# Spin-charge relation (condensed matter physics)

①

$\begin{cases} \text{spin: half-integral} & U(1)_{em} = \text{odd} \\ \text{spin: integral} & U(1)_{em} : \text{even.} \end{cases}$

( $\because$ microcopic theories are the systems of electrons )

note that this relation cannot be a fundamental property.
( e.g. neutrino, neutron )

② Since fermions carries odd electric charges,

one can place the system not only on ~~the~~ a spin mfd,

but also on a spin$_c$ mfd.

How? $\bar{\psi} i \gamma^\mu D_\mu \psi \longrightarrow D_\mu^{(g)} - i A_\mu$ → spin$_c$ connection.

s.t. $\frac{1}{2\pi}\int_{\mathcal{C}} A = \frac{1}{2\pi}\int_{\mathcal{C}} \omega_2$ modulo integer

$\omega_2$: second Stiefel-Whitney class.

③ • gapped system of fermions on spin$_c$

$\downarrow$ RG

• topological field theory "on spin$_c$"

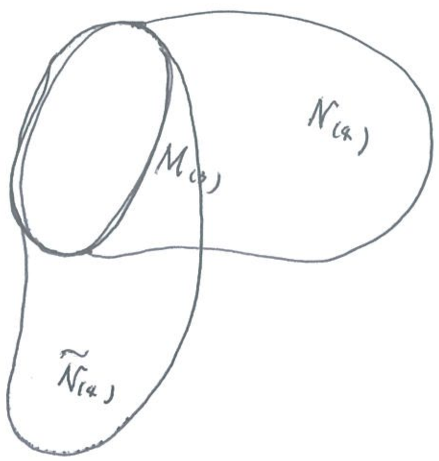
⟶ {constraints on low-energy effective theories.}

(i) Chern-Simons theory (even $k$)

$$\frac{k}{4\pi}\int_{M_{(3)}} A\wedge dA \equiv 2\pi k\cdot\frac{1}{2}\int_{N_{(4)}}\frac{F}{2\pi}\wedge\frac{F}{2\pi} \qquad \partial N_{(4)} = M_{(3)}$$

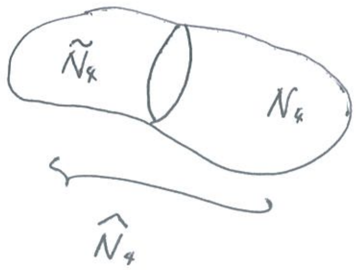
Q : extension possible?

Q : ~~is~~ does the above def. dep. on the choice of $N_{(4)}$ ~~extension~~




$$\frac{Z_{CS}[N_4]}{Z_{CS}[\tilde{N}_4]} = \int DA\; e^{2\pi i k \frac{1}{2}\int_{\hat{N}_{(4)}}\frac{F}{2\pi}\wedge\frac{F}{2\pi}}$$

no boundary




$\hat{N}_{(4)}$ (& $M_{(3)}$)

$$\frac{1}{2}\int_{\hat{N}_4}\frac{F}{2\pi}\wedge\frac{F}{2\pi} \in \mathbb{Z}$$ only for spin mfd (orientable)

otherwise $$\frac{1}{2}\int_{\hat{N}_4}\frac{F}{2\pi}\wedge\frac{F}{2\pi} \in \frac{1}{2}\mathbb{Z}.$$

$\Rightarrow$ $\frac{k}{4\pi}\int_{M_{(3)}} A\wedge dA$ can be defined modulo $2\pi$ on spin mfd

$\pi$ on other mfd.

$\Leftrightarrow$ "$k$ is restricted on by even integer" n.b. $\frac{1}{2}\int_{\hat{N}_4} \frac{F}{2\pi} \wedge \frac{F}{2\pi} \in \mathbb{Z}$ ? for spin mfd $\hat{N}_4$

Dirac operator $\quad i\gamma^\mu D_\mu = i\gamma^\mu D_\mu^{(g)} + \gamma^\mu A_\mu$

- $\mathrm{Index}[i\gamma^\mu D_\mu^{(g)}] = \int_{\hat{N}_{(4)}} \hat{A}(R) = \frac{1}{48}\int_{\hat{N}_{(4)}} \mathrm{tr} \frac{R\wedge R}{(2\pi)^2}$

$$\begin{cases} \frac{1}{48}\int_{\hat{N}_4} \mathrm{tr}\frac{R\wedge R}{(2\pi)^2} = \frac{1}{8}\sigma \quad \text{signature} \\ \mathrm{Index}[i\gamma^\mu D_\mu^{(g)}] \in 2\mathbb{Z} \quad (\exists \text{ pair of zero modes}) \end{cases}$$

zero modes ← Kramers doublet on spin mfd.

- $\mathrm{Index}[i\gamma^\mu D_\mu^{(g)}] = \int_{\hat{N}_{(4)}} \hat{A}(R)\, e^{F/2\pi}$

$$= \frac{1}{48}\int_{\hat{N}_{(4)}} \mathrm{tr}\frac{R\wedge R}{(2\pi)^2} + \frac{1}{2}\int_{\hat{N}_{(4)}} \frac{F}{2\pi}\wedge\frac{F}{2\pi}$$

LHS $\in \mathbb{Z}$

$\frac{1}{48}\int_{\hat{N}_{(4)}} \hat{A}(R) \in 2\mathbb{Z}$

$\therefore \frac{1}{2}\int_{\hat{N}_{(4)}} \frac{F}{2\pi}\wedge\frac{F}{2\pi} \in \mathbb{Z}$ on spin mfd $\hat{N}_{(4)}$

(ii) Chern-Simons theory (odd $k$)

On a spin mfd, $k\mathcal{L}_{CS}[A]$ with odd $k$

can be placed

But the actual value of $e^{ik\mathcal{L}_{CS}[A]}$ dep's

on the choice of spin str. of $M_{(3)}$

$k=1$. $\quad e^{\pi i \int_{N} \frac{F}{2\pi}\wedge\frac{F}{2\pi}}$ $\quad \int_{T^{(2)}} \frac{F}{2\pi} = \text{integer}.$

$M_{(3)} = T^{(2)} \times S^{(1)}$

$++ \quad -$

extension possible.

