MODEL

① Cast of characters

A_μ : $U(1)_A$ background EM field

a_μ : $U(1)_a$ dynamical gauge field (emergent)

χ : Dirac fermion of $(1, 2s)$ why even numbers? → will be discussed later

$U(1)_A$ $U(1)_a$

ω : Higgs field of $(0, 1)$

ϕ : Higgs field of $(2, 2s)$

consistent with

Spin / $U(1)_A$ charge relation. (NOT a fundamental property, (e.g. neutron, neutrino) but ...)

- states of odd $U(1)_A$ charge have half-integral spin
- states of even $U(1)_A$ charge have integer spin

this relation is only the case in the condensed matter physics since ~~many~~ such systems are made of electrons only!

(2) <u>interactions</u>

(i) Yukawa coupling

T-invariance?

$$\mathcal{L}_{Yuk} = i\,\underbrace{\phi^*}_{(-2,-4S)}\, \underbrace{\chi^T \gamma^0 \chi}_{(2,4S)} + c.c.$$

$T : \chi^T \gamma^0 \chi(t,\vec{x})$

$\mapsto \chi^T \gamma^0 \chi(-t,\vec{x})$

to have a T-inv. Yukawa coupling.

$$T : \phi(t,\vec{x}) \longmapsto -\phi(-t,\vec{x})$$

) scalar potential

$$V(\phi, \omega) = (|\phi|^2 - \xi_1^2)(|\omega|^2 - \xi_2^2) + |\omega|^2|\phi|^2$$

s.t. $\exists$ two broken phases in the IR

(a) phase I. $\langle \omega \rangle = \xi_2$ $\langle \phi \rangle = 0$

$(0,1)$

$T : \omega_{(t,x)} \longmapsto \omega(-t\cdot$

s.t.

$\langle \omega \rangle = \xi_2$ does n

breaks the T-in

--- $\cancel{U(1)}_a$ & ϕ becomes massive

χ remains massless

standard bo

state of

top'l insulat

(b) phase II $\langle \phi \rangle = \xi_1$. $\langle \omega \rangle = 0$.

$(2, 4S)$

- $U(1)_a \longrightarrow \mathbb{Z}_{4S}$ $\quad U(1)_A \longmapsto \mathbb{Z}_2$

- $U(1)_A$ remains unbroken if $\underbrace{\tilde{Q}_A = Q_A - Q_a/2S}$ dressed by the $U(1)_a$ gauge charge.

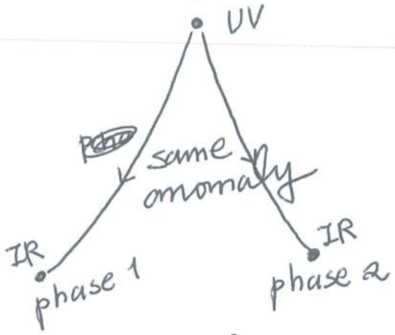
- T-symmetry ~~remains unbroken~~ appears broken ~~but~~

But $\tilde{T} = T \cdot \underbrace{e^{2\pi i/8S \cdot Q_a}}_{U(1)_a \text{ gauge}}$ is unbroken!

$$\tilde{T}^2 = T \cdot e^{2\pi i/8S \cdot Q_a}\, T\, e^{2\pi i/8S \cdot Q_a}$$

$$= T^2 = (-1)^F$$

- ω & x becomes massive.



$\Rightarrow$ $\therefore$ { phase II is a gapped phase, BUT SHARES the same anamaly with phase I. }

a (certain) top'l field theory describes the IR physics of the phase II.

