Chapter 13

Quantum Fields in Brief

If we begin to turn our attention to the matter part of the universe, these simpler types of fields need to be investigated. In this introductory chapter, we will start with classical field theories, understand dynamics of free fields, explore the symmetry principles and the consequence thereof, and finally justify the Wick rotation to the Euclidean time on which much of the following study in the volume depend on.

We have explored a little bit of quantum field theory in the last chapter of the first volume, just barely enough to talk about the Hawking effect. It is in fact quite amazing that such an important quantum phenomenon can be described with so little of quantum field theory. In this second volume, we will explore many more aspect of quantum field theories, with a view toward how quantum fields interact with curvatures, both gauge and space-time.

Although we have introduced and used the action principle repeatedly, we have not elaborated why such a principle should exist for classical dynamics beyond the mathematical fact that it actually works. A better way to motivate it physically would involve the quantum physics. If we elevate the classical theory to quantum level, the action is supposed to enter the path integral as,

$$\int [D\phi]\, e^{\mathrm{i}S(\phi)/\hbar} \,. \tag{13.0.1}$$

A classical solution with a definite ϕ etc becomes feasible out of such summation of quantum waves only if the nearby paths interfere constructively. The extremization of S, which implies a stationary phase condition, is a necessary condition for such constructive interference. In this rough sense, the action principle of classical physics had signaled the underlying quantum nature of the dynamics all along. In this preliminary chapter on quantum physics, we will make a lightening review of how one might handle such functional integrations.

13.1 Path Integral for Quantum Mechanics

Before we get to quantum field theory, it is instructive to browse how the Schrödinger quantum mechanics can be replaced by a path integral. Once this step becomes familiar, elevating them to field theory should be a quantitative extension, at least at a naive level. For this we will start with a standard mechanics Lagrangian, with a single degrees of freedom.

$$L = \frac{1}{2}m\dot{q}^2 - V(q) \ . \tag{13.1.1}$$

With the conjugate momentum,

$$p \equiv \frac{\delta L}{\delta \dot{q}} \ , \tag{13.1.2}$$

the Hamiltonian is

$$H(\pi, q) = \frac{p^2}{2m} + V(q) \ , \tag{13.1.3}$$

for the canonical version of the dynamics.

The Schrödinger quantum mechanics starts from this by replacing the Poisson bracket by a commutator,

$$\{q, p\}_{\text{P.B.}} = 1 \qquad \rightarrow \qquad [\mathbf{q}, \mathbf{p}] = \mathbb{i}\hbar \ . \tag{13.1.4}$$

elevating q and p to operators $\mathbf{q}$ and $\mathbf{p}$, with the standard representations, either

$$\mathbf{q} = q \ , \qquad \mathbf{p} = -\mathbb{i}\hbar\frac{\partial}{\partial q} \tag{13.1.5}$$

on the so-called coordinate basis, $|q\rangle$, or

$$\mathbf{q} = \mathbb{i}\hbar\frac{\partial}{\partial p} \ , \qquad \mathbf{p} = p \tag{13.1.6}$$

on the momentum basis counterpart, $|p\rangle$. The two sets of basis have the natural pairing $\langle q|p\rangle \sim e^{\mathbb{i}pq/\hbar}$. With q living on a real line, in particular, we have the canonical normalization,

$$\langle q'|q\rangle = \delta(q'-q) \ \& \ \langle p'|p\rangle = \delta(p'-p) \quad \rightarrow \quad \langle q|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{\mathbb{i}pq/\hbar} \ , \tag{13.1.7}$$

as can be inferred from the standard Fourier analysis.

The Hamiltonian is thus elevated to an operator,

$$\mathbf{H} = H(\mathbf{p},\mathbf{q}) \ , \tag{13.1.8}$$

with some carefully chosen ordering of the operators, as needed. The Schrödinger equation dictates the evolution of the wave function as

$$\mathbb{i}\hbar\frac{\partial}{\partial t}|\psi\rangle = \mathbf{H}|\psi\rangle \ , \tag{13.1.9}$$

or equivalently,

$$|\psi(T+t_0)\rangle = e^{-\mathbb{i}\mathbf{H}T/\hbar}|\psi\rangle_{t_0} \tag{13.1.10}$$

in the integrated form. The right hand side is in the so-called Heisenberg picture where the time-dependence is entirely encoded in the evolution operator.

The transition to the path integral starts from dividing the time interval T to N small ones, each of span T/N, and eventually taking $N \to \infty$, we may write

$$\langle q|\psi(T+t_0)\rangle = \langle q|\underbrace{e^{-\mathbb{i}\mathbf{H}T/N\hbar} \quad \cdots \quad e^{-\mathbb{i}\mathbf{H}T/N\hbar}}_{N\text{ times}}|\psi(t_0)\rangle \ , \tag{13.1.11}$$

for arbitrary natural number $N > 0$. Inserting a pair of identity operators, say,

$$1 = \int dp_k|p_k\rangle\langle p_k| \int dq_k|q_k\rangle\langle q_k| = \int \frac{dp_k\,dq_k}{\sqrt{2\pi\hbar}}\ e^{-\mathbb{i}p_kq_k/\hbar}|p_k\rangle\langle q_k| \tag{13.1.12}$$

between each adjacent pairs as well as in front of $|\psi(t_0)\rangle$, for $k = 1, \ldots, N$. We will also refer to q of $\langle q|$ above as q_{N+1} for a notational convenience.

This gives,

$$\langle q = q_{N+1}|\psi(T+t_0)\rangle$$
$$= \int \prod_{k=1}^{N} \frac{dp_k\, dq_k}{\sqrt{2\pi\hbar}} \left[\prod_{k=1}^{N} \langle q_{k+1}|e^{-\mathbb{i}\mathbf{H}T/N\hbar}|p_k\rangle e^{-\mathbb{i}p_k q_k/\hbar}\right] \langle q_1|\psi(t_0)\rangle \ . \quad (13.1.13)$$

As we take N very large the evolution operator for the short time span of T/N may be approximated as

$$e^{-\mathbb{i}\mathbf{H}T/N\hbar} \simeq e^{-\mathbb{i}V(\mathbf{q})T/N\hbar}\, e^{-\mathbb{i}(\mathbf{p}^2/2m)T/N\hbar} \quad (13.1.14)$$

to the leading order in T/N, resulting in

$$\langle q_{k+1}|e^{-\mathbb{i}\mathbf{H}T/N\hbar}|p_k\rangle e^{-\mathbb{i}p_k q_k/\hbar} \simeq \frac{1}{\sqrt{2\pi\hbar}} e^{\mathbb{i}p_k(q_{k+1}-q_k)/\hbar} e^{-\mathbb{i}H(p_k,q_{k+1})T/N\hbar} \ . \quad (13.1.15)$$

This statement becomes exact as $N \to \infty$ so

$$\lim_{N\to\infty} \int \prod_{k=1}^{N} \frac{dp_k\, dq_k}{\sqrt{2\pi\hbar}} \left[\prod_{k=1}^{N} \langle q_{k+1}|e^{-\mathbb{i}\mathbf{H}T/N\hbar}|p_k\rangle e^{-\mathbb{i}p_k q_k/\hbar}\right]$$
$$= \lim_{N\to\infty} \int \prod_{k=1}^{N} \frac{dp_k\, dq_k}{2\pi\hbar} e^{\mathbb{i}\sum_{k=1}^{N}[p_k(q_{k+1}-q_k)N/T - H(p_k,q_{k+1})]\times T/N\hbar} \ . \quad (13.1.16)$$

The final step is to take the continuum limit by replacing q_k's to $q(t)$, upon which we also replace $(q_{k+1}-q_k)N/T$ by $\dot{q}$. Then, the summation over k becomes an integration over dt, written schematically as

$$\to \quad \int [DpDq]\ e^{(\mathbb{i}/\hbar)\int dt\ (p\,\dot{q} - H(p,q))} \quad (13.1.17)$$

where $[DpDq]$ is a shorthand notation that covers all the subtle factors and limits of this functional integration. Note how the integral in the exponent is nothing but the action S

$$S(q) = \int dt\ L(q) \quad \to \quad S(p,q) = \int dt\ (p\,\dot{q} - H(p,q)) \ , \quad (13.1.18)$$

expressed in the canonical variables. To go back to the configuration space version,

$S(q)$, we merely need to extremize with respect to p.

Although one can stop here, we also have the option of integrating over the momentum. With our standard quadratic $p^2/2m$, this will at most generate a numerical factor while converting the exponent to the action in the configuration variables only,

$$\rightarrow \qquad \int [Dq]\; e^{(\mathbb{i}/\hbar)\int dt\, L(q)} \; . \tag{13.1.19}$$

If we did this before taking the infinite N limit, the integral would have become

$$\int \frac{dp_k\, dq_k}{2\pi\hbar} \quad \Rightarrow \quad \int \frac{dq_k}{\sqrt{2\pi\hbar/m}} \; , \tag{13.1.20}$$

modulo possible phase factors. At the end of the day, time evolution of the most general wavefunctions can be constructed out of the following pairing

$$\langle q; t | q_0; t_0 \rangle = \langle q | e^{-\mathbb{i}\mathbf{H}(t-t_0)/\hbar} | q_0 \rangle = \int_{q(t_0)=q_0}^{q(t)=q} [Dq]\; e^{\mathbb{i}S(q)/\hbar} \tag{13.1.21}$$

between position eigenstates at two distinct times.

Among those trajectories that contribute to the path integral are the classical ones, which are locally dominant saddle configurations that extremize the phase $S/\hbar$. The latter condition, nothing but the action principle, implies constructive interference and thus pile-up of the amplitude along such extremal paths. This is not enough to justify cleanly the transition to the classical physics, yet it does indicate clearly that the action principle has a quantum origin.

In particular, one can see that the amplitudes for such extremal paths are proportional to

$$e^{\mathbb{i}S_{\rm HJ}/\hbar} \tag{13.1.22}$$

with the Hamilton-Jacobi function we have introduced in Chapter 8. If we take this literally as the wavefunction and call it $\psi_{\rm HJ}$, the Schrödinger equation for $\psi_{\rm HJ}$ results in

$$\mathbb{i}\hbar\frac{\partial}{\partial t}\psi_{\rm HJ} = \mathbf{H}(\mathbf{p},\mathbf{q})\,\psi_{\rm HJ} \qquad \rightarrow \qquad -\partial_t S_{\rm HJ} = H(\partial_q S_{\rm HJ}, q) + O(\hbar^1) \tag{13.1.23}$$

The zero-th order in $\hbar$ of the latter is exactly the Hamilton-Jacobi equation for the classical physics, once again assuring us that classical physics should eventually emerge from quantum physics.

Coming back to the path integral, note how we became increasingly sloppy with the overall numerical factors toward the end of the process. A big part of this can be blamed on technical difficulties in handling an infinite product of common numerical factor such as 2π, as we take the continuum limit of $T/N \to 0$. While the path integral offers many qualitative insights and sometime simpler route to overall structure of the quantum theory, the usual Schrödinger deescription has the advantage when it comes to specific wavefunction and specific numbers. Often, the most effective way to fix the overall numerical factor, if physical, is to resort back to the operator side of the story.

For many purposes, on the other hand, this overall normalization of the path integral become irrelevant. For instance, we often compute quantities called correlators, which are the same path integral as above except "local operators" say, functions of $q(t)$ in the above example, inserted. Typical expressions are

$$\frac{\int [Dq]\; e^{\mathbb{i}S(q)/\hbar}\; [q(t)]^{n_1}\; [q(t')]^{n_2} \cdots}{\int [Dq]\; e^{\mathbb{i}S(q)/\hbar}} \;, \tag{13.1.24}$$

which are, as seen above, "normalized" by the pure path integral as the denominator. For such, the numerical factor that goes into the definition of $[Dq]$ is of course not important.

