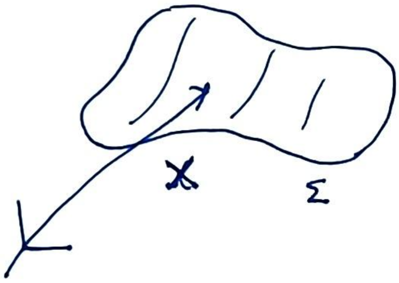
자유경계 극소곡면 (free boundary minimal surfaces) 소개 2023-1

Introduction to free boundary minimal surfaces (FBMS) in B^3

$$\Sigma \subset B^3 \ \text{FBMS} \iff \begin{cases} \text{(i)} \ \mathring{\Sigma} \text{ is a minimal surface} \\ \text{(ii)} \ \Sigma \perp S^2 \text{ along } \partial\Sigma \end{cases}$$

<u>Def</u> (minimal surface in $\mathbb{R}^3$)



2nd fundamental form $A : T\Sigma^2 \to N\Sigma$

$$A(e_i, e_j) = (\overline{\nabla}_{e_i} e_j)^{\perp} = h_{ij} N$$

mean curvature vector of Σ

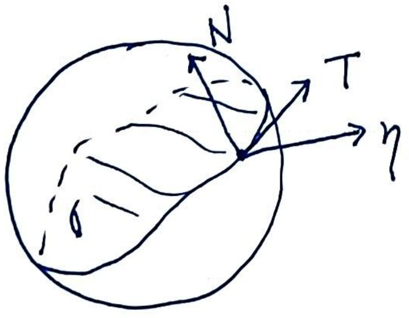
$$\vec{H} = \mathrm{tr}_g A = \sum_{i=1}^{2} A(e_i, e_i) = \sum_{i,j=1}^{2} g^{ij} A(\partial_i, \partial_j)$$

$\vec{H} = HN$, Σ is minimal $\iff$ $\vec{H} = \vec{0}$ or $H = 0$.

<u>Note 1</u> position vector $X = (x, y, z) = (x_1, x_2, x_3)$

$$\Delta_\Sigma X = (\Delta_\Sigma x, \Delta_\Sigma y, \Delta_\Sigma z) = \vec{H}$$

<u>Note 2</u> FBMS in B^3



conormal vector field η of Σ

$\{N, T\}$ ONF of S^2

η is a unit normal v.f. of S^2

$\eta = X$ for FBMS in B^3

<u>Note 3</u>

$S^3 \supset \Sigma$ minimal surface $\qquad 0 \times \Sigma$: Free boundary minimal 3-mfd in B^4

$$\boxed{\text{minimal surface in } S^3} \iff \boxed{\text{FBMS in } B^3}$$

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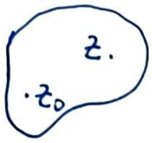
- <u>variational charcterization</u>

I minimal surfaces

① related to complex analysis

Ω : Riemann surface (simple case $\Omega \subset \mathbb{C}$ domain)

g : meromorphic fn on Ω, f: holomorphic fn on Ω



$$\mathrm{Re} \int_{z_0}^{z} \left(\frac{1}{2} f(1-g^2),\ \frac{i}{2} f(1+g^2),\ fg \right) \quad : \text{minimal surface}$$

· order of poles of g = 2 [order of zeros of f]

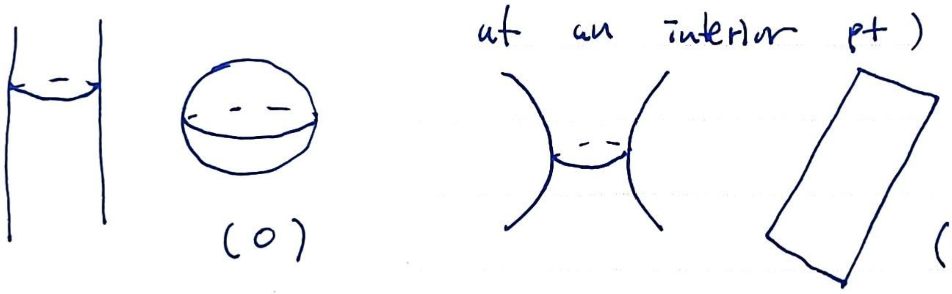
$\Omega = \mathbb{C}$, $g(z) = z$, $f(z) = 1$: Enneper's surface

② related to PDE

locally $\Sigma = \mathrm{Graph}(u)$ $\quad u : T_p\Sigma \to \mathbb{R}$

$$H = \mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}} \right) = 0$$ (u satisfies a qusilinear 2nd order elliptic PDE)

$\Rightarrow$ maximum principle (Any two minimal surfaces can't touch each other at an interior pt)