③ quasi-particles in the phase II

$\langle \phi \rangle \neq 0$ eaten by a_μ (Higgs mech.)

$$U(1)_A \times \cancel{U(1)_a} \longrightarrow \widetilde{U(1)}_A \times \mathbb{Z}_{4s}$$

$$\tilde{Q}_A = Q_A - Q_a/2s$$

(i) no vortex

a. χ : fermion of charge $\underset{U(1)_A \ U(1)_a}{(1, 2s)} \longrightarrow \underset{\widetilde{U(1)}_A ,\ \mathbb{Z}_{4s}}{(0, 2s)}$

ω : scalar of charge $(0, 1) \longrightarrow (-\frac{1}{2s}, 1)$

b. relations

bulk electron

$\chi^2 = 1$, $\bar{\omega}^{2s}\chi = e$, $\omega\bar{\omega} = 1$

$\underset{\widetilde{U(1)}_A \ \mathbb{Z}_{4s}}{(0, 0)}$ $\quad \underset{\text{spin}\frac{1}{2}}{(1, 0)} \quad \underset{\text{spin}\frac{1}{2}}{(1, 0)}$

(fermion mass term)
from the Yukawa coupling

also, $\underset{U(1)_A \ U(1)_a}{(1, 0)} \simeq (1, 0)$

mass term
Potential
$|\omega|^2|\phi|^2$ term

finally,

in the context of
low-energy
top'l field theory

$$(\bar{\omega}^{2s}\chi)^2 = \bar{\omega}^{4s} \cdot \cancelto{1}{\chi^2} = e^2 = 1$$

spin 0 mod $\mathbb{Z}$

$$\therefore \quad \omega^{4s} = \bar{\omega}^{4s} = 1$$

(iii) Vortex with $\nu = 2$

two fermionic zero modes $\quad \gamma^+$ spin $j' = \frac{1}{2}$

γ^- spin $j' = -\frac{1}{2}$

canonical quantization $\rightarrow$ $\{ \gamma_i, \gamma_j \} \propto \delta_{ij}$

<u>degenerate states</u> $\quad \gamma^+ | j' \rangle$ spin $j' + 1/2$

$| j' \rangle$ spin $+j'$ $\quad$ $\widetilde{CT}$-sym. spectrum.

$$-(j' + 1/2) = j' \quad \therefore \quad j' = -1/4$$

thus $\left\{ |-\tfrac{1}{4}\rangle, \ |+\tfrac{1}{4}\rangle \right\}$ semion.

action of $\mathcal{X}$

Other quasi-particles can be generated by the action of ω^n on the above two degenerate states

ω has spin $j' = \frac{1}{2S}$

$\omega^n \cdot | j' = \pm \frac{1}{4} \rangle$

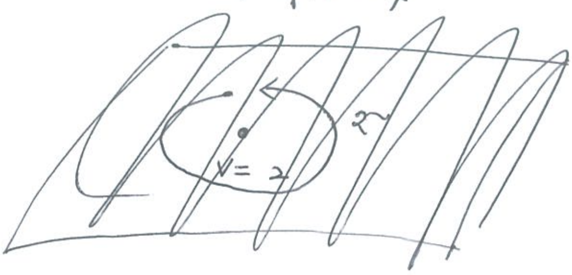
$\omega^n | -\frac{1}{4}$

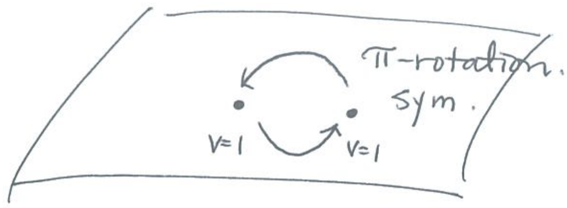
$$j' = \frac{n}{2S} \pm \frac{1}{4} \qquad \widetilde{U(1)}_A \text{ charge} = -\frac{n}{2S}$$

<u>remark</u> eigenvalues of braiding matrix B_{12}.

V=2 vortex can be slightly separated by two V=1 vortex.

two degenerate states of spin $\pm\frac{1}{4}$ $\therefore$ non-abelian statistic.