$N = T^{(2)} \times D^{(2)}$

disc

$\langle W\rangle = \mathrm{tr}_{\mathcal{H}_W}[1] = 1$

$W_1[S^1]$

one-dimensional

spin $\frac{1}{2}$.

no holonomy.

change soure on $T^{(2)}$

Wilson loop around $S^{(1)}$.

$$\int_{D^{(2)}} f = \int_{S^1} A = 2\pi\mathbb{Z}$$

$A$ flat-connection

locally

$dA = 0$

$$\int_N \frac{F}{2\pi}\wedge\frac{F}{2\pi} = 2\underbrace{\int\frac{F_{12}}{2\pi}}_{\mathbb{Z}}\underbrace{\int\frac{F_{34}}{2\pi}}_{\mathbb{Z}} dx^1\wedge dx^2\, dx^3\, dx^4 = 2\mathbb{Z}$$

$$e^{i\int_{S^1}A} = e^{i\int_{D^{(2)}}F}$$

$\therefore \quad e^{2\pi i} = 1$.

$$\langle W[S^1]\rangle = \mathrm{tr}_{\mathcal{H}_W}[(-1)^F]^{+++} = -1$$

fermionic.

(iii) What if $A$ is not a gauge field but a spin$_{\mathbb{C}}$ connection?

a. $$\underbrace{\mathrm{Index}\left[ i\gamma^\mu D^A_\mu \right]}_{\text{integer on spin}_{\mathbb{C}}\text{ mfd with spin}_{\mathbb{C}}\text{ connection } A} = \frac{1}{48}\int_{\hat{N}_4} \mathrm{tr}\,\frac{R\wedge R}{(2\pi)^2} + \frac{1}{2}\int_{\hat{N}_4} \frac{F}{2\pi}\wedge\frac{F}{2\pi}$$

none of them is guaranteed to be integer-valued.

(each of them $\in \frac{1}{8}\mathbb{Z}$).

well-defined U(1) field

b. $$\mathrm{Index}\left[ i\gamma^\mu D^{a+A}_\mu \right] = \frac{1}{48}\int_{\hat{N}_4} \mathrm{tr}\,\frac{R\wedge R}{(2\pi)^2} + \frac{1}{2}\int_{\hat{N}_4} \frac{f+F}{2\pi}\wedge\frac{f+F}{2\pi}$$

$f = da$

(b) − (a) $\Rightarrow$ $$\frac{1}{2}\int_{\hat{N}_4} \frac{f}{2\pi}\wedge\frac{f}{2\pi} + \int_{\hat{N}_4}\frac{f}{2\pi}\wedge\frac{F}{2\pi} \in \mathbb{Z}$$

This implies that, on spin$_{\mathbb{C}}$ mfd $M_{(3)}$,

spin$_{\mathbb{C}}$ connection.

$$\underbrace{\frac{1}{4\pi}\int_{M_{(3)}} a\wedge da + \frac{1}{2\pi}\int_{M_{(3)}} a\wedge dA}_{\text{well-defined}} \in 2\pi\mathbb{Z}$$

c. In general, when $k - q \in 2\mathbb{Z}$, one can define the top'l theory below on the spin$^c$ mfd

$$\frac{k}{4\pi}\int_{M_{(3)}} a \wedge da + \frac{q}{2\pi}\int_{M_{(3)}} A \wedge da$$

! a + A is not well-defined!

check : $a \to a + 2A$ ( field redefinition )

$$\frac{k}{4\pi}\int_{M_{(3)}} a \wedge da + \frac{q+2k}{2\pi}\int_{M_{(3)}} a \wedge dA + \frac{k}{4\pi}\int_{M_{(3)}} (2A) \wedge d(2A) + \frac{q}{4\pi}\int_{M_{(3)}} (2A) \wedge d(2A).$$

$k - (q + 2k) \in 2\mathbb{Z}$

$\therefore$ these two terms can be defined on the spin c mfd.

even integer

$$\frac{k-2q}{4\pi} + \frac{2q}{4\pi}$$

even integer

$$= \frac{k+q}{4\pi} \in 2\mathbb{Z}.$$

$$\therefore \frac{k+q}{4\pi}\int_{M_{(3)}} (2A) \wedge d(2A)$$

is well-defined

d. ~~[illegible]~~ Can we define the Chern-Simons coupling of A ( spin$_c$ connection ) without the gravitational CS coupling ?

$$\mathrm{Index}\,[\,i\gamma^\mu D_\mu\,] = \underbrace{\int_{N(4)} \hat{A}(R)}_{\in \frac{1}{8}\mathbb{Z}} + \frac{1}{2}\int_{N(4)} \frac{F}{2\pi}\wedge\frac{F}{2\pi}$$

$\therefore$ $\frac{8}{4\pi}\int_{M(3)} A\wedge dA$ can be well-defined on spin$_c$ mfd

"multiple of 8"

without the gravitational CS coupling.