenoid

D. Seo (2021) – three planes → two planes (single plane if horizontal)

Note No other proof exists for now.

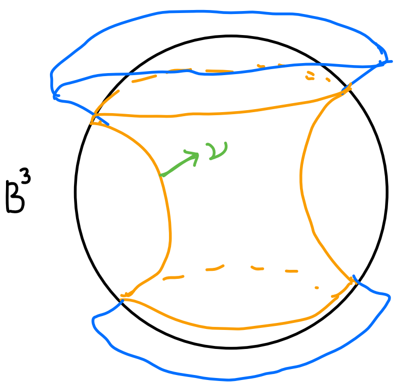
② Brendle's two point ftn

G. Huisken (1998) adopted a two-point ftn $W_t(x,y) = \dfrac{L(t)}{|F_t(x) - F_t(y)|} \sin\left(\dfrac{\pi d_t(x,y)}{L(t)}\right)$

B. Andrews (2002) further developed the theory

S. Brendle
$$\left[\begin{aligned} &\alpha := \sup_{\substack{x,y\in\Sigma,\\ x\neq y}} \frac{2\langle \nu(x), Y-X\rangle}{\Phi(x)|X-Y|^2} \qquad \left(\Phi(x) = \tfrac{1}{\sqrt{2}}|A|(x)\right) \\ &Z_\alpha(X,Y) := \frac{\alpha}{2}\Phi(x)|X-Y|^2 + \langle \nu(x), X-Y\rangle \end{aligned}\right]$$

applied Bony's maximum principle, proved Lawson's conjecture
↓
$X : \Sigma \to S^3$ embedded minimal torus → $F \simeq$ C.T.



$\Sigma \subset B^3$: free bdry minimal annulus

Σ를 B^3 밖으로 약간 확장하여 $\tilde{\Sigma}$ 라고하자

(Björling's problem), $\tilde{\Sigma}$의 내부로 향하는 unit normal ν

α, $Z_\alpha(X,Y)$ defined as above.

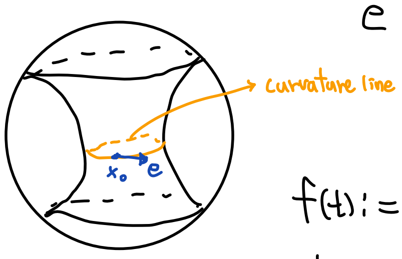
정의에 의해 $\tilde{\Sigma} \times \tilde{\Sigma}$ 에서 $Z_\alpha \geq 0$

Note $\Phi(x) = \dfrac{|A|}{\sqrt{2}}(x) = \dfrac{\alpha}{\lambda^2}$ $\quad (ds^2 = \lambda^2 |dz|^2)$

Critical catenoid : $\alpha = 1$

$\alpha \le 1$ 가정

$$\alpha \le 1 \Rightarrow Z_1(X,Y) = \frac{1}{2}\Phi(X)|X-Y|^2 + \langle \nu(X), X-Y\rangle \ge 0$$



e : principle direction s.t. $A(e,e) = \frac{1}{\sqrt{2}}|A|(x_0)$

$\gamma(t) \to \gamma(0) = x_0,\ \gamma'(0) = e$ 인 unit speed geodesic

$$f(t) := Z_1(x_0, \gamma(t))$$

$$f'(t) = \Phi(x_0)\langle x_0 - \gamma(t), -\dot\gamma(t)\rangle + \langle \nu(x_0), -\dot\gamma(t)\rangle$$

$$f''(t) = \Phi(x_0)\langle -\dot\gamma(t), -\dot\gamma(t)\rangle + \Phi(x_0)\langle x_0-\gamma(t), -\ddot\gamma(t)\rangle + \langle \nu(x_0), -\ddot\gamma(t)\rangle$$

$$= \Phi(x_0) + \Phi(x_0)\langle x_0-\gamma(t), -A(\dot\gamma(t),\dot\gamma(t))\nu(\gamma(t))\rangle - A(\dot\gamma(t),\dot\gamma(t))$$

$$f(0) = Z_1(x_0,x_0) = 0$$
$$f'(0) = \langle \nu(x_0), -e\rangle = 0$$
$$f''(0) = \Phi(x_0) - A(e,e) = \frac{1}{\sqrt{2}}|A|(x_0) - A(e,e) = 0$$

$$f(t) \ge 0,\ f(0) = f'(0) = f''(0) = 0 \Rightarrow f'''(0) = -D_e(A(e,e)) = 0$$

$\therefore$ $D_e|A|(x_0) = 0$ holds for all $x_0 \in \Sigma$

Σ 감는 모든 curv. line 에 대하여 principle curv. 상수

$\partial\Sigma$ 의 curv. 상수 $\to$ $\partial\Sigma$: circle $\to$ Σ is the critical catenoid

Rmk $\alpha > 1$: $Z_\alpha(x_0, y_0) = 0 \Rightarrow x_0 \in \partial\Sigma,\ y_0 \in \partial\Sigma$.

$$\Rightarrow \frac{\partial}{\partial r_{x_0}} Z_\alpha(X,Y)\Big|_{(x_0,y_0)} \overset{\text{strict}}{>} 0$$