13.2 Green's Functions in the Minkowskii

The oldest prototype of the field theory is the Maxwell theory. In fact, the concept of the field itself came from Faraday's proposal to think about the field lines, i.e., $F_{\mu\nu}$, themselves as physical entity, rather than as mathematical intermediary that enter the Newton's equation for the charged particle motion. Because of the Bianchi identity $dF = 0$, one realizes at some point that F is not fundamental either, and instead we should think about the vector potential A_μ, such that

$$F = dA \;, \qquad d * dA = J_e \tag{13.2.1}$$

for some 3-form electric current J_e as the source. So for this oldest example, A is the field upon which the rest of the Maxwell theory hinges. One of the earliest exercise one performs in graduate classes in Maxwell theory is to solve the field equation, given arbitrary source on the right hand side.

Let us first recall these, but in a simpler context of the scalar theory,

$$(-\nabla^2 + m^2)\,\phi(x) \;=\; \rho(x) \tag{13.2.2}$$

where we added a mass term for the generality. The action from which this equation comes from is

$$\int d^d x\,\sqrt{g}\,\left(-\frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 + \rho\,\phi\right) \tag{13.2.3}$$

In the flat Minkowskii space-time, this can be solved quite explicitly, and we will learn a few important lesson about relativistic fields along the way. The most economical way to solve such an equation is the Fourier analysis,*

$$\phi(x) = \int \frac{d^dk}{(2\pi)^d}\,\tilde\phi(k)\,e^{\mathbb{i}k_\mu x^\mu}\;, \qquad \tilde\phi(p) = \int d^dy\;\phi(y)\,e^{-\mathbb{i}p_\mu y^\mu} \tag{13.2.5}$$

whereby the above equation reduces to algebraic one, with $k^2 = \eta^{\mu\nu}k_\mu k_\nu = -w^2 + \mathbf{k}^2$ and $k_0 = -w$,

$$(k^2 + m^2)\,\tilde\phi(k) \;=\; \tilde\rho(k) \tag{13.2.6}$$

so that

$$\phi(x) \;=\; \int d^dx'\; G(x-x')\rho(x')\;, \tag{13.2.7}$$

where

$$G(x) \;=\; \int \frac{d^dk}{(2\pi)^d}\,\frac{1}{k^2+m^2}\,e^{\mathbb{i}k^\mu x_\mu}$$

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*The integration measures are

$$\begin{aligned} d^dx &= dt\wedge dx^1\wedge\cdots\wedge d^{d-1}x \\ d^dk &= dw\wedge dk_1\wedge\cdots\wedge dk_{d-1} \end{aligned} \tag{13.2.4}$$

despite the sign between k^0 and w. This follows from how the Fourier analysis works.

$$= \int_C \frac{dw}{2\pi} \int \frac{d^{d-1}\mathbf{k}}{(2\pi)^{d-1}} \frac{1}{-w^2+k^2+m^2} e^{-\mathbb{i}wt+\mathbb{i}\mathbf{k}\cdot\mathbf{x}} \tag{13.2.8}$$

is a Green's function. Green's function remains ambiguous until we specify the contour C in the w-plane since the integrand has poles at $w = w_\pm \equiv \pm\sqrt{\mathbf{k}^2+m^2}$ along the real axis of w.

One has four different contour choices. For instance, suppose we move the contour slightly above the real axis. The contour can be closed to a null result along the lower half infinity $w \to \mathbb{i}\infty$ if $t < 0$, upon which G_C vanishes. With this choice, therefore, the Green's function is such that ρ affects ϕ along the future direction only, resulting in the retarded Green's function, $G_{\rm ret}$. The opposite choice gives the advanced Green's function, $G_{\rm adv}$. The Green's function relevant for the quantization, G_F, is known as the Causal Greens function would be yet another combination, to which we will come back in the last part of this chapter.

It is convenient to define the basic building blocks,

$$\begin{aligned}\Delta_\pm(x) &= \mathbb{i}\int \frac{d^{d-1}\mathbf{k}}{(2\pi)^{d-1}} \frac{1}{2w_\pm(\mathbf{k},m)} e^{-\mathbb{i}w_\pm t+\mathbb{i}\mathbf{k}\cdot\mathbf{x}} \\ &= \pm\mathbb{i}\int \frac{K^{d-2}dK\, d\Omega_{d-2}}{(2\pi)^{d-1}} \frac{1}{2\sqrt{K^2+m^2}} e^{\mp\mathbb{i}\sqrt{K^2+m^2}\,t+\mathbb{i}Kr\cos\theta)} \end{aligned}\tag{13.2.9}$$

with $K \equiv \sqrt{\mathbf{k}^2}$, corresponding to $C_\pm$ small clock-wise circular contours around $w = w_\pm$, respectively, so that

$$G_{\rm ret} = \vartheta(t)\,(\Delta_+ + \Delta_-)\ , \qquad G_{\rm adv} = -\vartheta(-t)\,(\Delta_+ + \Delta_-) \tag{13.2.10}$$

for example, with the step function $\vartheta(z) = 1$ for $z \geq 1$ and $\vartheta(z) = 0$ for $z < 0$.

More explicit forms can be found for $d = 4$, as $d\Omega_2 = d(-\cos\theta)d\phi$,

$$\Delta_\pm^{(d=4)}(x) = \pm\int_0^\infty \frac{KdK}{8\pi^2 r\sqrt{K^2+m^2}} e^{\mp\mathbb{i}\sqrt{K^2+m^2}\,t}(e^{+\mathbb{i}Kr} - e^{-\mathbb{i}Kr}) \tag{13.2.11}$$

As such, the computation boils down to that of a single function

$$\begin{aligned}\Delta_+^{(d=4)} &= \frac{1}{4\pi r}\partial_r \int_{-\infty}^\infty \frac{-\mathbb{i}dK}{2\pi\sqrt{K^2+m^2}} e^{-\mathbb{i}\sqrt{K^2+m^2}\,t-\mathbb{i}Kr} = \frac{1}{4\pi r}\partial_r h(t,r)^* \\ \Delta_-^{(d=4)} &= \frac{1}{4\pi r}\partial_r \int_{-\infty}^\infty \frac{\mathbb{i}dK}{2\pi\sqrt{K^2+m^2}} e^{\mathbb{i}\sqrt{K^2+m^2}\,t+\mathbb{i}Kr} = \frac{1}{4\pi r}\partial_r h(t,r) \end{aligned}\tag{13.2.12}$$

where, with $K = m\sinh\xi$,

$$h(t,r) \equiv \frac{\mathbb{i}}{2\pi}\int_{-\infty}^{\infty} d\xi\; e^{\mathbb{i}t\cosh\xi + \mathbb{i}r\sinh\xi} \tag{13.2.13}$$

h can be written via integral representations of the Henkel functions, and thus eventually the Bessel functions J_λ and Neuman functions N_λ.

The end result depends on the sign of $\tau^2 \equiv t^2 - r^2$ and on the sign of t. We quote the results from a classic text by Bogoliubov and Shirkov,

$$h(t,r) = \begin{cases} \frac{1}{2}\left(-J_0(m\tau) - \mathbb{i}N_0(m\tau)\right) & \tau^2 > 0 \quad t > 0 \\ \\ \frac{\mathbb{i}}{\pi}K_0(m\sqrt{-\tau^2}) & \tau^2 < 0 \\ \\ \frac{1}{2}\left(J_0(m\tau) - \mathbb{i}N_0(m\tau)\right) & \tau^2 > 0 \quad t < 0 \end{cases} \tag{13.2.14}$$

where we see discontinuity at $\tau^2 = 0$.

The discontinuity induces a delta function at the light-cone on $\Delta_\pm$ such that[†]

$$\begin{aligned} \Delta_+^{(d=4)} &= \frac{1}{4\pi}(\vartheta(t) - \vartheta(-t))\left(\delta(\tau^2) - \vartheta(\tau^2)\frac{m}{2\tau^2}J_1(m\tau)\right) \\ &\quad + \frac{m\mathbb{i}}{8\pi\tau}\vartheta(\tau^2)N_1(m\tau) + \frac{m\mathbb{i}}{4\pi^2\sqrt{-\tau^2}}\vartheta(-\tau^2)K_1(m\sqrt{-\tau^2}) \\ \Delta_-^{(d=4)} &= \left(\Delta_+^{(d=4)}\right)^* \end{aligned} \tag{13.2.15}$$

from which we read off

$$\begin{aligned} G_{\rm ret}^{(d=4)} &= \vartheta(t)\left(\frac{1}{2\pi}\delta(t^2 - \mathbf{x}^2) - \vartheta(\tau^2)\frac{m}{2\tau}J_1(m\tau)\right) \\ G_{\rm adv}^{(d=4)} &= -\vartheta(-t)\left(\frac{1}{2\pi}\delta(t^2 - \mathbf{x}^2) - \vartheta(\tau^2)\frac{m}{2\tau}J_1(m\tau)\right) \end{aligned} \tag{13.2.16}$$

which manifestly vanish outside the future and the past light-cone, respectively.

The second terms drop out in the massless limit $m \to 0$, with $J_1(z) = z + O(z^2)$,

[†]The complex conjugation leaves $\tau \equiv \sqrt{\tau^2}$ and $\sqrt{-\tau^2}$ intact; the arguments of the square roots here are always nonnegative by construction.

leaving us with

$$\lim_{m\to 0} G^{(d=4)}_{\text{ret,adv}} = \pm\vartheta(\pm t)\,\frac{\delta(t^2-\mathbf{x}^2)}{2\pi}\ , \tag{13.2.17}$$

which one encounters while solving classical electromagnetic fields. For instance, we have

$$-\partial^2 A_\mu = j_\mu \tag{13.2.18}$$

in the covariant gauge $\partial^\mu A_\mu = 0$, solved by

$$\begin{aligned} A_\mu(t,\mathbf{x})^{\text{ret}} &= \int d^4x'\,\frac{\delta((t-t')^2-(\mathbf{x}-\mathbf{x}')^2)}{2\pi}\,j_\mu(t',\mathbf{x}') \\ &= \int d^3\mathbf{x}'\,\frac{j_\mu(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}')}{4\pi|\mathbf{x}-\mathbf{x}'|} \end{aligned} \tag{13.2.19}$$

as expected.

13.3 Path Integral and Causal Green's Function

Now let us come back to the path integral of a real scalar field, for which we have computed the retarded and the advanced Green's function earlier. One way to quantize such a theory is via the canonical quantization, which we made a brief overview in the last chapter of the Volume I.

In this volume, we will instead proceed with the path integral viewpoint, recalling the canonical counterpart at places. The path integral in the flat Minkwoskii spacetime is,

$$\int [D\phi]\; e^{\mathbb{i}S_{\text{free}}/\hbar} \tag{13.3.1}$$

with

$$S_{\text{free}} \equiv \int d^dx\,\left(-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2\right) \tag{13.3.2}$$

Exactly how one might perform such an integration and what are physical quantities to be extracted constitute the bulk of the quantum field theory framework.

We hope to address such questions at least partially as we develop later in the volume, although the effort would come nowhere near the comprehensive. We will be content to offer more of conceptual frameworks in this volume, enough to impart the essential features of quantum field theory such as the renormalization and the symmetry principles. In this preliminary chapter, in particular, we will assume a sensible formulation of such a path integral exists and explore the most rudimentary consequences.