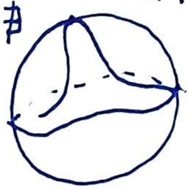


(o) (x)

Recall $\Delta_\Sigma X = \vec{H}$

① $\Delta_\Sigma |x|^2 = \sum_{i=1}^{3} \left(2x_i \Delta x_i + 2|\nabla^\Sigma x_i|^2 \right) = 4 > 0$ subharmonic (No interior maximum pt)

$\nexists$



② $4|\Sigma| = \int_\Sigma \Delta_\Sigma |x|^2 = \int_{\partial\Sigma} \langle \overbrace{\nabla^\Sigma |x|^2}^{2X}, \overbrace{\eta}^{X} \rangle = 2|\partial\Sigma| \quad (2|\Sigma| = |\partial\Sigma|)$

③ $\int_\Sigma \Delta_\Sigma X = 0 = \int_{\partial\Sigma} X \quad \left(\int_{\partial\Sigma} x = \int_{\partial\Sigma} y = \int_{\partial\Sigma} z = 0 \right)$

(Lima-Menezes) Every plane passing through the origin divides an embedded FBMS in two connected surfaces.

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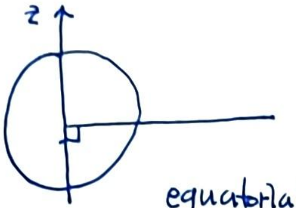
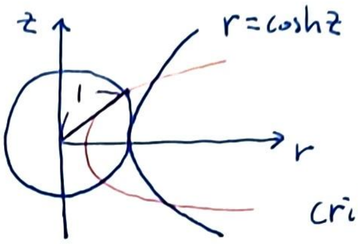
II Existence (examples)

① rotationally symmetric case



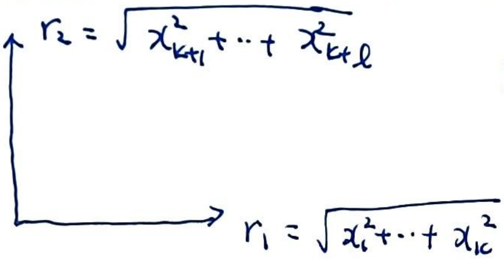
equatorial disk



critical catenoid

② $O(k) \times O(\ell)$ invariant surface (Fredin - Gulian - McGrath 2017)

$\mathbb{R}^{k+\ell}$



Steklov eigenvalue (Σ is a surface with $\partial\Sigma$)

$\varphi \in C^\infty(\partial\Sigma)$, $\hat{\varphi}$ is a harmonic extension of φ if $\begin{cases} \Delta\hat{\varphi} = 0 & \text{in } \Sigma \\ \hat{\varphi} = \varphi & \text{on } \partial\Sigma \end{cases}$

$\frac{\partial\hat{\varphi}}{\partial\eta} = \lambda\varphi$ on $\partial\Sigma$ $\Leftrightarrow$ λ : steklov eigenvalue, φ : steklov eigenfn.

Σ : FBMS in B^3 $\quad X = (x_1, x_2, x_3)$ position vector

$\Delta x_i = 0$ & $\frac{\partial x_i}{\partial\eta} = 1 \cdot x_i$ i.e. 1 : steklov eigenvalue x_i : steklov fns.

③ (Fraser - Schoen 2016, 2011)

Σ : compact with ∂. g_0 is a smooth metric on Σ s.t.

$$\sigma_1(g_0)\,|\partial\Sigma|_{g_0} = \max_g \sigma_1(g)\,|\partial\Sigma|_g$$

Then, $\exists\, u_1, u_2, u_3$ are the first steklov eigenfns s.t.

$(u_1, u_2, u_3) : \Sigma \longrightarrow \mathbb{B}^3$ minimal immersion

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④ gluing techniques

- desingularizing (Kapouleas - Martin 2021) $\left\{\begin{array}{l}\text{critical catenoid} \cup \text{equatorial disk} \\ \text{Scherk's surface} \quad \cos x_3 = \sinh x_1 \sinh x_2\end{array}\right.$

- doubling (Folha-Pacard-Zolotareva. 2017)
  - two equatorial disks $\cup$ half catenoid
  - " $\cup$ " $\cup$ one catenoid

- tripling (Kapouleas-Wiygul 2017 arXiv)

⑤ min-max method (Martin 2015, Carlotto - Franz - Schulz 2020)

FBMS with connected boundary and arbitrary genus.

etc ...

Ⅲ Uniqueness

Nitsche (1985)

Any free boundary minimal disk in B^3 is an equatorial disk.

