TOPOLOGICAL INSULATOR (Introduction)

Insulator

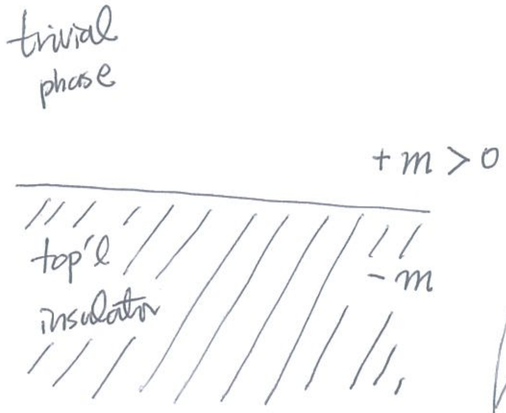
1. $U(1)_A$ is conserved
   electromagnetic

2. gapped phase

(3. time reversal sym.) topological Insulator

remark if a gapped system breaks the $U(1)_A$,
then it becomes a superconductor

example 4-dim'l massive Dirac fermion



$$\mathcal{L}_{\text{outside}} = i \bar{\Psi} \Gamma^\mu D_\mu \Psi + i m \bar{\Psi} \Psi$$

$$\mathcal{L}_{\text{inside}} = i \bar{\Psi} \Gamma^\mu D_\mu \Psi - i m \bar{\Psi} \Psi$$

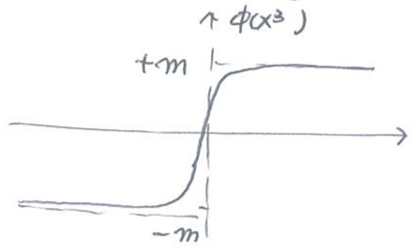
(Classically,) $\exists$ time reversal symmetry

(i) Suppose that the system outside the top'l insulator is in the <u>trivial</u> gapped phase.

(ii) ~~[illegible]~~ note that the mass changes the sign near the surface of the material

$\to$ $\exists$ { <u>massless</u> Dirac fermion on the boundary of the material }

(~ massless Dirac fermions in the vicinity of
3-dim'l
Domain wall $\phi(x)$



)

characteristic of standard ~~to~~ top'l insulator.

(iii) One can also describe the top'l insulator as follows.

Suppose that our Universe is in the phase of

top'l insulator, i.e. the Dirac mass has negative sign.

Perform the chiral rotation by $\frac{\pi}{2}$ angle;

chiral rotation : $\Psi \to e^{i\alpha\Gamma_c}\Psi$

$$\Rightarrow \quad \bar{\Psi}\Psi \longmapsto \bar{\Psi}(\cos 2\alpha + i\Gamma_c \sin 2\alpha)\Psi$$

$$\overset{\alpha = \pi/2}{=} -\bar{\Psi}\Psi$$

~~the same sign~~
the sign is flipped

Namely, $\mathcal{L}_{\mathrm{TI}} = i\bar{\Psi}\Gamma^{\mu}D_{\mu}\Psi + im\bar{\Psi}\Psi$

$$\underbrace{+ \; i\pi \int_{\text{Universe}} \hat{A}(R) + \frac{1}{2}\frac{F\wedge F}{(2\pi)^2}}_{\text{topological term.}}$$

remark

$$\mathcal{L}_{top} = \frac{\vartheta}{8\pi^2}\int_M F\wedge F$$

when M has no boundary,

$\vartheta \cong \vartheta + 2\pi$ periodic.

$\Rightarrow$ when $\vartheta = 0$ or π,

the time reversal symmetry is preserved.

$T: \vartheta \to -\vartheta$

but $-\vartheta \cong -\vartheta + 2\pi$

$\therefore \vartheta = -\vartheta + 2\pi n$

for T-invariance

thus $\vartheta = 0$: trivial phase

$\vartheta = \pi$: non-trivial phase (top'l insulator)

(iv) We now understand that, a system of free fermion has a massless fermion on the boundary.

massive

$\boxed{Q}$ What if we introduce to the free system (weak) interactions ?

3d Dirac fermion.

$$\mathcal{L}_{boundary} = i\bar{\psi}\gamma^\mu D_\mu^A \psi$$

$\Leftarrow$ Classically: $U(1)_A$ & T-invariance !!

However, one can see from the ~~parti~~ partition function that T-symmetry is broken quantum mechanically!

$$Z_{\psi}[A] = |Z_{\psi}[A]|\, \underbrace{e^{-\frac{\pi}{2} i \eta(A, N)}}_{\text{non-trivial phase}}$$

$\partial M = N$

eta invariant

see the section of "standard boundary state"

① { T-symmetry is preserved } $\longleftrightarrow$ { partition function has to be real }

② Why ~~T-symmetry~~ ?

$\leftarrow \nexists$ regularization preserving both $U(1)_A$ & T-symmetry

(a part of defining QFT)

③ However, the total system again is T-invariant!

$$Z^{tot} = |Z^{tot}|\, \underbrace{e^{-\frac{\pi}{2} i \eta(A, N)}}_{\text{boundary}} \times \underbrace{e^{\pi i \left(\int_M \hat{A}(R) + \frac{1}{2}\left(\frac{F}{2\pi}\right)^2 \right)}}_{\text{bulk}}$$

$N = \partial M$

$\Rightarrow$ no longer T-invariant when M has boundaries

Index $(i\Gamma^\mu D_\mu)$ $\boxed{\text{APS INDEX THM}}$

$$= \int_M \left[\hat{A}(R) + \frac{1}{2}\left(\frac{F}{2\pi}\right)^2\right] - \frac{1}{2}\eta(A, N)$$

thus, one can say

$$Z^{tot} = |Z^{tot}|\, e^{\pi i \underbrace{\text{Index}(i\Gamma^\mu D_\mu)}_{\text{real-valued}}}$$

④ the physics on the boundary & in the bulk are controlled by the symmetry, anomaly & index thm.

⇒ massless fermion on ∂

robust under small sym-preserving deformation.

($\because$ index is integer-valued)

$\boxed{\text{Q}}$ what about LARGE symmetry-preserving deformation?

Can it gap out the massless fermion on ∂ & leave a top'l field theory that contribute the same anomaly?