In the quantum field theory, the basic building block is the two-point function via the path integral,

$$\sim \int [D\phi]\ \phi(x)\phi(x')e^{\mathbb{i}S_{\rm free}(\phi)/\hbar}\ , \tag{13.3.3}$$

modulo a normalizing factor. When we express this from the Hamiltonian side, it should be related to the expectation value of type

$$\langle 0|\,\boldsymbol{\phi}(x)\boldsymbol{\phi}(x')\,|0\rangle \tag{13.3.4}$$

in the Schödinger picture, with the two vacuum states on the left and the right, respectively, at the future and the past infinity. Coming back to the Heisenberg picture, this should be written as

$$\langle 0|\,e^{-\mathbb{i}/\hbar\int_t^\infty dt\,\mathbf{H}}\,\boldsymbol{\phi}(x)\,e^{-\mathbb{i}/\hbar\int_{t'}^t dt\,\mathbf{H}}\,\boldsymbol{\phi}(x')\,e^{-\mathbb{i}/\hbar\int_{-\infty}^{t'} dt\,\mathbf{H}}\,|0\rangle \tag{13.3.5}$$

Recall how the path integral came from sandwiching the time-evolution operator $e^{-\mathbb{i}\mathbf{H}t/\hbar}$ between the initial and the final state. The only difference here is the insertion of $\boldsymbol{\phi}(x)$ and $\boldsymbol{\phi}(x')$.

Although it looks as if the two insertions of ϕ in the path integral (13.3.3) do not care about time-ordering, the equivalent operator picture tells us that this two-point function actually computes a time-ordered quantity in the Hamiltonian side

$$\big\langle\, T(\boldsymbol{\phi}(x)\boldsymbol{\phi}(x'))\,\big\rangle \equiv \langle 0|\ \vartheta(t-t')\,\boldsymbol{\phi}(x)\,\boldsymbol{\phi}(x') + \vartheta(t'-t)\,\boldsymbol{\phi}(x')\,\boldsymbol{\phi}(x)\ |0\rangle \tag{13.3.6}$$

This time-ordered 2-point function is computed by the causal Green's function

$$G_C(x) \equiv \vartheta(t)\Delta_+(x-x') - \vartheta(-t)\Delta_-(x) \tag{13.3.7}$$

modulo an overall factor of $\mathbb{i}\hbar$ and also other normalizing factor to be discussed in next section. In next section, it should become clearer why one of the Green's function is the right answer.

The causal Green's function has a simple momentum space interpretation. From the definition of $\Delta_\pm$, we can see that the contour would be mostly along the real line of w, except it pass above $w = w_+$ and below w_-. Equivalently, the same is achieved by giving a slightly negative imaginary part to w_+ and a slightly positive imaginary part to w_-, which can be instituted by deforming the momentum space propagator as

$$\frac{1}{k^2+m^2} \quad \rightarrow \quad \frac{1}{k^2+m^2-\mathbb{i}\varepsilon} \tag{13.3.8}$$

for an infinitesimal real positive number ε. In other words,

$$G_C(x) \;=\; \int \frac{d^dk}{(2\pi)^d}\, \frac{1}{k^2+m^2-\mathbb{i}\varepsilon}\, e^{\mathbb{i}k^\mu x_\mu} \tag{13.3.9}$$

with w integral now strictly along the real line.

Again quoting the results from Bogoliubov and Shirkov, the causal Green's function in $d = 4$ has the form in the Minkowskii spacetime has the form,

$$\begin{aligned} G_C^{(d=4)}(x) \;=\; & \frac{1}{4\pi}\delta(\tau^2) - \frac{m}{8\pi\sqrt{\tau^2}}\vartheta(\tau^2)\left(J_1(m\tau) - iN_1(m\tau)\right) \\ & + \frac{m\mathbb{i}}{4\pi^2\sqrt{-\tau^2}}\vartheta(-\tau^2)K_1(m\sqrt{-\tau^2}) \end{aligned} \tag{13.3.10}$$

Note how the step functions with $\pm t$ as the argument disappeared altogether. Oddly enough this "causal" Green's function does not vanish outside the light-cones, as can be seen from the last part. The same can be seen more clearly from its massless limit

$$\lim_{m\to 0} G_C^{(d=4)}(x) \;=\; \frac{1}{4\pi}\delta(\tau^2) + \frac{1}{4\pi^2\mathbb{i}}\frac{1}{\tau^2} \tag{13.3.11}$$

whose second term is actually something one is very familiar with once we make an analytical continuation to an imaginary time and change the problem to that on the Euclidean space $\mathbb{R}^4$, as we will see below.

13.3.1 Generating Functional and the Feynman Propagator

This causal Green's function is one of the main building block for field theory path integrals in the perturbative approximation. We will not attempt to derive the full arsenal of the Feynman diagram techniques here, in favor of their Euclidean cousins in a later chapter, but still one can see easily why the causal Green's function play the pivotal role here and how it is trivially connected to the Feynman propagator.

We again start from the path integral,

$$Z[\mathbb{j}\,] = \int [D\phi]\; e^{\mathbb{i}S(\phi)/\hbar + \mathbb{j}\cdot\phi} \tag{13.3.12}$$

where $\mathbb{j}$ is an external field with a short-hand notation,

$$\mathbb{j}\cdot\phi \equiv \int d^dx\; \mathbb{j}(x)\phi(x) \tag{13.3.13}$$

If S is free, we may perform the Gaussian integral, at least formally,

$$Z_{\rm free}[\mathbb{j}\,] \;=\; \int [D\phi]\; e^{-\frac{1}{2}\int_k \tilde\phi(-k)[\mathbb{i}(k^2+m^2)/\hbar]\tilde\phi(k) + \int_k \tilde{\mathbb{j}}(-k)\tilde\phi(k)} \tag{13.3.14}$$

in the momentum space with $\int_k \equiv \int d^dk/(2\pi)^d$, which gives

$$\begin{aligned} Z_{\rm free}[\mathbb{j}\,] \;&=\; Z_{\rm free}[\mathbb{j}=0\,]\times e^{\frac{1}{2}\int_k \tilde{\mathbb{j}}(-k)[\mathbb{i}(k^2+m^2)/\hbar]^{-1}\tilde{\mathbb{j}}(k)} \\ &=\; Z_{\rm free}[\mathbb{j}=0\,]\times e^{\frac{1}{2}\int_x\int_{x'} \mathbb{j}(x)[-\mathbb{i}\hbar\, G_C(x-x')]\,\mathbb{j}(x')} \end{aligned} \tag{13.3.15}$$

again with $\int_x \equiv \int d^dx$.

The appearance of the causal, rather than the retarded or the advanced, Green's function is justified by

$$\frac{\delta^2}{\delta\mathbb{j}(x)\delta\mathbb{j}(x')}\log Z_{\rm free}[\mathbb{j}\,] \;=\; -\mathbb{i}\hbar\, G_C(x-x') \tag{13.3.16}$$

which gives

$$-\mathbb{i}\hbar\, G_C(x-x') \;=\; \frac{\int [D\phi]\;\phi(x)\,\phi(x')\,e^{\mathbb{i}S_{\rm free}(\phi)/\hbar}}{\int [D\phi]\; e^{\mathbb{i}S_{\rm free}(\phi)/\hbar}} \tag{13.3.17}$$

In the momentum space, the same may be written as

$$\frac{-\mathbb{i}\hbar}{k^2+m^2-i\varepsilon} \;=\; \frac{\mathbb{i}\hbar}{w^2-\mathbf{k}^2-m^2+i\varepsilon} \tag{13.3.18}$$

which is the well-known Feynman propagator.

What do we do when the action is not free? Suppose we have the following simple for

$$S(\phi)=S_{\rm free}(\phi)-\int V(\phi) \tag{13.3.19}$$

where $V(\phi)$ is a polynomial in ϕ. If we can treat this additional term in the exponent by a Taylor expansion,

$$\begin{aligned} Z[\,\mathbb{j}\,] &= \int[d\phi]\,\sum_n \frac{1}{n!}\left(-\mathbb{i}\int V(\phi)/\hbar\right)^n \,e^{iS_{\rm free}(\phi)/\hbar+\int \mathbb{j}\phi} \\ &= \int[d\phi]\,\sum_n \frac{1}{n!}\left(-\mathbb{i}\int V(\delta/\delta\mathbb{j})/\hbar\right)^n \,e^{iS_{\rm free}(\phi)/\hbar+\int \mathbb{j}\phi} \\ &= e^{-\mathbb{i}\int V(\delta/\delta\mathbb{j})/\hbar} Z_{\rm free}[\,\mathbb{j}\,] \end{aligned} \tag{13.3.20}$$

at least formally. Once this computation makes sense, one would find

$$\frac{\delta^2}{\delta\mathbb{j}(x)\delta\mathbb{j}(x')}\log Z[\,\mathbb{j}\,] \tag{13.3.21}$$

a quantum corrected version of the propagator, for instance. Actual computation for such would proceed via a triple perturbation, order by order in powers of $\mathbb{j}$, of derivatives, and of $\hbar$.

An extensive discussion is needed for details of such a perturbation scheme where we compute a formal series expansion via the above Feynman propagators and the monomials in V, commonly called the interaction vertices. There do exist many superb texts that deal with this perturbation scheme, which is very much necessary if one is interested in high order estimate of scattering amplitudes and such. We will not attempt to repeat this here in its full glory although we do cover the most essential ingredients in a later chapter, for a Wick-rotated scalar theory as well as for a simpler prototype of the matrix integral toy model.

Instead, for the bulk of the presentation, we choose to stay as close as possible to the Gaussian path integral by performing the path integrals in multiple steps. Effectively this would treat part of the fields on equal footing as $\mathbb{j}$ above, although in a far more complicated coupling to the analog of ϕ's. Part of the fields would be treated as classical background, like $\mathbb{j}$ above, while some others would be treated quantum and path integrated over, like ϕ above.

The distinction may come naturally by treating gauge fields or gravity as classical while treating matter fields at quantum level. More generally, we would also separate a single field into two parts, with the quantum part built from modes with high momentum or large eigenvalues, while the rest, relatively slowly varying part, would be treated as the background, yet to be integrated. This is known as the background field method and offers a clean picture of the phenomenon of runnning couplings, hence the renormalization.

Our choice here is in part motivated by the desire to separate the heavy machinery of the perturbation expansion from the qualitative discussion of the renormalization and running coupling. When we start with the heavy machinery of the Feynman diagrams first and encounter the apparently divergent Feynman integrals term by term in Z at a time, one can easily start with the false impression that the renormalization is a "trick" that "miraculously" allows us to bypass "the disease." Nothing would be further from the truth.

Instead, the latter type of phenomena tell us that the framework of the quantum field theories must not be what one naively thought it would be, namely an infinite number of harmonic oscillators, interacting among themselves, that collectively describe fields in continuous space-time. The latter is, although very much understandable, by and large a wishful thinking by the pioneers of the subject. Models that achieve the continuum limit do exist and in fact play the crucial roles for the fundamental forces of nature, as we will see later, but such special corners among the vast landscape of quantum field theories do not mean that a generic quantum field theory should have such a well-defined continuum limit.

13.3.2 Wick Rotation (I)

The deformation of the momentum space propagator by $-\mathbb{i}\varepsilon$ that results in G_C and equivalently in the Feynman propagator suggests a different way of performing many of the field theory computations down the road. Note that if no contribution arises from the asymptotic arcs on the complex w-plane, we may deform w contour further counter-clock-wise from the real line to the imaginary line,

$$w \quad \to \quad \mathbb{i} w_E \;\equiv\; \mathbb{i} k_E^d \tag{13.3.22}$$

as this avoids both of the poles $w_\pm$ after $-\mathbb{i}\varepsilon$ shifts.