B_{12}

$$e^{\pi i\left(2p \pm \frac{1}{4}\right)} \doteq \text{eigenvalues of } B_{12}$$

~~see the meaning of B_{12}~~

j' is not conserved, but conserved mod 2 ($\because$ the system is sym. under the π-rotation)

(iv) $V = 3$ vortex

3 fermionic zero modes $\xrightarrow[\text{quantize them}]{}$ $\begin{cases} \gamma^+ & \text{spin } +1 \\ \gamma^0 & \text{spin } 0. \\ \gamma^- & \text{spin } -1 \end{cases}$

two-dim'l rep. $|+\tfrac{1}{2}\rangle = \gamma^+ |-\tfrac{1}{2}\rangle$ spin increased by +1

$|-\tfrac{1}{2}\rangle$

exchanged by the action χ

this mode has v.e.v. in the vortex state

$\gamma^0 \doteq \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

it induces the non-abelian statistics

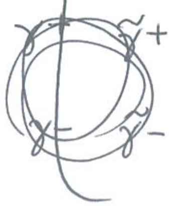
other quasi-particles are generated by action w/ ω^n

	$\widetilde{U(1)}_A$	j'
$\omega^n \lvert j' = \pm\tfrac{1}{2}\rangle$	$-\dfrac{n}{25}$	$\pm\tfrac{1}{2} + 3n/45$

(v) $V = 4$ vortex

4 fermionic zero modes

$\begin{cases} \gamma^+ & \text{spin } +\tfrac{3}{2} \\ \tilde{\gamma}^+ & \text{spin } +\tfrac{1}{2} \\ \tilde{\gamma}^- & \text{spin } -\tfrac{1}{2} \\ \gamma^- & \text{spin } -\tfrac{3}{2} \end{cases}$



four dim'l rep. : $\left\{ \begin{array}{l} |j_0+2\rangle \\ |j_0+1/2\rangle,\ |j_0+3/2\rangle \\ |j_0\rangle \end{array} \right\}$

must be inv. under $\widetilde{CT}$ transf.

$$-(j_0+2) = j_0 \rightarrow j_0 = -1$$

consistent

$$-(j_0+\tfrac{3}{2}) = j_0+1/2 \rightarrow j_0 = -1$$

quasi-particles

$\underbrace{\{|-1\rangle, |1\rangle\}}_{\text{bosons}} \cup \underbrace{\{|-\tfrac{1}{2}\rangle, |+\tfrac{1}{2}\rangle\}}_{\text{fermions.}}$

& $\omega^n \{ |-1\rangle, |1\rangle, |-\tfrac{1}{2}\rangle, |+\tfrac{1}{2}\rangle \}$

$\widetilde{U(1)_A}$ $\quad -\frac{n}{2S}$

j' $\quad \pm 1 + \frac{n}{S}$ & $\pm\frac{1}{2} + \frac{n}{S}$.

<u>remark</u>

$V = 0, 1, 2, 3, 4, 5, 6, 7, 8$ <u>symmetric</u> in the condensed matter physics.

$\boxed{\therefore\ V = 0, 1, 2, 3, 4 \quad \text{Done!!}}$

$\nu \cong \nu + 8$ for $C\tilde{T}$ invariant system.

$\nu \cong 8 - \nu$ because $\tilde{T} : \nu \to -\nu$

(flux gets ~~fllped~~ flipped under τ)

see the section of monopole op's !!

WHY ?

(i) One can argue that the theory of our interest also has a monopole op. $\underbrace{\mathcal{O} \ \& \ \mathcal{O}^{+}}_{\text{scalar op's}}$

$\Rightarrow \Delta\mathcal{L} = \mathcal{O} + \mathcal{O}^{+}$

one can add to the theory $\Delta\mathcal{L}$ interaction

it can creates or annihilate the vortex of $\nu = 4S$ that carries the unit flux

thus,

$$\boxed{\nu \cong \nu + 4S} \ !!$$

even

~~further reduction~~

(ii) $C\tilde{T}$-invariant perturbation $\boxed{\nu \cong \nu + 8}$

$\Delta\mathcal{L} = \sum_{i=1}^{8} p_{ijkl}\, \gamma^{i}\gamma^{j}\gamma^{k}\gamma^{l}$, where γ^{i} are fermion zero modes

④ topological field theory describing the phase II in the IR limit.

