Weighted scalar curvature rigidity for weighted area-minimizing hypersurface

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Introduction

- First variation formula
  Let $F : \Sigma \times (-\epsilon, \epsilon) \to M$ be a variation of Σ with compact support.

$$\frac{d}{dt}_{t=0} Vol(F(\Sigma, t)) = \int_\Sigma \langle F_t, H \rangle dvol_\Sigma,$$

  where F_t is the variational vector field and H is the mean curvature.

Definition 1

A submanifold Σ is said to be a *minimal submanifold* if its mean curvature vanishes, $H = 0$. In other words, Σ is a critical point of the volume functional. Moreover, if Σ is $(n-1)$-dimensional, then Σ is called a *minimal hypersurface*.

- Second variation formula

$$\frac{d^2}{dt^2}_{t=0} Vol(F(\Sigma, t)) = \int_\Sigma |\nabla^\Sigma \varphi|^2 - \varphi^2 \left(Ric^M(N, N) + |B|^2 \right) dvol_\Sigma,$$

where N is the normal vector on Σ, B is the second fundamental form of Σ, and $\varphi \in C^\infty(\Sigma)$.

Definition 2

We say that Σ is *stable* if

$$\frac{d^2}{dt^2}_{t=0} Vol(F(\Sigma, t)) \geq 0.$$

3-manifolds

- (Schoen and Yau (1979)) Let M be a complete 3-dimensional Riemannian manifold with $R^M > 0$. If M contains a closed, two-sided, immersed, stable minimal hypersurface Σ, then the hypersurface must have genus 0.

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- (Fischer-Colbrie and Schoen (1980)) If $R^M \geq 0$, then the genus of Σ must be zero or one. Moreover, if the genus is one, then
  (i) Σ is totally geodesic,
  (ii) the normal Ric^M vanish all along Σ.

- (Shen and Zhu (1997)) If $R^M \geq R_0$, then the area of any closed, two-sided, stable minimal surface Σ with genus $\beta \neq 1$, satisfies

$$\begin{cases} A(\Sigma) \leq 4\pi & \textit{if} \quad R_0 = 2 \quad \textit{and} \quad \beta = 0 \\ A(\Sigma) \geq 4\pi(\beta - 1) & \textit{if} \quad R_0 = -2 \quad \textit{and} \quad \beta \geq 2. \end{cases}$$

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- (Bray, Brendle, Neves (2010)) If

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then
(i) Σ is totally geodesic,
(ii) the normal Ric^M vanishes along Σ.
Moreover, if Σ is area-minimizng and $A(\Sigma) = 4\pi$, then M is isometric to $\mathbb{S}^2 \times (-\epsilon, \epsilon)$ in a neighborhood of Σ.

Theorem 1 (Micallef and Moraru (2015))

Let M be a complete 3-dimensional Riemannian manifold with $R^M \geq R_0$. Assume that M contains a closed, two-sided, embedded, area-minimizing hypersurface Σ.

(1) Suppose that $R_0 = 2$ and $A(\Sigma) = 4\pi$. Then Σ has $\beta = 0$ and it has a neighborhood which is isometric to the product $g_1 + dt^2$ on $\mathbb{S}^2 \times (-\epsilon, \epsilon)$.

(2) Suppose that $R_0 = 0$ and Σ has $\beta = 1$. Then Σ has a neighborhood which is isometric to the product $g_0 + dt^2$ on $\mathbb{T}^2 \times (-\epsilon, \epsilon)$.

(3) Suppose that $R_0 = -2$ and that Σ has $\beta \geq 2$ and $A(\Sigma) = 4\pi(\beta - 1)$. Then Σ has a neighborhood which is isometric to the product $g_{-1} + dt^2$ on $\Sigma \times (-\epsilon, \epsilon)$.

Proof of Theorem 1. By second variation of area for minimal hypersurface, we have

$$0 \leq \int_{\Sigma} |\nabla^{\Sigma}\varphi|^2 - \varphi^2 \left(Ric^M(N,N) + |B|^2 \right) dvol_{\Sigma},$$

where ∇^{Σ} and $dvol_{\Sigma}$ are the gradient and the area element of Σ.