(proof) $X : \underset{(u,v)}{\underset{\Psi}{D}} \longrightarrow \mathbb{B}^3$

$z = u + iv$

$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right) \qquad ds^2 = \lambda^2(du^2 + dv^2)$

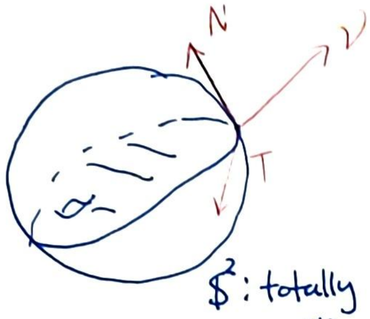
$$\Phi(z) := h\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) = \frac{1}{4}\left\{h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) - h\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) - 2i\,h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)\right\}$$

$$\Phi_{\bar z} = \frac{1}{8}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)\left\{h_{11} - h_{22} - 2i h_{12}\right)$$

$$8\Phi_{\bar z} = \left(h_{11,1} - h_{22,1} + 2h_{12,2}\right) + i\left(h_{11,2} - h_{22,2} - 2h_{12,1}\right)$$

$$\underset{\text{Codazzi}}{\overset{\uparrow}{=}} \left(h_{11,1} + h_{22,1}\right) - i\left(h_{11,2} + h_{22,2}\right)$$

$h_{11} + h_{22} = 0 \iff \Phi$: holomorphic



S^2: totally umbilic

$0 = \langle N, \nu \rangle$

KT
$\parallel$ K B^3 umbilic

$$0 = T\langle N, \nu\rangle = \langle \overbrace{\bar\nabla_T N}^{\alpha T + \beta \nu}, \nu \rangle + \langle N, \bar\nabla_T \nu \rangle$$

$\beta = 0 \Rightarrow \partial\Sigma$ is a line of curvature of Σ.

i.e. $h_{12}(z) = 0$ on $\partial\Sigma \overset{\text{harmonicity}}{\Rightarrow} h_{12}(z) \equiv 0$ on Σ.

$H = 0$ $(h_{11} + h_{22} = 0)$ $\quad h_{11}(z) = \text{Const} = C$ on Σ.

$C = 0 \Rightarrow \Sigma$: plane

$C \neq 0 \Rightarrow \nexists$ umbilic pt.

By Poincaré-Hopf index thm $\sum_i \mathrm{ind}(p_i) = \chi(D)$ $\cancel{\#}$.

IV Area estimates (Fraser-Schoen 2016, Brendle 2012)

Σ: FBMS in B^3, $|\Sigma| \geq \pi$ $\quad (|\Sigma^k| \geq |D^k|$ in general$)$

(pf) $W(x) = \frac{1}{2}x - \frac{x-y}{|x-y|^2}$ for a fixed pt $y \in \partial B^3$.

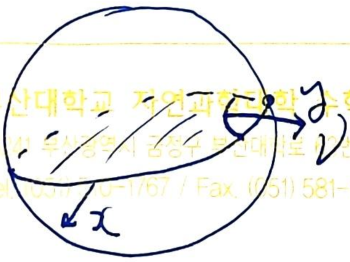
A $\operatorname{div}_\Sigma W = \sum_{i=1}^{2} \langle \nabla_{e_i} W, e_i \rangle \leq 1$

B $\langle W(x), x \rangle = 0$ i.e. W is tangential to S^2.

C $W(x) = -\frac{x-y}{|x-y|^2} + o\left(\frac{1}{|x-y|}\right)$ as $x \to y$

Let's fix a pt $y \in \partial\Sigma$,

$$0 \underset{(A)}{\leq} \int_{\Sigma \setminus B_\varepsilon(y)} (1 - \operatorname{div}_\Sigma W) = |\Sigma \setminus B_\varepsilon(y)| - \int_{\Sigma \cap \partial B_\varepsilon(y)} \langle W, \nu \rangle - \int_{\partial\Sigma \setminus B_\varepsilon(y)} \langle W, x \rangle$$



$\nu = -\frac{x-y}{|x-y|} + o(1)$

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since Ⓑ, $\int_{\partial \Sigma \setminus B_\varepsilon(y)} \langle W, x \rangle = 0$

since Ⓒ, $\langle W, \nu \rangle = \frac{1}{\varepsilon} + o\left(\frac{1}{\varepsilon}\right)$

since $|\Sigma \cap \partial B_\varepsilon(y)| \approx \pi\varepsilon$, $\lim_{\varepsilon \to 0} \int_{\Sigma \cap \partial B_\varepsilon(y)} \langle W, \nu \rangle = \pi$

Hence, $|\Sigma| \geq \pi$.

"=" holds $\Leftrightarrow$ Σ : equatorial disk.

(Mcgrath - Zou)

$\Sigma \subset B^3$ FBMS with genus 0 and at least 2 - boundary components.

$2\pi < |\Sigma| < 4\pi$

This is sharp asymptotically.

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