(i) Ising TFT $\times$ $\mathbb{Z}_{4S,2S^2}$ Dijkgraaf-Witten theory $/ \mathbb{Z}_2^{diag}$.

Ising TFT observables

W_1 spin 0 (identity)

W_ψ spin $\frac{1}{2}$

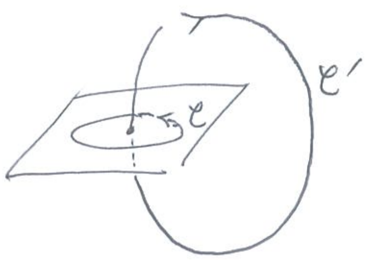
W_σ spin $\frac{1}{16}$

they satisfy the following relations

$$\begin{cases} W_\psi^2 = 1 \\ W_\sigma^2 = W_1 + W_\psi \end{cases} \qquad W_\sigma W_\psi = W_\sigma$$

fusion algebra
non-abelian statistics.

Finally, note that the Ising TFT has the $\mathbb{Z}_2$ symmetry ;



$$W_\psi[\mathcal{C}]\, W_\psi[\mathcal{C}'] = (+1)\, W_\psi[\mathcal{C}']$$

$$W_\psi[\mathcal{C}]\, W_\sigma[\mathcal{C}'] = (-1)\, W_\sigma[\mathcal{C}']$$

$$W_\psi[\mathcal{C}]\, W_1[\mathcal{C}'] = (+1)\, W_1[\mathcal{C}']$$

$U_{g=-1}[\mathcal{C}] = W_\psi[\mathcal{C}]$: $\mathbb{Z}_2$ generator

$\mathbb{Z}_{4s,\,2s^2}$ Dijkgraaf-Witten theory

$$\frac{4s}{2\pi}\int b \wedge da \quad + \quad \frac{2s^2}{4\pi}\int a \wedge da$$

$$W_{(n_a,n_b)}[\mathcal{C}] = e^{\,i n_a \int_{\mathcal{C}} a \,+\, i n_b \int_{\mathcal{C}} b}$$

then, it is easy to show that

$$W_{(n_a,n_b)}[\mathcal{C}]\, W_{(n_a',n_b')}[\mathcal{C}']$$

$$= \exp\left[2\pi i \left\{ \frac{n_a n_b'}{4s} + \frac{n_b n_a'}{4s} - \frac{n_b n_b'}{8} \right\}\right] W_{(n_a',n_b')}[\mathcal{C}']$$

$$n_a = \frac{2s^2}{2\pi} da + \frac{4s}{2\pi} db$$

$$n_b = \frac{4s}{2\pi} da$$

$$\therefore \begin{cases} \dfrac{1}{2\pi} da = \dfrac{n_b}{4s} \\[2ex] \dfrac{1}{2\pi} db = \dfrac{n_a}{4s} - \dfrac{n_b}{8} \end{cases}$$

$\cancel{W_{4s}}$

$$\begin{cases} W_{n_a,n_b}[\mathcal{C}] \quad \text{spin} \quad \dfrac{n_a n_b}{4s} - \dfrac{n_b^2}{\cancel{8}\,16} \\[2ex] W_{(4s,0)}[\mathcal{C}] = 1 \end{cases}$$

the above Dijkgraaf-Witten theory has

$\mathbb{Z}_2$ symmetry

$\cancel{W\ [\mathcal{C}]\ W_{n_a,n_b}[\mathcal{C}']}$

$$W_{(2s,0)}[\mathcal{C}]\; W_{(n_a,n_b)}[\mathcal{C}'] = e^{2\pi i \frac{2s}{4s} n_b} \cdot W_{(n_a,n_b)}[\mathcal{C}']$$