This same analytic continuation of the frequency is also achieved by rotating the time coordinate itself as

$$t \quad \to \quad -\mathbb{i} t_E \;\equiv\; -\mathbb{i} x_E^d \tag{13.3.23}$$

whereby we also have the Minkowskii metric become

$$\eta_{\mu\nu} \quad \to \quad \delta_{\mu\nu} \tag{13.3.24}$$

The Green's function in this Euclidean regime is unique,

$$G_E(x) \quad = \quad \int \frac{d^d k_E}{(2\pi)^d} \, \frac{1}{k_E^2 + m^2} \, e^{\mathbb{i} k_E^\mu x^\beta \delta_{\mu\beta}} \tag{13.3.25}$$

and solves

$$(-\partial_E^2 + m^2)\, \phi(x_E) \;=\; \rho(x_E) \tag{13.3.26}$$

G^E is an analytic continuation of G_C modulo a factor of $-\mathbb{i}$

$$G_E(x_E) = -\mathbb{i} G_C(x) \Bigg|_{t \,\to\, -\mathbb{i} t_E} \tag{13.3.27}$$

but this $-\mathbb{i}$ is unnecessary as the mutually canceling such factors enter in the Fourier analysis when converting to k_E and to x_E.

For instance, we see that, with the $d = 4$ massless limit above, the first term $\delta(\tau^2) \to \delta(-x_E^2)$ piece drops out as it does not contribute upon the x_E-integration for $d \geq 3$, while the other gives

$$\lim_{m\to 0} G_E^{(d=4)}(x_E) = \frac{1}{2\pi^2} \cdot \frac{1}{2x_E^2} \tag{13.3.28}$$

which is the well-known solution to $-\partial_E^2 \, \phi = \delta^{(4)}(x_E)$.

This analytic continuation to the Euclidean time is called the Wick rotation and for many computation brings about enormous simplifications. In principle there can be obstruction to the analytic continuations that hinder this shortcut but, happily, for computations we present in this volume do not suffer from such problems. We will resort to the Wick rotation beginning with the next chapter quite freely, and henceforth, we may often skip the subscript E that stand for the "Euclidean signature" whenever the confusion is unlikely.

13.4 Dirac Spinors and Dirac Brackets

One of more profound fact of life with the quantum mechanics is the dichotomy of bosons and fermions, or the spin-statistics relation and the notion of identical particles. Recall that in building up atoms, we must consider all electrons as indistinguishable from one another and impose the Fermi statistics on their wavefunctions. Even though Fermi and Bose statistics can be consistently imposed at this level, its true origin cannot be found in quantum mechanics itself, where we follow existing particles individually and solve for their wavefunctions. The latter determines the probability distribution of the particles but does not recognize that individual particles are themselves wavepackets from the more fundamental quantum fields. For fermions, this eventually leads to the spinor fields.

Although we will eventually get to quantum field theories of spinors down the road, first we need to learn a quantum mechanics prototype for such quantum fields that generate fermions. By this, we do not mean how we usually compute the electron wavefunction with a half-integer spin attached as a tensor product. Rather, the quantum fields for spinors must be built up from different kind of numbers called Grassman number whose product is anti-commuting instead of commuting. A Fermionic field would be composed of multi-component wavefunctions based on regular complex numbers, often called c-numbers, multiplied by such Grassman coefficients. The former carries the spin representation under the Lorentz group while the latter ensures the necessary Fermi statistics built-in.

13.4.1 Fermionic Quantum Mechanics

Let us start with a quantum mechanics with Grassman numbers. The simplest possible Lagrangian with real $\chi = \chi^\dagger$ has only one time-derivative,

$$L_{\rm real} \;=\; \frac{\mathbb{i}}{2}k^{ab}\chi_a\dot{\chi}_b - \frac{\mathbb{i}}{2}m^{ab}\chi_a\chi_b \tag{13.4.1}$$

with symmetric k and anti-symmetric m, which follows from

$$\chi_a\chi_b = -\chi_b\chi_a \tag{13.4.2}$$

and an integration by part. The reality is another matter. A convention of the complex conjugation on product of multiple Grassman numbers comes with a sign ambiguity, due to the choice of the ordering after the conjugation. Here we choose to use

$$(\xi\zeta)^\dagger = \zeta^\dagger\xi^\dagger \tag{13.4.3}$$

whereby real k and real m lead to real $L_{\rm real}$.

A more familiar complex form results when $k = K \otimes 1_{2\times 2}$ and $m = M \otimes \epsilon_{2\times 2}$. We may then rotate χ_{2S-1}'s into χ_{2S}'s and vice versa and define

$$\psi_S = \frac{1}{\sqrt{2}}(\chi_{a=2S-1} + \mathbb{i}\chi_{a=2S}) \tag{13.4.4}$$

The Lagrangian is then

$$L_{\rm complex} \;=\; \mathbb{i}K^{ST}\psi_S^\dagger\dot{\psi}_T - M^{ST}\psi_S^\dagger\psi_T \tag{13.4.5}$$

where, for reality, K and M are now both Hermitian.

Since there is only one time derivative in the Lagrangian, we should expect second-class constraints to emerge, reducing the number of degrees of freedom. We need to decide first how to take variation with respect to the Grassman number, and will take the derivative acting from the right so that,

$$\varphi^b \;\equiv\; \pi^b - L_{\rm real}\frac{\overleftarrow{\delta}}{\delta\dot{\chi}_b} \;=\; \pi^b - \frac{\mathbb{i}}{2}k^{ab}\chi_a \;\approx\; 0 \tag{13.4.6}$$

Similarly

$$\Phi^T \equiv \Pi^T - L_{\text{complex}}\frac{\overleftarrow{\delta}}{\delta\dot{\psi}_T} = \Pi^T - \mathbb{i}K^{ST}\psi^\dagger_S \approx 0 \tag{13.4.7}$$

whereas the conjugate of $\psi^\dagger_S$ is weakly zero $(\Phi^\dagger)^S \equiv (\Pi^\dagger)^S \approx 0$ by itself.

Note how the canonical momentum $\Pi^\dagger$ of $\psi^\dagger$, is not complex conjugate of Π despite the notation. This has something to do with how we did not write the kinetic term democratically between ψ and $\psi^\dagger$. The potential confusion from this is washed out by the time we reach the Dirac bracket below, thankfully. Another oddity is how the Poisson bracket becomes symmetric rather than anti-symmetric, which we distinguish by using { , } in place of [,], such that

$$\{\,\chi_a, \pi^b\,\}_{\text{P.B.}} = \delta_a{}^b\ ,$$

$$\{\,\psi_S, \Pi^T\,\}_{\text{P.B.}} = \delta_S{}^T\ , \qquad \{\,\psi^\dagger_S, (\Pi^\dagger)^T\,\}_{\text{P.B.}} = \delta_S{}^T \tag{13.4.8}$$

Since $\{\,\varphi^a, \varphi^b\,\}_{\text{P.B.}}$ and $\{\,(\Phi^\dagger)^S, \Phi^T\,\}_{\text{P.B.}}$ are both nonsingular, the Dirac bracket would be a convenient starting point.

After some gymnastics, we arrive at

$$\{\,\chi_a, \chi_b\,\}_{\text{Dirac}} = -\mathbb{i}(k^{-1})_{ab}\ , \qquad \{\,\psi_S, \psi^\dagger_T\,\}_{\text{Dirac}} = -\mathbb{i}(K^{-1})_{ST} \tag{13.4.9}$$

In the former, a naive factor 2, which one would have gotten for real fermions under a preemptive application of the constraint followed by the Poisson bracket, is not present. A well-practiced, if not recommendable, strategy for reaching the first is by starting from the second, by decomposing each complex fermion into two real fermions with the conventional $1/\sqrt{2}$ normalizing factor. In turn, with the latter complex fermions, the situation is quite analogous to the Chern-Simons theory; Depending on how the kinetic term is written, the naive procedure based on premature application of the constraints may or may not give the right answer.

Given the anti-commuting nature of the fundamental variables, there is a potential sign ambiguity in defining the conjugate momenta, but our choice above can be seen to be consistent with the equation of motion. With the standard canonical form of the evolution,

$$\dot{f} \;=\; \{\, f, H \,\}_{\text{Dirac}} \tag{13.4.10}$$

and the Hamiltonians

$$H_{\text{real}} = \frac{\mathbb{i}}{2} m^{ab} \chi_a \chi_b \ , \qquad H_{\text{complex}} = M^{AB} \psi_A^\dagger \psi_T \tag{13.4.11}$$

we see that the equation of motions

$$\dot{\chi}_a = (k^{-1} m)_a{}^b \chi_b \ , \qquad \dot{\psi}_A = -\mathbb{i} (K^{-1} M)_A{}^T \chi_T \tag{13.4.12}$$

are reproduced. Finally, the quantization rule is to convert the Dirac bracket to the canonical (anti-)commutator by multiplying $\mathbb{i}\hbar$ on the right hand side, so we arrive at

$$\{\, \boldsymbol{\chi}_a, \boldsymbol{\chi}_b \,\} \;=\; (k^{-1})_{ab} \hbar \ , \qquad \{\, \boldsymbol{\psi}_A, \boldsymbol{\psi}_T^\dagger \,\} \;=\; (K^{-1})_{AB} \hbar \tag{13.4.13}$$

13.4.2 Fermionic Quantum Fields

For fermionic quantum fields, we need to understand the spinor representation under the Lorentz group. The discussion starts with the Clifford algebra

$$\gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab} \tag{13.4.14}$$

with $(\gamma^i)^\dagger = \gamma^i$ and $(\gamma^0)^\dagger = -\gamma^0$. The covariant derivative has the form,

$$\mathbf{D}_\mu = D_\mu + \frac{1}{4} w_{\mu ab} \gamma^{ab} \tag{13.4.15}$$

where D_μ parallel transport all tensor indices while the spinor part of the connection displayed separately. We are using the notation,

$$\gamma^{a_1 a_2 \cdots a_k} = \frac{1}{k!} \sum_{\sigma\text{: permutations}} (-1)^{|\sigma|} \gamma^{a_{\sigma_1}} \gamma^{a_{\sigma_2}} \cdots \gamma^{a_{\sigma_k}} \tag{13.4.16}$$

Because of how the Lorentz boost generator is hermitian instead of anti-hermitian, the naive inner product $\Psi^\dagger\Psi$ of a Dirac spinor is not a scalar. Rather, with $\bar{\Psi} \equiv \Psi^\dagger\gamma^0$

$$\bar{\Psi}\Psi \tag{13.4.17}$$

is a scalar quantity. The action for a single Dirac fermion is then, with real m,

$$S = -\int d^4x \, \left(\bar{\Psi}\mathbb{i}\gamma^a\mathbf{D}_a\Psi + \mathbb{i}m\bar{\Psi}\Psi\right) \tag{13.4.18}$$

The reality of the action is ensured by

$$(\Psi^\dagger\gamma^0\Psi)^\dagger = \Psi^\dagger(\gamma^0)^\dagger\Psi = -\Psi^\dagger\gamma^0\Psi \tag{13.4.19}$$

for the mass term, and for the kinetic term the Clifford algebra will do the rest.

The Green's function follows from the same procedure with the scalar field. The analog of $-\partial^2 + m^2$ here is

$$\mathbb{i}\gamma^a\partial_a + \mathbb{i}m \tag{13.4.20}$$

so the Green's function in the momentums space

$$(-\gamma^a k_a + \mathbb{i}m)^{-1} = \frac{-\gamma^a k_a - \mathbb{i}m}{k^2 + m^2} \tag{13.4.21}$$

This results in the retarded and the advanced Green's function for spinor

$$\left(\mathbb{i}\gamma^a_{IJ}\partial_a - \mathbb{i}m\delta_{IJ}\right) G_{\text{ret,adv}}(x) \tag{13.4.22}$$

where $G_{\text{ret,adv}}(x)$ are those of the scalar field we obtained in an earlier section.