III

Immersed condition

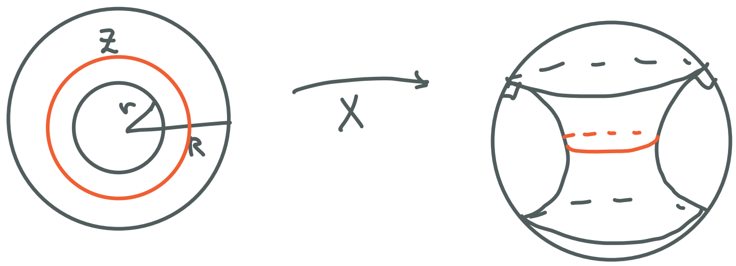
Recall Weierstrass data $(g, dh) = (g, g\omega) = (g, gf\,dz)$

g : meromorphic ftn, $g\omega$: holomorphic height differential

$\mathrm{Re}\int \left[\frac{1}{2}(1-g^2)\omega,\ \frac{i}{2}(1+g^2)\omega,\ g\omega \right]$ give rise to the minimal immersion.

$A(r,R) : \{ z \mid r<|z|<R \}$

$A(r,R) \xrightarrow[X]{} \Sigma$: emb. F.B.M.A. in B^3



One could try $\rightarrow$ $z^2 g' f = \alpha$ (constant)

$g = \cdots + C_{-2}\frac{1}{z^2} + C_{-1}\frac{1}{z} + C_0 + C_1 z + C_2 z^2 + \cdots$

$f = \cdots + d_{-2}\frac{1}{z^2} + \cdots + d_2 z^2 + \cdots$

We know

$\cdots + C_{-2}d_1 + C_{-1}d_0 + C_1 d_{-2} + \cdots = 0 \quad \cdots$

Note (J. Lee, E. Yoon)

$X : A(1,R) \to \mathbb{R}^3$ be a conformal harmonic immersion.

Lewy's result implies that X can be extended as an immersion

$X : A(1-\epsilon, R+\epsilon) \to \mathbb{R}^3$. Abuse of the notation.

Recall, $\sigma = \mathrm{Re}\left(\frac{C_0}{z^2} dz^2\right)$

$\left(\Phi(z)\,dz^2 \rightarrow \text{Hopf diff.},\quad z^2\Phi(z) = \overset{C_0}{\overset{\parallel}{\alpha}} + i\beta \right)$

$|\sigma|^2 = \frac{2C_0^2}{r^4\lambda^4} \quad (ds^2 = \lambda^2|dz|^2)$

Simons' identity $\Delta_\Sigma \log|\sigma|^2 = -2|\sigma|^2$

$$\rightarrow \frac{1}{\lambda^2}\Delta \log \frac{2C_0^2}{r^4\lambda^4} = -\frac{4C_0^2}{r^4\lambda^4} \qquad \left(\Delta = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar z}\right)$$

$v := \log \frac{1}{r^4\lambda^4}$.

As $\log r$ is a harmonic ftn, we get the Liouville type eq. in $A(1-\varepsilon, R+\varepsilon)$

$$\Delta v + 2C_0^2 e^{v} = 0.$$

Moreover, geodesic curvature being 1 along the bdry curves yields

$$\left[\begin{array}{l} -\frac{1}{r\lambda}\left(1+\frac{r}{\lambda}\frac{\partial \lambda}{\partial r}\right) = 1 \text{ if } r=1, \\ -\frac{1}{r\lambda}\left(1+\frac{r}{\lambda}\frac{\partial \lambda}{\partial r}\right) = -1 \text{ if } r=R. \end{array}\right.$$

$$\Rightarrow \left[\begin{array}{l} \frac{\partial v}{\partial n} = 2e^{-\frac{1}{2}v} - 2 \quad \text{if } r=1, \\ \frac{\partial v}{\partial n} = \frac{2}{R^2}e^{-\frac{1}{2}v} + \frac{2}{R} \quad \text{if } r=R. \end{array}\right.$$

$$\left[\begin{array}{l} \Delta v + 2C_0^2 e^{v} = 0. \\ \frac{\partial v}{\partial n} = 2e^{-\frac{1}{2}v} - 2 \quad \text{if } r=1, \\ \frac{\partial v}{\partial n} = \frac{2}{R^2}e^{-\frac{1}{2}v} + \frac{2}{R} \quad \text{if } r=R. \end{array}\right. \qquad E[R, \varepsilon, C_0]$$

$\uparrow$

Liouville type bdry value problem.

<u>Thm</u> $X : A(r,R) \to \Sigma$ be a free bdry minimal annulus in B^3 whose bdry curves $X(\partial A_1) =: \partial\Sigma_1$, $X(\partial A_2) =: \partial\Sigma_2$ are embedded, where $\partial A_1, \partial A_2$ denote $\{|z|=r\}$ & $\{|z|=R\}$ resp.

$g : A(r,R) \to \mathbb{C}$ of Σ is an one to one map that sends $A(r,R)$ to a doubly connected region bdd by $g(\partial A_1)$ & $g(\partial A_2)$.

Wente (1986) ; Construction of closed CMC tori in $\mathbb{R}^3$

Studied surfaces foliated by spherical lines of curvature

line of curv. on a surface lies on a sphere → constant contact angle