$\mathbb{Z}_2$ generator

$$= (-1)^{n_b}\, W_{(n_a,n_b)}[\mathcal{C}']$$

$\mathbb{Z}_2^{diag}$ gauging

Both theories have $\mathbb{Z}_2$ one-form global symmetry

$\mathbb{Z}_2^{diag}$ generator $\quad U_{g=-1}[\mathcal{C}] = W_{(25,0)}[\mathcal{C}]\, W_\psi[\mathcal{C}]$

After $\mathbb{Z}_2^{diag}$ gauging. $\quad \dfrac{\text{Ising TFT} \times \mathbb{Z}_{45,252}\ \text{DW}}{\mathbb{Z}_2^{diag}}$

has the following observables ;

$\mathbb{Z}_2^{diag}$ invariants

$$\begin{cases} W_{(n_a,n_b)}[\mathcal{C}] & n_b = \text{even} \quad \text{spin} = \dfrac{n_a n_b}{45} - \dfrac{n_b^2}{16} \mod \mathbb{Z} \\ W_{(n_a,n_b)}[\mathcal{C}]\, W_\psi[\mathcal{C}] & n_b = \text{even} \quad \text{spin} = \dfrac{n_a n_b}{45} - \dfrac{n_b^2}{16} + \dfrac{1}{2} \mod \mathbb{Z} \\ W_{(n_a,n_b)}[\mathcal{C}]\, W_\sigma[\mathcal{C}] & n_b = \text{odd} \quad \text{spin} = \dfrac{n_a n_b}{45} - \dfrac{n_b^2}{16} + \dfrac{1}{16} \mod \mathbb{Z} \end{cases}$$

@

$W_{(1,0)}[\mathcal{C}]$ is identified ~~with~~ as ω

spin 0.

check $\quad \left(W_{(1,0)}[\mathcal{C}]\right)^{45} = W_{(45,0)}[\mathcal{C}] = 1$

$\sim \quad \omega^{45} = 1$

(b) $W_\psi[\ell]$ can be identified as χ ($\hat{\nu}=0$)

spin $1/2$

check $\quad (W_\psi[\ell])^2 = 1 \quad \sim \quad \chi^2 = 1$

(c) $W_{(n_a,0)} W_\psi[\ell] \quad \sim \quad \omega^{n_a}\chi \quad (\hat{\nu}=0)$

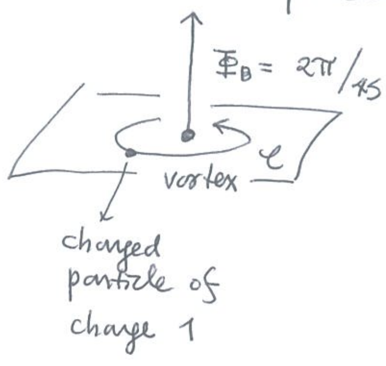
(d) $W_{(0,1)} W_\sigma[\ell]$ single Majorana zero-mode $\quad$ spin $= 0$

this charge can be identified as the vorticity $\hat{\nu}$

? Why ?

$$W_{(1,0)}[\ell]\, W_{(0,1)}[\ell'] = e^{2\pi i \left(\frac{1}{4S}\right)} W_{(0,1)}[\ell']$$

charged particle $\quad$ vortex with $\hat{\nu}=1$ $\quad$ Aharanov–Bohm phase



check

$$\left(W_{(0,1)} W_\sigma[\ell]\right)^2 = W_{(0,2)}\left(W_1 + W_\psi\right)[\ell]$$

$$= W_{(0,2)}[\ell] + W_{(0,2)} W_\psi[\ell]$$

state $|-\frac{1}{4}\rangle$ in the backround $\hat{\nu}=2$ $\quad$ spin $-\frac{1}{4}$ $\quad$ spin $+\frac{1}{4}$ $\quad$ $|+\frac{1}{4}\rangle \sim \chi|-\frac{1}{4}\rangle$