For quantization, we concentrate on the time-derivative part of the action,

$$S = \int d^4x \, \left(\mathbb{i}(\Psi_I)^\dagger\partial_t\Psi_I + \cdots\right) \tag{13.4.23}$$

with the 4-component spinor index I. Comparing to the mechanics prototype, we see that the Dirac bracket,

$$\{\, \Psi_I(x), \Psi^\dagger_J(y) \,\}_{\text{Dirac}} = -\mathbb{i}\, \delta_{IJ}\delta^{(d-1)}(\mathbf{x} - \mathbf{y}) \tag{13.4.24}$$

and the canonical anti-commutator,

$$\{\,\mathbf{\Psi}_I(x), \mathbf{\Psi}_J^\dagger(y)\,\} = \hbar\,\delta_{IJ}\delta^{(d-1)}(\mathbf{x}-\mathbf{y}) \tag{13.4.25}$$

at quantum level. The Feynman propagator also follows as

$$\frac{\mathbb{i}\hbar(\gamma^a k_a + \mathbb{i}m)}{k^2+m^2-\mathbb{i}\varepsilon} \tag{13.4.26}$$

As with the above scalar example, we will not foray into the dictionary of the Feynman diagrams here either, and instead later focus on what we can compute with the Euclidean path integrals at Gaussian level.

In the Appendix A.4, we catalog and classify spinors in arbitrary dimensions and signatures, but here we confine ourselves to $d=4$ space-time with the Lorentzian signature. See the next chapter for its Euclidean alalog. One well-known representation starts with the Dirac matrices,

$$\gamma^{1,2,3} = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}\,, \qquad \gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{13.4.27}$$

The rotation generator for the ij plane, for example, is

$$\frac{\gamma^{ij}}{2} = \frac{1}{2}\begin{pmatrix} \mathbb{i}\epsilon^{ijk}\sigma_k & 0 \\ 0 & \mathbb{i}\epsilon^{ijk}\sigma_k \end{pmatrix}\,, \tag{13.4.28}$$

while the Lorentz boosts correspond to

$$\frac{\gamma^{0i}}{2} = \frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}\,, \tag{13.4.29}$$

A special property of even dimensions is that one can build the so-called chirality operator Γ out of the Dirac matrices as

$$\Gamma \equiv -\mathbb{i}\gamma^0\cdots\gamma^3 = \begin{pmatrix} 1_{2\times2} & 0 \\ 0 & -1_{2\times2} \end{pmatrix} \tag{13.4.30}$$

which anti-commute with all γ^a's and commute with the SO rotation generators $\gamma^{ab}/4$'s. As such, the four-component Dirac spinor consists of two irreducible representations of $SO(1,3)$, which are called chiral or anti-chiral, depending on the ±1

eigenvalue under Γ,

$$\Psi = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} . \tag{13.4.31}$$

Each half $\Psi_\pm$ of the spinor is called the Weyl spinor as opposed to the Dirac spinor. In the Dirac fermion action, we can see how the kinetic term pair up $\Psi_\pm$ with $\Psi^\dagger_\pm$ while the mass term pair up $\Psi_\pm$ with $\Psi^\dagger_\mp$ instead. This means that with $m = 0$ one can write down a theory of Weyl fermions as the smallest building block. Indeed, this is how the Standard Model of particles physics is constructed where the chiral and anti-chiral fermions come in different gauge representations.

13.5 Symmetries and Conserved Currents

Given a field theory with an action,

$$S(\phi) \;=\; \int d^d x \; \mathcal{L}(\phi, \partial_\mu \phi) \tag{13.5.1}$$

two immediate information we obtain are the usual Euler-Lagrange equation of motion,

$$\partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \right) = \frac{\delta \mathcal{L}}{\delta \phi} \tag{13.5.2}$$

and the celebrated Noether current,

$$J^\mu_\theta \equiv \sum_\phi \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \boldsymbol{\delta}_\theta \phi \tag{13.5.3}$$

where $\boldsymbol{\delta}_\theta \phi$ is an infinitesimal and uniform symmetry transformation of ϕ that leaves the action invariant. From $\boldsymbol{\delta}_\theta S = 0$, it is straightforward to show

$$\partial_\mu J^\mu_\theta = 0 \tag{13.5.4}$$

with the equation of motion for ϕ imposed. With uniform θ, the symmetry is said to be global.

Let us make things a little more concrete. For the moment, let us consider the transformation that can be written as a left multiplication,

$$\boldsymbol{\delta}_\theta \phi \;=\; \mathbb{i}\theta\phi \tag{13.5.5}$$

where $\theta = \sum_C \theta^C t^C$ with hermitian t^C's and ϕ is a multi-component column vector. Such an infinitesimal transformation would accumulate to a finite θ as well,

$$\phi \quad \to \quad e^{\mathbb{i}\theta}\phi \tag{13.5.6}$$

so sometimes we also use the same notation for infinitesimal θ.

Gauging such a symmetry means introducing a gauge connection $A = \sum_C A^C t^C$,

$$\partial_\mu \phi \quad \to \quad D_\mu \phi \equiv (\partial_\mu - \mathbb{i}A_\mu)\phi \tag{13.5.7}$$

With the additional transformation

$$(\partial_\mu - \mathbb{i}A_\mu) \quad \to \quad e^{\mathbb{i}\theta}(\partial_\mu - \mathbb{i}A_\mu)e^{-\mathbb{i}\theta} \tag{13.5.8}$$

the action

$$S(\phi, A) \;=\; \int d^d x\, \mathcal{L}(\phi, D_\mu\phi) \tag{13.5.9}$$

is now invariant

$$S(\phi, A) = S(\phi + \boldsymbol{\delta}_\theta\phi, A + \boldsymbol{\delta}_\theta A), \tag{13.5.10}$$

under space-time dependent θ as well. This is called the local gauge symmetry. The equation of motion for ϕ becomes covariant,

$$D_\mu \left(\frac{\delta\mathcal{L}(\phi, D_\mu\phi)}{\delta(D_\mu\phi)} \right) = \frac{\delta\mathcal{L}(\phi, D_\mu\phi)}{\delta\phi} \tag{13.5.11}$$

Whether or not we treat A as a dynamical field is left undecided for now. If we choose to take A as dynamical, the matter action would be accompanied by some gauge kinetic term, such as,

$$-\frac{1}{4\mathbb{g}^2}\sum_C F^C_{\mu\nu}F^{C\mu\nu} \tag{13.5.12}$$

for some coupling constant $\mathbb{g}$, but more importantly the path integral would involve integration over A as well. The equation of motion for A would be

$$\frac{1}{\mathbb{g}^2} D_\mu F^{C\mu\nu} = -\frac{\delta \mathcal{L}}{\delta A^C_\nu} \tag{13.5.13}$$

where the gauge field in D_μ acts on $F \equiv \sum_C \mathcal{T}^C F^C$ in the adjoint representation, which is to say, via a commutator $-\mathbb{i}[A, F]$ if $\mathcal{T}^C$ are symmetry generators in the defining representation.

The alternate definition of the current sitting on the right hand side here

$$\left(J^C\right)^\mu \equiv -\frac{\delta \mathcal{L}}{\delta A^C_\mu} \tag{13.5.14}$$

is also conserved,

$$0 = D_\mu J^\mu = \partial_\mu J^\mu - \mathbb{i}(A_\mu J^\mu - J^\mu A_\mu) \tag{13.5.15}$$

with $J \equiv \sum_C \mathcal{T}^C J^C$. With dynamical A, we can see this same fact differently from the A equation of motion, since

$$D^\nu D^\mu F_{\mu\nu} = \frac{1}{2}[D^\nu, D^\mu] F_{\mu\nu} = -\frac{\mathbb{i}}{2}[F^{\nu\mu}, F_{\mu\nu}] = 0 \tag{13.5.16}$$

as a mathematical identity, with $F \equiv \sum_C \mathcal{T}^C F^C$.

Note that

$$\sum_C \theta^C J^\mu_C = -\sum_C \sum_\phi \frac{\delta \mathcal{L}}{\delta(D_\mu \phi)} \frac{\theta^C \delta(D_\mu \phi)}{\delta A^C_\mu} = \sum_\phi \frac{\delta \mathcal{L}(\phi, D_\mu \phi)}{\delta(D_\mu \phi)} \boldsymbol{\delta}_\theta \phi \tag{13.5.17}$$

The right hand side is nothing but the covariantized version of the Noether current. We should emphasize that this conservation law holds regardless of whether we treat A as dynamical or external.

In most of what follows, we will often bypass the factor $\mathbb{i}$ by adopting the anti-hermitian notation such that

$$\mathcal{T}^C = -\mathbb{i} t^C \ , \qquad \Theta = -\mathbb{i}\theta = \sum_C \theta^C \mathcal{T}^C \ , \qquad \mathcal{A} = -\mathbb{i} A = \sum_C A^C \mathcal{T}^C \tag{13.5.18}$$

Apart from the notational simplicity, this convention aligns better with the case of the gravitational connection where the natural rotational generators are written as anti-symmetric $d\times d$ matrices for the orthonormal Lorentz indices or as anti-hermitian $\sim \gamma^{ab}$ for spinors. In this anti-hermitian convention, the transformation rules are,

$$
\begin{aligned}
\phi &\rightarrow e^{-\Theta}\phi \\
(d+\mathcal{A}) &\rightarrow e^{-\Theta}(d+\mathcal{A})e^{\Theta}
\end{aligned}
\tag{13.5.19}
$$

and, in particular,

$$
\boldsymbol{\delta}_\Theta \mathcal{A}_\mu = \partial_\mu \Theta + \mathcal{A}_\mu \Theta - \Theta \mathcal{A}_\mu \tag{13.5.20}
$$

for an infinitesimal Θ. Also natural to define is, for a purely notational convenience,

$$
\mathcal{J}^\mu_C \equiv -\frac{\delta \mathcal{L}}{\delta \mathcal{A}^C_\mu} \ , \tag{13.5.21}
$$

obeying the same conservation law. Later when we discuss anomaly of continuous symmetries, this is the form of the current we use, albeit in the Euclidean signature after a Wick rotation.

13.5.1 Energy-Momentum Tensor

All of above elevate to curved space-time almost verbatim, with the covariantized action,

$$
S(\phi) \ = \ \int d^dx \, \sqrt{g} \, \mathcal{L}(\phi, \nabla_\mu \phi) \ = \ \int \mathcal{V} \, \mathcal{L}(\phi, \nabla_\mu \phi) \tag{13.5.22}
$$

where $\mathcal{V}$ is the volume form. This covariantizes ∂ to ∇, but as long as $\boldsymbol{\delta}$ do not transform the metric, nothing else changes. Alternatively, one can choose to treat $\mathcal{V}\mathcal{L}$ itself as the d-form Lagrangian density. In the latter option, the same Noether procedure on $\mathcal{V}\mathcal{L}$ would produce the $(d-1)$-form current, or a contraction between $\mathcal{V}$ and the vectorial $\mathcal{J}$, with

$$
0 = d_{\mathcal{A}}(\mathcal{V} \lrcorner \mathcal{J}) \tag{13.5.23}
$$

as the conservaton law, where $d_{\mathcal{A}}$ is the covariantized exterior derivative. For simplicity we will use the common notation $\mathcal{J}$ to denote either form of the current, from now on.