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$$\begin{aligned} 0 &\leq \int_{\Sigma} -\frac{R^M}{2} + K^{\Sigma} - \frac{|B|^2}{2} dvol_{\Sigma} \\ &\leq -\frac{R_0}{2} A(\Sigma) + \int_{\Sigma} K^{\Sigma} dvol_{\Sigma}. \end{aligned}$$

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By Gauss-Bonnet theorem, we get

$$R_0 A(\Sigma) \leq 4\pi\chi(\Sigma) = 8\pi(1-\beta),$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ.

If $R_0 A(\Sigma) = 8\pi(1-\beta)$, then every inequality above is in fact an equality. So we have $Ric(N,N) = 0$ and $B = 0$ on Σ.

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Then we can use Jacobi equation $L = \Delta_\Sigma + Ric^M + |B|^2$ to construct a constant mean curvature foliation in a neighborhood of Σ.
(i) $R_0 = 2$ and $A(\Sigma) = 4\pi$,
(ii) $R_0 = 0$ and $\beta = 1$,
(iii) $R_0 = -2$, $\beta \geq 2$ and $A(\Sigma) = 4\pi(\beta - 1)$.
Using the Jacobi equation,We can show that a constant mean curvature foliation is an area-minimizing surface in all cases.

n-manifolds

- Einstein-Hilbert functional

$$Y(g) := \frac{\int_M R^M dvol}{Vol(M)^{(n-2)/n}}.$$

- Writting $\tilde{g} = u^{\frac{4}{n-2}} g$ for a positive function u on M, then

$$Y_g(u) = \frac{\int_M \left(\frac{4(n-1)}{n-2} |\nabla u|^2 + R^M u^2 \right) dvol}{\left(\int_M u^{2n/(n-2)} dvol \right)^{(n-2)/n}}$$

- Yamabe invariant

$$Q_g(M) := \inf_{u>0} Y_g(u)$$

- σ-constant (or Yamabe constant)

$$\sigma(M) := \sup_{[g] \in \mathcal{C}} Q_g(M),$$

where $\mathcal{C}$ is the space of conformal classes on M.

Remark 1

When $n = 3$ and $R_0 = -2$, then $\sigma(\Sigma) = 4\pi\chi(\Sigma) = 8\pi(1 - \beta)$, where $\chi(\Sigma)$ and β are the Euler characteristic and genus of Σ. i.e., in some sense, the σ-constant can be view as a generalisation of the Euler characteristic to higher dimensions.

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Theorem 2 (Moraru (2016))

Let M be an n-dimensional Riemannian manifold ($n \geq 4$) with $R^M \geq R_0$. Assume that Σ be a closed, two-sided, embedded, area-minimizing hypersurface.

(1) If $R_0 < 0$ and $\sigma(\Sigma) < 0$, then $R_0 A(\Sigma)^{\frac{2}{n-1}} \leq \sigma(\Sigma)$. Moreover, if equality holds, then M is isometric to $\Sigma \times (-\epsilon, \epsilon)$ in a neighborhood of Σ.

(2) If $R_0 = 0$ and $\sigma(\Sigma) \leq 0$, then $\sigma(\Sigma) = 0$ and M is isometric to $\Sigma \times (-\epsilon, \epsilon)$ in a neighborhood of Σ.

Proof of Theorem 2. By second variation of area for minimal hypersurface, we have

$$0 \leq \int_\Sigma |\nabla^\Sigma \varphi|^2 - \varphi^2 \left(Ric^M(N,N) + |B|^2 \right) dvol_\Sigma,$$

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where ∇^Σ and $dvol_\Sigma$ are the gradient and the area element of Σ. From the Gauss equation, we obtain

$$\begin{aligned} 0 &\leq \int_\Sigma 2|\nabla^\Sigma \varphi|^2 + (R^\Sigma - R^M - |B|^2)\varphi^2 dvol_\Sigma \\ &\leq \int_\Sigma \left(\frac{4(n-2)}{n-3} |\nabla^\Sigma \varphi|^2 + R^\Sigma \varphi^2 \right) dvol_\Sigma - \int_\Sigma R^M \varphi^2 dvol_\Sigma, \end{aligned}$$

where in the last inequality we have used that $2 < \frac{4(n-2)}{n-3}$ for all $n \geq 4$.