$W_{(0,2)}[\mathcal{C}]$ & $W_{(0,2)} W_\psi [\mathcal{C}]$; doublet in the background with $\hat{\mathbb{V}} = 2$

e $\quad W_{(n_a,1)} W_\sigma [\mathcal{C}] \quad \sim \quad \omega^n |j'=0\rangle_{\hat{\mathbb{V}}=1}$

<u>remark</u> $\quad W_{(0,1)} W_\sigma [\mathcal{C}] \times W_\psi [\mathcal{C}] \simeq W_{(0,1)} W_\sigma [\mathcal{C}]$

this is consistent with the fact that

adding χ to the vortex with $\hat{\mathbb{V}} = 1$ is <u>trivial</u>

$\hat{\mathbb{V}} = 1$ has a single state & fermion zero mode $\gamma \doteq \pm 1$

$\vdots$

$$\therefore \left\{ \begin{array}{l} \text{quasi-particles} \\ \text{in the phase II} \end{array} \right\} = \left\{ \begin{array}{l} \text{observables of} \\ \text{Ising} \times \mathbb{Z}_{45,25^\circ}\ \text{DW} / \mathbb{Z}_2^{\text{dog}} \end{array} \right\} \ !!$$

⑤ Why $\text{Ising} \times \mathbb{Z}_{(4s,\,2s^2)}\,\text{DW} \big/ \mathbb{Z}_2^{\text{diag}}$. ?

the corresponding action is

$$S_{TFT} = -\frac{\pi}{2}\,\eta\,(A + 2sa) + \int_N \frac{2s^2}{\cancel{4}\pi}\, a \wedge da + \frac{s}{2\pi} A \wedge da$$

$2sa$: $\mathbb{Z}_2$ gauge field.

(i) where does it come from?

In the ~~pal~~ phase II, χ ~~only~~ contributes ~~to~~ the T-violating phase to the partition function!

$$D_\mu \chi = (\partial_\mu - \cancel{2}\, i A_\mu - \underbrace{i(2s)a_\mu}_{\mathbb{Z}_2 \text{ gauge field}})\,\chi$$

&

$U(1)_a \to \mathbb{Z}_{4s}$

$$\Rightarrow \; Z_{\text{phase II}} = |Z_{\text{phase II}}|\, e^{-\frac{\pi}{2} i \eta (A + \overset{\mathbb{Z}_2 \text{ gauge field}}{2sa})}\, \underset{??}{(\cdots)}$$

(ii) to determine $(\cdots)$, let's consider ~~the~~ ~~an~~ ~~(expected)~~ the APS index thm again, which is crucial to make the total partition function <u>real</u>!

(17)

$$Z'_{tot} = |Z_{tot}|\, e^{-\frac{\pi}{2} i \eta (A+2sa) + \underbrace{i\pi \int_M \hat{A}(R) + \frac{1}{2}\left(\frac{dA+2s\,da}{2\pi}\right)^2}}$$

$$= \underbrace{e^{i\pi \int_M \hat{A}(R) + \frac{1}{2}\left(\frac{F}{2\pi}\right)^2}}_{\text{top'l insulator}} \times$$

$$e^{i\pi \int_M 2s^2 \cdot \left(\frac{da}{2\pi}\right)^2 + 2s\left(\frac{dA'}{2\pi}\right)\wedge\left(\frac{da}{2\pi}\right)}$$

"even" to place the TFT on Spin$_c$ mfd

$$= |Z_{tot}| \exp\Big[\underbrace{-\frac{\pi}{2} i \eta (A+2sa) + i\int_N \frac{2s^2}{4\pi} a \wedge da + \frac{s}{2\pi} A \wedge da}_{\text{the contribution from the boundary}}\Big]$$

3d gauge field

$$\times \underbrace{\exp\left[i\pi \int_M \hat{A}(R) + \frac{1}{2}\left(\frac{F}{2\pi}\right)^2\right]}_{\text{bulk contribution}}$$