With such a covariant coupling to the general metric, we know of another conserved quantity, namely the energy-momentum tensor in a direct analogy with the current defined from the variation of the gauge field as in Eq. (13.5.14),

$$T_{\mu\nu} \;\equiv\; -\frac{2}{\sqrt{g}}\frac{\delta(\sqrt{g}\,\mathcal{L})}{\delta g^{\mu\nu}} \tag{13.5.24}$$

This quantity would sit in the Einstein equation,

$$G_{\mu\nu} \;=\; 8\pi G_N T_{\mu\nu} \tag{13.5.25}$$

opposite to the Einstein tensor $G_{\mu\nu}$ whose divergence vanishes identically as a mathematical identity. As such, $T_{\mu\nu}$ is also a conserved quantity,

$$\nabla^\mu T_{\mu\nu} \;=\; 0 \tag{13.5.26}$$

regardless of $\mathcal{L}$ as long as this matter Lagrangian obeys the general covariance.

We have seen and used examples of $T_{\mu\nu}$ in the first volume, so let us remind ourselves of the simplest example of a real scalar

$$\begin{aligned} \mathcal{L}(\phi, \nabla_\mu\phi; g) \;&=\; \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi) \quad \rightarrow \\ T_{\mu\nu} \;&=\; \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2 + g_{\mu\nu}V(\phi) \end{aligned} \tag{13.5.27}$$

The conservation law holds,

$$\nabla^\mu T_{\mu\nu} \;=\; \nabla^2\phi\nabla_\nu\phi + V'(\phi)\nabla_\nu\phi \;=\; \big(\nabla^2\phi + V'(\phi)\big)\nabla_\nu\phi \;=\; 0 \tag{13.5.28}$$

again upon the equation of motion.

Noether Energy-Momentum

In the Minkowskii space-time, on the other hand, the time translation and the spatial translations are isometries, so the Noether procedure would generate conserved current with two space-time indices, say, the Noether energy-momentum tensor. We have seen how the Noether current of a global symmetry elevates to the gauge current via a simple covariantization. Is the same true for the two energy-momentum tensors?

The Noether procedure for space-time translations differ from that of the above internal symmetry, as they affect the space-time coordinates which in turn affect $\mathcal{L}$ as well. Let us consider how $\mathcal{L}(\phi, \partial_\mu\phi)$ is nominally affected by $\phi(x) \to \phi(x+\xi)$,

$$\begin{aligned}\boldsymbol{\delta}_\xi \mathcal{L} &= \frac{\delta\mathcal{L}}{\delta\phi}\xi^\alpha\partial_\alpha\phi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\partial_\mu(\xi^\alpha\partial_\alpha\phi) \;=\; \xi^\alpha\partial_\alpha\mathcal{L} + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\partial_\alpha\phi\,(\partial_\mu\xi^\alpha) \\ &\to (\partial_\mu\xi^\alpha)\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\partial_\alpha\phi - \delta^\mu{}_\alpha\mathcal{L}\right) \\ &\to \xi^\alpha\,\partial_\mu\left(-\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\partial_\alpha\phi + \delta^\mu{}_\alpha\mathcal{L}\right) \end{aligned} \tag{13.5.29}$$

upon integrations by part. This leads us to define the Noether energy-momentum,

$$\mathbb{T}^\mu{}_\alpha \;\equiv\; -\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\partial_\alpha\phi + \delta^\mu_\alpha\mathcal{L} \tag{13.5.30}$$

with its conservation law,

$$\partial_\mu\mathbb{T}^\mu{}_\alpha \;=\; 0 \tag{13.5.31}$$

For scalars, it is easy to see that $T_{\mu\nu}$ is a covariantization of $\mathbb{T}_{\mu\nu}$.

For gauge field, however, the simplest implementation of the above famously gives

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad\to\quad \mathbb{T}^\mu{}_\alpha \;=\; F^{\mu\lambda}\partial_\alpha A_\lambda - \frac{1}{4}\delta^\mu{}_\alpha F^2 \tag{13.5.32}$$

which not only differs from the flat limit of T but also is not gauge-invariant. There are various known remedies for this discrepancy. Perhaps the oldest such is adding $-\partial_\lambda(F^{\mu\lambda}A_\alpha)$ which is automatically conserved and then taking into account, if any, the charged matter contributions to $\mathbb{T}$. All these eventually bring us to the symmetric energy-momentum tensor, $T_{\mu\nu}$, however, so in this note, we will happily take T as the one and only sensible definition of the energy-momentum tensor, referring readers to literature for the above apparent tension and resolutions.

13.5.2 Ward Identities and Anomalies

We come back to the statement of how the action is invariant under the gauge transformation, and explore its consequences in the quantum regime. Note how the conservation of the Noether current,

$$d_{\mathcal{A}}\mathcal{J} = 0 \tag{13.5.33}$$

relies on the equation of motion, so it may appear that this useful fact would be lost in the quantum regime. However, this is not so, as we can see from the path integral representation of the quantum physics.

At quantum level, the most basic quantities to consider are correlators,

$$\langle\, T(\boldsymbol{\mathcal{O}}\cdots\boldsymbol{\mathcal{O}}')\,\rangle \;=\; \frac{\int[D\mathcal{A}][D\phi]\;\mathcal{O}\cdots\mathcal{O}' e^{\mathbb{i}S(\phi,\mathcal{A})/\hbar}}{\int[D\mathcal{A}][D\phi]\;e^{\mathbb{i}S(\phi,\mathcal{A})/\hbar}} \tag{13.5.34}$$

where the left hand side is by definition time-ordered expectation, and here we are yet to integrate over the gauge field. Now suppose we chose to integrate over ϕ first, leaving $\mathcal{A}$ as a classical background,

$$\langle\, \mathcal{O}\cdots\mathcal{O}'\,\rangle_{\mathcal{A}} \;=\; \frac{\int[D\phi]\;\mathcal{O}\cdots\mathcal{O}' e^{\mathbb{i}S(\phi,\mathcal{A})/\hbar}}{\int[D\phi]\;e^{\mathbb{i}S(\phi,\mathcal{A})/\hbar}} \tag{13.5.35}$$

where we also simplified the notation on the left side with the time ordering and quantization of $\mathcal{O}$'s taken for granted.

If the conservation law makes sense even at quantum level, we would expect

$$\langle\, (d_{\mathcal{A}}\mathcal{J})\,\mathcal{O}\cdots\mathcal{O}'\,\rangle_{\mathcal{A}} = 0 \tag{13.5.36}$$

for carefully chosen set of operators $\mathcal{O}$'s and at least

$$\langle\, d_{\mathcal{A}}\mathcal{J}\,\rangle_{\mathcal{A}} = 0 \tag{13.5.37}$$

as a consequence of

$$S(\phi,\mathcal{A}) = S(\phi+\boldsymbol{\delta}_\Theta\phi,\mathcal{A}+\boldsymbol{\delta}_\Theta\mathcal{A})\ . \tag{13.5.38}$$

The key question here is whether such a symmetry transformation leaves the integration measure intact.

Let us suppose for now that the path integral measure is also inert under such a change of variable, $\phi \to \phi + \boldsymbol{\delta}_\Theta \phi$, modulo a field-independent Jacobian. If this is the case, we should find

$$\begin{aligned} \int [D\phi]\; e^{\mathrm{i}S(\phi,\mathcal{A})/\hbar} &= \int [D\phi]\; e^{\mathrm{i}S(\phi+\boldsymbol{\delta}_\Theta\phi,\mathcal{A}+\boldsymbol{\delta}_\Theta\mathcal{A})/\hbar} \\ &= \int [D(\phi+\boldsymbol{\delta}_\Theta\phi)]\; e^{\mathrm{i}S(\phi+\boldsymbol{\delta}_\Theta\phi,\mathcal{A}+\boldsymbol{\delta}_\Theta\mathcal{A})/\hbar} \\ &= \int [D\phi]\; e^{\mathrm{i}S(\phi,\mathcal{A}+\boldsymbol{\delta}_\Theta\mathcal{A})/\hbar} \end{aligned} \tag{13.5.39}$$

modulo a field-independent Jacobian in the second step, while for the last step we used the fact that ϕ is a dummy variable for the integration.

Expanding the last expression in small but otherwise arbitrary Θ and using $\boldsymbol{\delta}_\Theta\mathcal{A} = d_\mathcal{A}\Theta$, this gives

$$\langle\, d_\mathcal{A}\mathcal{J}\,\rangle_\mathcal{A} \;=\; 0 \tag{13.5.40}$$

One can see that the same works with insertions of $\mathcal{O}$'s

$$\langle\, (d_\mathcal{A}\mathcal{J})\,\mathcal{O}\cdots\mathcal{O}'\,\rangle_\mathcal{A} = 0 \tag{13.5.41}$$

as long as $\mathcal{O}$'s are all gauge-invariant operators and constructed entirely out of ϕ. These quantum identities from the conserved current are called the Ward identities.

Let us come back to the question of how the conservation laws of the Noether current elevate to quantum statements. We see that three key observations are responsible for this. One is that we are integrating over the field ϕ, which then become dummy variables. The second key is the assumed invariance of the path integral measure, entirely separate from that of ϕ in the action. With these two combined, the equation of motion for ϕ has no place for the proof of the Ward identity. The last is that the divergence of the current in the Ward identity really results from the gauge variation of $\mathcal{A}$, rather than from the variation of ϕ, inside the action. In a sense, although we can of course eventually trace it to Noether's conservation laws, the Ward identity is more immediately the conservation law of the gauge current.

In reality, however, Ward identities can fail in the second step. One can for instance imagine highly nonlinear transformation rule, where $\boldsymbol{\delta}_\Theta\phi$ involves ϕ^2, $\phi\nabla\phi$. Whether or not the change of the integration variable could induce a Jacobian needs to be carefully considered first. A linear transformation such as our gauge rotations,

$$\boldsymbol{\delta}_\Theta\phi = -\Theta\phi \tag{13.5.42}$$

is seemingly safe enough for the measure since, often, both ϕ and its complex conjugate ϕ^* would enter the measure on equal footing,

$$[D\phi^* D\phi] \tag{13.5.43}$$

However, one does encounter a nontrivial phase shift of the measure, linear in Θ,

$$[D(e^{\Theta}\phi^*)D(e^{-\Theta}\phi)] = [D\phi^* D\phi]\, e^{\mathrm{i}(\cdots)} \tag{13.5.44}$$

which depends not on ϕ but on $\mathcal{A}$.

This way, given nontrivial $\mathcal{A}$ background the naive invariance under the gauge transformation can fail. Such a field-dependent phase throws a term on the right hand side of the Ward identities, say,

$$\langle\, d_{\mathcal{A}}\mathcal{J}\,\rangle_{\mathcal{A}} \;\neq\; 0 \tag{13.5.45}$$

Quantity on the right hand side are called the anomaly. This happens when ϕ's include some net number of so-called chiral fields; We have encountered an example of chiral fields in the previous section while discussing spinors, which will indeed give us the prototype example of anomalous Ward identities in a later chapter.

Gravitational Ward Identity

The same would apply for symmetries associated with the space-time. The local Lorentz transformation, which rotates the orthonormal frame and transforms the spin connection as a gauge field, is no different. For general coordinate transformations, however, some of details differ as its infinitesimal transformation rules are not quite the simple left-multiplication by a matrix. Rather we have

$$\boldsymbol{\delta}_\xi\phi = \mathfrak{L}_\xi\phi\ , \qquad \boldsymbol{\delta}_\xi g_{\mu\nu} = \mathfrak{L}_\xi g_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu \tag{13.5.46}$$

Transformation rules on the spin/Christoffel connections can be derived in principle from the metric transformation, as we have done in the previous volume.