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where in the last inequality we have used that $2 < \frac{4(n-2)}{n-3}$ for all $n \geq 4$. By the assumption $R^M \geq R_0$ and hence, by Holder inequality, the above inequality gives

$$\begin{aligned} 0 \;\; &\leq \;\; \int_\Sigma \left(\frac{4(n-2)}{n-3} |\nabla^\Sigma \varphi|^2 + R^\Sigma \varphi^2 \right) dvol_\Sigma \\ &\quad - R_0 A(\Sigma)^{\frac{2}{n-1}} \left(\int_\Sigma \varphi^{\frac{2(n-1)}{n-3}} dvol_\Sigma \right)^{\frac{n-3}{n-1}} . \end{aligned}$$

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Dividing the last inequality by $\left(\int_\Sigma \varphi^{\frac{2(n-1)}{n-3}} dvol_\Sigma \right)^{\frac{n-3}{n-1}} > 0$, we get

$$R_0 A(\Sigma)^{\frac{2}{n-1}} \leq \frac{\int_\Sigma \left(\frac{4(n-2)}{n-3} |\nabla^\Sigma \varphi|^2 + R^\Sigma \varphi^2 \right) dvol_\Sigma}{\left(\int_\Sigma \varphi^{\frac{2(n-1)}{n-3}} dvol_\Sigma \right)^{\frac{n-3}{n-1}}}.$$

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Therefore,

$$R_0 A(\Sigma)^{\frac{2}{n-1}} \leq \sigma(\Sigma).$$

Remark 2

Let $\Sigma := \mathbb{S}^{n-2} \times \mathbb{S}^1(\ell)$, where $\mathbb{S}^{n-2}$ is the $(n-2)$-dimensional unit sphere and $S^1(\ell)$ is the circle of radius ℓ. Let $M := \Sigma \times \mathbb{S}^1$ with the product metric. Then $R^M = (n-2)(n-3) := R_0^+ > 0$ and $\sigma(\Sigma) = \sigma(\mathbb{S}^{n-1})$. That is, $\sigma(\Sigma)$ is independent of both R_0^+ and ℓ. Therefore, the area of Σ is arbitrarily large when ℓ increases.

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- (Mendes (2019)) Let M^5 be a Riemannian manifold with $R^M \geq R_0 > 0$ and $Ric^M \geq 0$. If Σ^4 is a two-sided, closed, embedded, area-minimizing hypersurface, then

$$A(\Sigma)\left(\frac{R_0}{12}\right)^2 \leq A(\mathbb{S}^4) + \frac{1}{12}\int_\Sigma |\mathring{Ric}|^2 dvol_\Sigma,$$

where $\mathring{Ric}$ is the traceless Ricci curvature. If equality holds, then M is isometric to $\mathbb{S}^4 \times (-\epsilon, \epsilon)$ in a neighborhood of Σ.

- Gauss-Bonnet-Chern formula: When $n = 5$

$$8\pi\chi(\Sigma) = \int_\Sigma \left(\frac{1}{4}|W^\Sigma|^2 + \frac{1}{24}|R^\Sigma|^2 - \frac{1}{2}|\mathring{Ric}^\Sigma|^2 \right) dvol_\Sigma,$$

where W^Σ and $\mathring{Ric}$ are the Weyl and the traceless Ricci tensor of Σ.

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- (Deng (2021)) Let M be an n-dimensional Riemannian manifold ($n \geq 4$) with $R^M \geq R_0 > 0$ and $Ric^M \geq 0$. If Σ^{n-1} is a two-sided, closed, embedded, area-minimizing hypersurface with $Ric^\Sigma = \frac{R^\Sigma}{n} g_\Sigma$, then

$$A(\Sigma)^{\frac{2}{n}} R_0 \leq n(n-1)A(\mathbb{S}^{n-1})^{\frac{2}{n}}.$$

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- The author assumed that the Σ is Einstein.

