U(2) CS theory v.s. SU(2) × U(1) CS theory ①

(Ising TFT as well)

① Start with $SU(2)_{k_1} \times U(1)_{k_2}$ CS theory

$$\mathcal{L} = \underbrace{\frac{k_1}{4\pi} \mathrm{tr}\left[b \wedge db - \frac{2}{3} i\, b \wedge b \wedge b \right]} + \underbrace{\frac{k_2}{4\pi} a \wedge da}$$

Wilson line W^b_j ($j = 0, 1/2, \cdots, k_1/2$) W^a_n (k_2: even, $n = 0, \pm 1, \cdots, \pm\left(\frac{k_2}{2}\right)$; k_2: odd, $n = 0, \pm 1, \cdots, k_2$)

$\uparrow$ spin

spin $\dfrac{j(j+1)}{k_1 + 2}$ spin $\dfrac{n^2}{2k_2}$

② $U(2) \cong SU(2) \times U(1) / \mathbb{Z}_2$

(i) $\exists\ g \in SU(2)$, $e^{i\varphi} \in U(1)$

one can construct a $U(2)$ group element as

$$\tilde{g} = g \cdot e^{i\varphi}$$

(ii) note that the above map is two-to-one

$$\tilde{g} = g \cdot e^{i\varphi} = g \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}_{\text{center of } SU(2)} \cdot e^{i\varphi}(-1)$$

②

(iii) $SU(2)_{k_1}$ CS theory : $\exists\ \mathbb{Z}_2$ one-form symmetry

(center)

$U(1)_{k_2}$ CS theory : $\exists\ \tilde{\mathbb{Z}}_2$ one-form symmetry

each of them could be anomalous, BUT

the diagonal ~~subgroup~~ part has to be non-anomalous

otherwise, one cannot gauge it.

(iv) $\mathbb{Z}_2$ generator $W^b_{j=\frac{k_1}{2}}$ spin $\frac{\frac{k_1}{2}\cdot\frac{1}{2}(\cancel{k_1+2})}{\cancel{k_1+2}} = \frac{k_1}{4}$

$\left(\because \langle W^{b2}_{j=\frac{k_1}{2}} \rangle = 1\right)$

anomalous for odd k_1

$\tilde{\mathbb{Z}}_2$ generator $W^a_{n=k_2/2}$ spin $\frac{1}{2k_2}\left(\frac{k_2}{2}\right)^2 = \frac{k_2}{8}$

(k_2 : even) $\left(\because \langle W^{a2}_{n=\frac{k_2}{2}} \rangle = 1\right)$

anomalous for $k_2 = 2, 6, \cdots$

diagonal part $\mathbb{Z}_2$ can be generated by

$W^b_{j=k_1/2} \cdot W^a_{n=k_2/2}$ spin. $\frac{k_1 + k_2/2}{4} \in \frac{1}{2}\mathbb{Z}$

$k_1 + k_2/2 \in 2\mathbb{Z}$

$$\begin{cases} 2k_1 + k_2 \in 4\mathbb{Z} & \to \text{ one can gauge it.} \\ 2k_1 + k_2 \in 8\mathbb{Z} & \to \end{cases}$$

$SU(2)_{k_1} \times U(1)_{k_2} / \mathbb{Z}_2$ can be defined only on spin mfd.

~~U(2) theory~~ $SU(2)_{k_1} \times U(1)_{k_2} / \mathbb{Z}_2$ can be defined on any mfd.

(V) $\mathbb{Z}_2$ & $\tilde{\mathbb{Z}}_2$ action.