Following the same line of logics as above, the would-be Ward identity is

$$\langle\, \nabla^\mu T_{\mu\nu} \,\cdots\, \rangle_g = 0 \tag{13.5.47}$$

for certain set of operators in the ellipsis, since the analog of $\boldsymbol{\delta}_\Theta \mathcal{A} \wedge \mathcal{J} = d_\mathcal{A}\Theta \wedge \mathcal{J}$ is

$$(\mathfrak{L}_\xi g^{\mu\nu})\, T_{\mu\nu}/2 \;=\; (\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu)\, T_{\mu\nu}/2 \;=\; \nabla^\mu \xi^\nu\, T_{\mu\nu} \;. \tag{13.5.48}$$

Again in the presence of "chiral" ϕ's, this Ward identity is also known to fail. In particular,

$$\langle\, \nabla^\mu T_{\mu\nu} \,\rangle_g \neq 0 \tag{13.5.49}$$

acquires a local expression via made from the metric data on the right hand side, called the diffeomorphism anomaly.

For the diffeomorphism Ward identity, there is an additional issue we need to keep in mind. Unlike gauge transformations, which acts within a particular gauge sectors, the diffeomorphim is universal in that it shifts not only the metric but all fields in the theory. If gauge fields are present, this induces additional variations of the action in the path integral and brings down other gauge current, in addition to the energy-momentum piece,

$$\sum \boldsymbol{\delta}_\xi \mathcal{A}_\mu \cdot \mathcal{J}^\mu \tag{13.5.50}$$

where the sum refers to gauge fields, regardless of internal or external while $\cdot$ here is used as a shorthand notation for space-time integration as well as the summation over internal gauge indices.

This shows that the left hand side of the Ward identity acquires addition terms as

$$\xi_\nu \cdot \langle\, \nabla^\mu T_{\mu\nu} \,\rangle_g \quad\rightarrow\quad \xi_\nu \cdot \langle \nabla_\mu \mathcal{T}^{\mu\nu} \rangle_{g,\mathcal{A}} - \sum_\mathcal{A} \boldsymbol{\delta}_\xi \mathcal{A}_\mu \cdot \langle \mathcal{J}^\mu \rangle_{g,\mathcal{A}} \tag{13.5.51}$$

Using

$$\mathcal{F} \;=\; d\mathcal{A}+\mathcal{A}^2 \;, \qquad \boldsymbol{\delta}_\xi \mathcal{A} \;=\; \mathfrak{L}_\xi \mathcal{A} \;=\; \xi \lrcorner\, d\mathcal{A} + d(\xi \lrcorner\, \mathcal{A}) \tag{13.5.52}$$

and performing another integration by part, the above translates to

$$\rightarrow \quad \xi_\nu \cdot \langle \nabla_\mu \mathcal{T}^{\mu\nu} \rangle_{g,\mathcal{A}} - \sum_{\mathcal{A}} \Big((\xi \lrcorner\, \mathcal{F})_\mu \cdot \langle \mathcal{J}^\mu \rangle_{g,\mathcal{A}} - (\xi \lrcorner\, \mathcal{A}) \cdot \langle D_\mu \mathcal{J}^\mu \rangle_{g,\mathcal{A}} \Big) \tag{13.5.53}$$

to be equated against the diffeomorphism anomaly.

The last expression inside the summation has a strange feature when $\mathcal{J}$ happens to be anomalous itself. In such cases, there are three contributing pieces here that are not covariant under the internal gauge transformations, namely,[‡]

$$\mathcal{J}^\mu \;, \qquad \xi \lrcorner\, \mathcal{A} \;, \qquad D_\mu \mathcal{J}^\mu \tag{13.5.54}$$

The middle one is obviously non-covarant, while the gauge current proves to be non-covariant when its own Ward identity is anomalous. If the diffoemorphism anomaly is absent, the above should be equal to zero, yet this would relate the energy-momentum, manifestly inert under the internal gauge transformations, to the gauge currents that transforms nontrivially and anomalously.

As such, even before we worry about the diffeomorphism anomaly that may appear on the right hand side, the left hand side of the Ward identity appears as if it is inconsistent by itself. The diffeomorphism anomaly is itself invariant under these internal gauge transformations, as it will turn out, so the same question remains regardless of whether the diffeomorphism Ward identity is anomalous or not.

This odd quandary has a satisfying resolution once we understand the general structure of the gauge anomaly $\langle D_\mu \mathcal{J}^\mu \rangle$. To make the long story short, the 2nd term and the 3rd term combine together such that

$$\xi_\nu \cdot \langle\, \nabla^\mu T_{\mu\nu} \,\rangle_g \quad \rightarrow \quad \xi_\nu \cdot \langle \nabla_\mu \mathcal{T}^{\mu\nu} \rangle_{g,\mathcal{A}} - \sum_{\mathcal{A}} (\xi \lrcorner\, \mathcal{F})_\mu \cdot \langle \mathcal{J}^\mu_{\rm cov} \rangle_{g,\mathcal{A}} \tag{13.5.55}$$

where $\mathcal{J}_{\rm cov}$ is the so-called covariant current, $\mathcal{J}^\mu_{\rm cov} = \mathcal{J}^\mu + \cdots$, with the ellipsis being a certain local expression of $\mathcal{A}$ called the Bardeen-Zumino current, constructed from

[‡]If the gauge anomaly cancels out, the last two drops out. Only $\langle \mathcal{J}^\mu \rangle_{g,\mathcal{A}}$ survives and the naive covariance of the current survives; the potential issue we note here removes itself.

the gauge anomaly of $\mathcal{J}$. As the name suggests, $\mathcal{J}_{\rm cov}$ transforms covariantly even in the presence of the anomaly, telling us that the left hand side was secretly invariant under internal gauge transformations, all along. This is one place where this so-called covariant current makes appearance in physical quantities. We will get back to this in a later chapter devoted to anomaly of continuous symmetries where we discuss both internal gauge/flavor symmetries and general coordinate invariance comprehensively.

Chapter 14

Euclidean Path Integrals

The fact that the path integral measure is pure phase adds difficulties. The path integral side of quantum theory becomes often far easier to handle as we move away from real time by analytic continuation whereby the integrand becomes more controllable. In fact, for the bulk of this note, we will deal with pure imaginary t and often confine our attention to the Gaussian functional integration thereof. Although the analytic continuation back to real time is fraught with dangers, a lot can be gained from considering quantum theory with such imaginary time. In this chapter, we will outline how this so-called "Wick rotation" is performed at the level of path integrals, serving as a cornerstone for many computations in the following chapters.

14.1 Wick-Rotation and Euclidean Metric

A path integral are ill-defined unless regularized and renormalized carefully; in particular most form of quantum field theories does not have a continuum limit, meaning an ability to remove the so-called ultraviolet cut-off and maintain sensible theory in the end. The gravity is a well-known such example, so even if we view the Einstein-Hilbert action as a part of a path integral,

$$\int [Dg]\; e^{\mathbb{i} S_{\rm EH}(g)/\hbar}\;, \tag{14.1.1}$$

such a form of quantum gravity has no hope of being consistent for several well known reasons. At best we should be view $S_{\rm EH}(g)$ as relevant saddle values of some unknown quantum gravity theory.

On the other hand, there are other quantum computations that can be performed with g given. For instance, consider the following formal expression,

$$e^{\mathbb{i}S_{\rm EH}(g)/\hbar}\times\int[D\phi]\;e^{\mathbb{i}S(\phi;g)/\hbar}\;, \tag{14.1.2}$$

and imagine that one managed to perform ϕ path integral completely. The result would be a functional of the metric g, say,

$$e^{\mathbb{i}S_{\rm EH}(g)/\hbar}\times e^{\mathbb{i}W_{\rm eff}(g)}\;. \tag{14.1.3}$$

More generally, let us consider gauge fields as well

$$e^{\mathbb{i}(S_{\rm EH}(g)+S_{\rm gauge}(\mathcal{A};g)/\hbar}\times\int[D\phi]\;e^{\mathbb{i}S(\phi;\mathcal{A},g)/\hbar}\;, \tag{14.1.4}$$

whereby we will find the action for the gravity and the gauge fields is shifted to

$$S_{\rm EH}(g)+S_{\rm gauge}(\mathcal{A};g)\quad\rightarrow\quad S_{\rm EH}(g)+S_{\rm gauge}(\mathcal{A};g)+\hbar W_{\rm eff}(\mathcal{A};g)\;, \tag{14.1.5}$$

due to "integrating out" the quantum field ϕ. The content of next few chapters would be mostly about $W_{\rm eff}$ and its analog where the path integral pf ϕ is performed incompletely and also where part of $\mathcal{A}$ is treated as quantum fluctuations on equal footing as ϕ.

Of course, computing $W_{\rm eff}(\mathcal{A};g)$ precisely is all but impossible task in general. However there are certain circumstances where part of its form is computable and leads to well-defined and unambiguous physical effect. One well-known such is the matter of anoamlies, failure of classical symmetry to elevate to the quantum level. These prove to be important handles in exploring superstring theories and the space-time effective theories thereof in diverse dimensions. Another such example is when the space-time has $d=2$, in which case the effective action due to quantum matter can be computed exactly. This also leads to a new derivation of the above Hawking radiance, as we will later demonstrate in this note.

More than anything else, the notion of the effective action proves to be quite useful when we try to understand the quantum field theories themselves, in the sense of the renormalization flow. Later in this part, we will rely on the effective action as part of data necessary to define what we mean by quantum field theory, although we will limit our scope to Gaussian approximations in favor of the conceptual simplicity.

14.1.1 Wick Rotation (II)

Much of the path integral computations can be carried out in a simpler manner by analytically continuing to the Euclidean signature. Since this inevitably replaces real time by an imaginary one, we need to familiarize ourselves to the general idea and how it works for geometry. Let us recall how the Wick rotation should be performed for matter fields, say a single real scalar ϕ. The process involves replacing a real time coordinate t by an imaginary time t_E, by identifying $t = -\mathbb{i}t_E$ such that the metric become of positive definite signature.

Let us concentrate on factorized form of the metric, for simplicity

$$g_{tt}(x)dt^2 + g_{ij}(x)d\mathbf{x}^i d\mathbf{x}^j \ , \qquad g_{tt} < 0 \ . \tag{14.1.6}$$

In the Lorentzian signature, the exponent of the path integral integrand is

$$\mathbb{i}S_{\text{matter}} = \mathbb{i}\int dtd^{d-1}\mathbf{x}\ \sqrt{-\det g}\left(-\frac{1}{2}(\nabla\phi)^2 - V(\phi)\right) \ , \tag{14.1.7}$$

where the sign of the kinetic follows from the $(-,+,\cdots,+)$ signature. The Wick rotation is then performed by $t = -\mathbb{i}t_E$,

$$\mathbb{i}S_{\text{matter}} \quad \to \quad -S^E_{\text{matter}} = -\int dt_E d^{d-1}\mathbf{x}\ \sqrt{\det g^E}\left(\frac{1}{2}(\nabla^E\phi)^2 + V(\phi)\right) \tag{14.1.8}$$

which makes sense For the Lagrangian density, the replacement $\partial_t \to \mathbb{i}\partial_{t_E}$ has the same effect as $\partial_t \to \partial_{t_E}$ and $g_{tt} \to g_{t_E t_E} = -g_{tt}$. We denote this new metric of Euclidean signature by g^E and the resulting covariant derivative by ∇^E.