Weighted manifolds

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- Weighted scalar curvature

$$\begin{cases} R_f^m := R^M + 2\Delta f - \frac{m+1}{m} |\nabla f|^2, \\ R_f := R^M + 2\Delta f - |\nabla f|^2. \end{cases}$$

- Weighted mean curvature

$$H_f := H - \langle \nabla f, N \rangle.$$

- Weighted Laplacian

$$\Delta_f := \Delta - \langle \nabla f, \nabla \rangle.$$

- Weighted area

$$A_f(\Sigma) := \int_\Sigma e^{-f} dvol.$$

- First variation for weighted area

$$\frac{d}{ds}\Big|_{s=0} A_f(\Sigma_s) = \int_\Sigma \varphi H_f \, e^{-f} dvol.$$

- Second variation for weighted area

$$\frac{d^2}{ds^2}\Big|_{s=0} A_f(\Sigma_s) = \int_\Sigma |\nabla^\Sigma \varphi|^2 - \varphi^2 \left(Ric_f(N,N) + |B|^2 \right) e^{-f}\, dvol$$

- Weighted stable minimal hypersurface (or f-stable minimal hypersurface)

$$\frac{d^2}{ds^2}\Big|_{s=0} A_f(\Sigma_s) \geq 0.$$

Theorem 3 (M. Fan (2008))

Let M be a complete 3-dimensional weighted Riemannian manifold with $R_f \geq C_0$ for some positive constant C_0. If M contains closed, two-sided, immersed, f-stable minimal hypersurfaces Σ, then the genus of Σ is zero.

Theorem 4 (Castro and Rosales (2014))

Let M be a complete 3-dimensional weighted Riemannian manifold with $R_f \geq C_0 e^{-f}$. Consider a closed, two-sided, embedded, f-area minimizing hypersurface Σ.

(1) If $C_0 < 0$ and $A_f(\Sigma) = \frac{8\pi(1-\beta)}{C_0}$ for $\beta \geq 2$, then there is a neighborhood of Σ in M which is isometric to a Riemannian product $\Sigma \times (-\epsilon, \epsilon)$.

(2) If $C_0 = 0$ and $\beta = 1$, then Σ is a flat torus and there is a neighborhood of Σ in M which is isometric to a Riemannian product $\Sigma \times (-\epsilon, \epsilon)$.

(3) If $C_0 > 0$ and $A_f(\Sigma) = \frac{8\pi}{C_0}$, then $\beta = 0$ and there is a neighborhood of Σ in M isometric to a Riemannian product $\Sigma \times (-\epsilon, \epsilon)$.

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Remark 3

If we replace R_f to R_f^m, then we can get similar results.

Theorem 5 (Lee, Park, and Pyo (2022))

Let M be a complete n-dimensional weighted Riemannian manifold ($n \geq 4$) with $R_f^m \geq C_0 e^{-\frac{2f}{n-1}}$ for $m \in (0, \frac{1}{n-3}]$. Assume that M contains an $(n-1)$-dimensional, closed, two-sided, embedded, f-area minimizing hypersurface Σ.

(1) If $C_0 < 0$ and $\sigma(\Sigma) < 0$, then we have

$$C_0 A_f(\Sigma)^{\frac{2}{n-1}} \leq \sigma(\Sigma).$$

Moreover, if equality holds, then M splits isometrically as a product in a neighborhood of Σ.

(2) If $C_0 = 0$ and $\sigma(\Sigma) \leq 0$, then $\sigma(\Sigma) = 0$ and M splits isometrically as a product in a neighborhood of Σ.

Theorem 6 (Lee, Park, and Pyo (2022))

Let M be a complete n-dimensional weighted Riemannian manifold $(n \geq 4)$. If $R_f^m \geq C_0^+$ for some positive constant C_0^+, then there is no $(n-1)$-dimensional, closed, two-sided, immersed, f-stable hypersurface Σ with $\sigma(\Sigma) \leq 0$.

Theorem 6 (Lee, Park, and Pyo (2022))

Let M be a complete n-dimensional weighted Riemannian manifold ($n \geq 4$). If $R_f^m \geq C_0^+$ for some positive constant C_0^+, then there is no $(n-1)$-dimensional, closed, two-sided, immersed, f-stable hypersurface Σ with $\sigma(\Sigma) \leq 0$.

Remark 4

If we change R_f^m to R_f, then we can get same result.

Thank you!

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