$SU(2)_{k_1}$

$$W_j(\mathcal{C})\, W_{j'}(\mathcal{C}') = S_{jj'} / S_{0j'}\; W_{j'}(\mathcal{C})$$

$$S_{jj'} = \sqrt{\frac{2}{k_1+2}}\, \sin\left[\frac{\pi}{k_1+2}(2j+1)(2j'+1)\right]$$

$$\Rightarrow \; S_{\frac{k_1}{2}, j'} / S_{0j'} = \sin\left[\frac{\pi}{k_1+2}(k_1+1)(2j'+1)\right] \Big/ \sin\left[\frac{\pi}{k_1+2}(2j'+1)\right]$$

$$\left(\begin{aligned} &\frac{k_1+2-1}{k_1+2}(2j'+1) \\ &= \underbrace{(2j'+1)}_{a} + \underbrace{\frac{-1}{k_1+2}(2j'+1)}_{b} \\ &\therefore \sin[\pi(\underset{\substack{\uparrow \\ \text{integer}}}{a} - b)] = -\cos\pi a \sin\pi b. \end{aligned}\right)$$

$$= -\cos(\pi(2j') + \pi) = \underline{\underline{(-1)^{2j'}}}.$$

$$\therefore \left(W^b_{j=k_1/2}\right)^2 = 1$$

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$U(1)_{k_2}$

$$W^a_n(\mathcal{C})\, W^a_m(\mathcal{C}') = e^{2\pi i \frac{nm}{k_2}}\, W^a_m(\mathcal{C}')$$

$n = \frac{k_2}{2}$

Aharonov-Bohm phase

$$= e^{\pi i m}\, W_m(\mathcal{C}').$$

$$\therefore \left(W^a_{n=k_2/2}(\mathcal{C}) \right)^2 = 1.$$

③ $\mathbb{Z}_2^{diag}$ gauging

$$\frac{k_1}{4\pi}\, tr\left[(b+\cdot)\wedge\left(db + \frac{\pi}{2}\delta\right) - \frac{2}{3} i b^3 \right] + \frac{k_2}{4\pi} (a+\cdot)\wedge(da + \pi\delta)$$

$db + \underbrace{da + \pi\delta}_{= \frac{dA}{2}}$ — da: ill-defined; db: ill-defined

$db + \ldots = dc$ ← well-defined

$da + \pi\delta$: $dA = 2da + 2\pi\delta$ ← well-defined gauge

$$= \frac{k_1}{4\pi}\, tr\left(c\wedge dc - \frac{2}{3} i c^3 \right) \underbrace{- \frac{k_1}{4\pi}\frac{1}{2} A\wedge dA + \frac{k_2}{4\pi}\frac{1}{4} A\wedge dA}$$

$$= +\frac{1}{4\pi}\left(\underbrace{\frac{k_2}{4} - \frac{k_1}{2}} \right)$$

$\in 2\mathbb{Z}$ for any mfd

$\in \mathbb{Z}$ for spin mfd

$$\left(\begin{array}{ll} k_2 - 2k_1 \in 8\mathbb{Z} & \text{non-spin} \\ 4\mathbb{Z} & \text{spin} \end{array} \right)$$

e.g. $\underset{\text{even}}{k_2} = 2k_1$ (i.e., $2k_1 + k_2 = 4k_1 \Rightarrow$ one can gauge $\mathbb{Z}_2^{diag}$)

$k_1 = 2p$ is even better.

$$\left(W^b_{j=\frac{k_1}{2}} W^a_{n=\frac{k_2}{2}} \right)(\mathcal{C})\, W^b_{j'} W^a_m (\mathcal{C}')$$

$$= (-1)^{2j'} (-1)^m \cdot W^b_{j'} W^a_m (\mathcal{C}')$$

if $2j' + m \in 2\mathbb{Z}$, $W^b_{j'} W^a_m (\mathcal{C}')$ is neutral under $\mathbb{Z}_2^{diag}$.

U(2) Wilson lines.

condition for U(2) rep.