What happens to the Einstein-Hilbert action along this procedure? Not much, actually. The curvature tensor would analytically continue under $g_{tt} \to g_{t_E t_E} = -g_{tt}$ naturally, so the curvature tensor remain the same expression constructed from g^E. $\sqrt{-\det g}$ becomes $\sqrt{\det g^E}$ as well, so the only explicit change in the action is $\mathbb{i}t \to t_E$. This means the exponent of the path integral becomes

$$\mathbb{i}S_{\text{EH}} \quad = \quad \frac{1}{16\pi G_N}\int dt\, d^{d-1}\mathbf{x}\ \sqrt{-\det g}\ R$$

$$\rightarrow \qquad -S^E_{\rm EH} \;=\; \frac{1}{16\pi G_N}\int dt_E\, d^{d-1}\mathbf{x}\,\sqrt{\det g^E}\, R^E\ . \tag{14.1.9}$$

This means the Einstein-Hilbert action respond to the Wick rotation much like the potential term in the matter action.

In fact, the same should work for any action so that

$$S = -S^E\Big|_{g^E\to g,\; t_E\to t}\ , \tag{14.1.10}$$

where we blindly replaced the metric and the time coordinate of the Euclidean side to those of the Lorentzian side. This also means that the same happens for the effective action, so once it is computed in Euclidean signature,

$$W^E_{\rm eff}(g^E)\ , \tag{14.1.11}$$

its Lorentzian counterpart is determined as

$$W_{\rm eff}(g) = -W^E_{\rm eff}(g^E)\Big|_{g^E\to g,\; t_E\to t}\ . \tag{14.1.12}$$

For most part of this chapter, we work with the Wick rotated theory; sometimes we may drop the superscript E from the metric g^E, occasionally, when the distinction is clear from the context.

Now we need to be careful. Given the Lorentzian signature, the operator is hyperbolic and does not really accept a conventional eigenvalue problem. Also the exponent is $\mathbb{i}S/\hbar$, so the integrand is not suppressed at large values of S or at large values of ϕ. Many of these problems have a simple solution via the so-called Wick rotation, which, in the simplest possible settings, takes $t\to -\mathbb{i}t_E$ and maps the geometry to one with the Euclidean signature. This also shifts,

$$\mathbb{i}\int dt\,(\cdots) \qquad \rightarrow \qquad -\int dt_E\,(\cdots)\ , \tag{14.1.13}$$

so the Wick rotated action S_E appears in the path integral as

$$\int [D\phi]\; e^{-S^E(\phi;g^E)/\hbar}\ , \tag{14.1.14}$$

where g^E is, as before, of the Euclidean signature.

Now the eigenvalue problem can be well-defined, modulo further subtleties coming from the infinite volumes of the space-time. Denoting the result of such an integration as

$$e^{-W^E_{\text{eff}}(g^E)}\ , \tag{14.1.15}$$

we now have a shift of the Einstein-Hilbert action,

$$S^E_{\text{EH}}(g^E) \quad\rightarrow\quad S^E_{\text{EH}}(g^E) + \hbar W^E_{\text{eff}}(g^E) \tag{14.1.16}$$

Extension of the above discussion as well as what follows to scalars coupled to a gauge field $\mathcal{A}$ is straightforward. All we need to do is to use the new covariant derivative,

$$\nabla_\mu \quad\rightarrow\quad D_\mu \equiv \nabla_\mu + \mathcal{A}_\mu \tag{14.1.17}$$

and put ϕ in some representation under the gauge algebra. We would be computing the shift of the action as

$$S^E_{\text{EH}}(g^E) + S^E_{\text{gauge}}(\mathcal{A};g^E) \quad\rightarrow\quad S^E_{\text{EH}}(g^E) + S^E_{\text{gauge}}(\mathcal{A};g^E) + \hbar W^E_{\text{eff}}(\mathcal{A};g^E) \tag{14.1.18}$$

We will come back to this more general class of theories in the next chapter, accompanied by a little more detailed discussion of Yang-Mills gauge group and the representations thereof. Henceforth, we may often drop the superscript E that stand for the "Euclidean signature" whenever the confusion is unlikely.

At the end of this chapter, we will consider the simplest class of quantum mechanics and demonstrate how the Wick rotated path integral reproduces the more familiar Schrödinger result precisely, including the overall normalization. The quantity we compute there is the partition function, for which the imaginary time span is nothing but the inverse temperature in unit of $\hbar$. We will also see how the question of overall numerical factors are often effortless built-in on the canonical, or Schrödinger side while obtaining the same on the path integral side often requires further work. Nevertheless, this toy model would serve well in persuading readers how the two viewpoints are equivalent.

14.2 Gaussian Functional Integral

The question is then how one might compute such one-loop effect under general curved background. In this section, we will consider a single real scalar coupled minimally to the external metric g. In principle we may resort to Feynman diagrams, to be introduced later, but for this particular one-loop effect, there is a more coherent method that can accommodate the geometrical properties cleanly and covariantly.

This method can be illustrated, again via a ordinary Gaussian integration as

$$e^{-W_{\rm toy}} \equiv \int [d\mathfrak{b}]\; e^{-\mathfrak{b}^T M\mathfrak{b}/2} = \frac{(2\pi)^{N/2}}{({\rm Det}M)^{1/2}}\ , \tag{14.2.1}$$

for $N\times N$ Hermitian matrix M and a column vector $\mathfrak{b}$ of rank N. The key formula is

$$W_{\rm toy} = \frac{1}{2}\log\left({\rm det}M\right) = \frac{1}{2}{\rm tr}\log\left(M\right)\ , \tag{14.2.2}$$

which is then represented by an integral

$${\rm tr}\log\left(M\right) \simeq -{\rm Tr}\int_0^\infty \frac{ds}{s}\; e^{-sM} = -\sum\int_0^\infty \frac{ds}{s}e^{-s\lambda}\ . \tag{14.2.3}$$

with

$$M = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ 0 & 0 & \lambda_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}\ . \tag{14.2.4}$$

In the formula above, $\simeq$ means that we drop a log divergence at the lower end of integral which is independent of eigenvalue λ's.

To see this better, let us take a λ derivative and then perform an indefinite integration thereof,

$$\int d\lambda\, \partial_\lambda\left(-\int_0^\infty \frac{ds}{s}e^{-s\lambda}\right) = \int d\lambda\ \int_0^\infty ds\; e^{-s\lambda} = \int d\lambda\,\frac{1}{\lambda} = \log\lambda \tag{14.2.5}$$

which implies

$$W_{\text{toy}} = -\frac{1}{2}\int_0^\infty \frac{ds}{s}\ \text{tr}\ e^{-sM}\ , \tag{14.2.6}$$

modulo a divergence independent of M.

When we elevate this to the functional integral, the basic concept remain the same but of course the matrix M is replaced by an operator with an infinite number of eigenvalues. If the column vector $\mathfrak{b}$ is replaced by a massive field $\phi(x)$, the natural thing to do is to expand ϕ in an appropriate Fourier basis,

$$\phi(x) = \sum_p \mathfrak{b}_p f_p(x)\ , \tag{14.2.7}$$

such that

$$S^E(\phi) \to \frac{1}{2}\mathfrak{b}^T M \mathfrak{b} \tag{14.2.8}$$

for some infinite dimensional matrix M. With

$$S^E(\phi; g) = \frac{1}{2}\int \sqrt{g}\ \phi(-\nabla^2 + m^2)\phi\ , \tag{14.2.9}$$

we have

$$M_{pq} = \int \sqrt{g}\, f_p^*(-\nabla^2 + m^2) f_q\ . \tag{14.2.10}$$

This formal expression is valid for any operator with positive eigenvalues only. When the operator admits a finite dimensional kernel, we can still extend the formulation easily. For now, we will pretend that the kernel is null.

This leads to

$$W_{\text{eff}}^E = -\frac{1}{2}\int_0^\infty \frac{ds}{s}\ \text{Tr}\ e^{-s(-\nabla^2+m^2)}\ , \tag{14.2.11}$$

where we used the capital Tr to emphasize that we are performing a functional trace. Since eigenvalues are unbounded from above, the above formula does not really make sense due to divergence associated with the lower end of ds integration. One can deal with this divergences by introducing an "ultraviolet cut-off" and defining instead

$$-\frac{1}{2}\int_{\epsilon}^{\infty}\frac{ds}{s}\,\mathrm{Tr}\,e^{-s(-\nabla^2+m^2)}\;. \tag{14.2.12}$$

For those who have studied quantum field theory, this ϵ would be eventually related to the momentum cut-off $\mathbf{\Lambda}$ as $\epsilon = 1/\mathbf{\Lambda}^2$. What this cut-off of the eigenvalues means need further consideration, to be discussed in due time.

For now, we will simply say that the path integral does not make sense without such a cut-off. In particular, one should NOT consider it a mathematical artefact to be removed later. In fact, the cut-off should be considered a part of the definition of the theory. More generally, we will need compute

$$-\frac{1}{2}\int_{1/\mathbf{\Lambda}^2}^{1/\boldsymbol{\mu}^2}\frac{ds}{s}\,\mathrm{Tr}\,e^{-s(-\nabla^2+m^2)}\;, \tag{14.2.13}$$

with another cut-off at the low energy side.

$W^E_{\rm eff}$ is a non-local functional of g and, in general, impossible to compute precisely. A more feasible task is to expand the operator $e^{-s(-\nabla^2+m^2)}$ systematically in power of s, which is is called the Heat Kernel expansion, or Schwinger-de Witt expansion. More concretely, we may adopt the position basis in the usual quantum mechanical sense,

$$|x\rangle\;, \tag{14.2.14}$$

with the target being the Euclidean space-time. The trace Tr may be understood as the one over the quantum mechanics Hilbert space relevant for ∇^2 as the Hamiltonian, and can be evaluated as

$$\mathrm{Tr}\,(\cdots) = \int d^dx\,\langle x|\cdots|x\rangle\;, \tag{14.2.15}$$

A key tool for this is the heat kernel

$$G_s(x;y) \equiv \langle x|e^{-sQ}|y\rangle = \sum_{n=0}^{\infty} G_s^{(n)}(x;y) \tag{14.2.16}$$

with $\mathcal{Q} \equiv -\nabla^2 + m^2$ for the current example, which obeys

$$-\partial_s G_s(x;y) = \mathcal{Q} G_s(x,y) \ , \tag{14.2.17}$$

where the hermitian $\mathcal{Q}$ can act either on x or on y.

To find a consistent expansion, we need to split the operator in the exponent into two parts,

$$\mathcal{Q} = \bar{\mathcal{Q}} + \delta\mathcal{Q} \ , \tag{14.2.18}$$

which gives the recursion relation,

$$-\partial_s G_s^{(n+1)}(x;y) = \bar{\mathcal{Q}}\, G_s^{(n+1)}(x,y) + \delta\mathcal{Q}\, G_s^{(n)}(x,y) \tag{14.2.19}$$

with

$$-\partial_s G_s^{(0)}(x;y) = \bar{\mathcal{Q}}\, G_s^{(0)}(x,y) \ , \qquad \lim_{s\to 0} G_s^{(0)}(x,y) = \delta^d(x-y) \ . \tag{14.2.20}$$

This recursion relation can be integrated into

$$G_s^{(n)}(x,y) = -\int_0^s dt \int_z G_{s-t}^{(0)}(x;z)\, \delta\mathcal{Q}\, G_t^{(n-1)} G_t^{(n)}(z;y) \ , \tag{14.2.21}$$

and can be in turn iterated into as

$$\begin{aligned} G_s^{(n)}(x;y) \;\; = \;\; & (-1)^n \int_0^s dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \int_{z_1} \cdots \int_{z_n} \\ & G_{s-t_1}^{(0)}(x;z_1)\, \delta\mathcal{Q}\, G_{t_1-t_2}(z_1;z_2) \, \cdots \, \delta\mathcal{Q}\, G_{t_n}(z_n;y) \ , \end{aligned} \tag{14.2.22}$$

which, in principle, gives a complete solution.