In fact, $j + \frac{n}{2} \in \mathbb{Z}$ ~~is a condition~~

~~that the corresponding~~

$$\cancel{W^b_{j=1/2} W^a_{m=1}(\mathcal{C}) = \mathrm{tr}_{\square}\, e^{i\oint b}\, e^{i\oint a}}$$

~~SU(2) fundamental~~

$$\cancel{= \mathrm{tr}_{U(2)}\, e^{i\oint_{\mathcal{C}} \left(b + \frac{A}{2}\right)}}$$

④ Ising TFT

Let's consider $SU(2)_k \times U(1)_{-2k}$ where $k \in \mathbb{Z}$.

In particular, ~~k~~ we choose $k=2$

(i) $SU(2)_2$

observables $W_{j=0}[\mathcal{C}]$, $W_{j=\frac{1}{2}}[\mathcal{C}]$, $W_{j=1}[\mathcal{C}]$

spin	0	$\frac{3}{16}$	$\frac{1}{2}$

$W_{j=1}[\mathcal{C}]$ → $\mathbb{Z}_2$ generator

(fusion) algebra

$$\begin{cases} \left(W_{j=\frac{1}{2}}[\mathcal{C}]\right)^2 = W_{j=0}[\mathcal{C}] + W_{j=1}[\mathcal{C}] \\ W_{j=\frac{1}{2}}[\mathcal{C}] \cdot W_{j=1}[\mathcal{C}] = W_{j=\frac{1}{2}}[\mathcal{C}] \\ W^2_{j=1}[\mathcal{C}] = W_{j=0}[\mathcal{C}] \end{cases}$$

$\mathbb{Z}_2$ action ~~$W_{j=1} W_{j'}[\mathcal{C}']$~~

$$W_{j=1}[\mathcal{C}]\, W_{j'}[\mathcal{C}'] = (-1)^{2j'} W_{j'}[\mathcal{C}']$$

(ii) $U(1)_{-4}$

observables $\widetilde{W}_n[\mathcal{C}]$ $n = 0, \pm 1, 2$

spin $0, -\frac{1}{8}, -\frac{1}{2}$

$\mathbb{Z}_2$ generator $\widetilde{W}_{n=2}[\mathcal{C}]$

$$\widetilde{W}_{n=2}[\mathcal{C}]\,\widetilde{W}_m[\mathcal{C}'] = e^{2\pi i\left(-\frac{2\cdot m}{4}\right)}\,\widetilde{W}_m[\mathcal{C}']$$

$$= e^{-\frac{\pi}{1} i\cdot m}\,\widetilde{W}_m[\mathcal{C}']$$

(iii) $\dfrac{SU(2)_2 \times U(1)_{-4}}{\mathbb{Z}_2^{diag}}$ generator $\underbrace{W_{j=1}[\mathcal{C}]\cdot\widetilde{W}_{n=2}[\mathcal{C}]}_{\text{spin } 0}$

observables leftover:

$\widetilde{W}_{n=0}[\mathcal{C}]$, $\widetilde{W}_{n=2}[\mathcal{C}]$ identified

spin 0, $\frac{1}{2}$

identified

$W_{j=1}[\mathcal{C}]\,\widetilde{W}_{n=0}[\mathcal{C}]$ $W_{j=1}[\mathcal{C}]\,\widetilde{W}_{n=2}[\mathcal{C}]$

spin $\frac{1}{2}$ 0

$W_{j=\frac{1}{2}}[\mathcal{C}]\,\widetilde{W}_{n=1}[\mathcal{C}]$ spin $\frac{1}{16}$.

note that, after gauging $\mathbb{Z}_2$ symmetry,

$W_{j=1}[\mathcal{C}]\tilde{W}_{n=2}[\mathcal{C}]$ can be identified as the identity op.

($\because$ it does not induce any holonomy & carries no spin)

$$\Rightarrow \quad W_j[\mathcal{C}]\tilde{W}_m[\mathcal{C}] \cong W_{1-j}[\mathcal{C}]\tilde{W}_{m+2}[\mathcal{C}]$$

thus, $\dfrac{SU(2)_2 \times U(1)_{-4}}{\mathbb{Z}_2}$ has observables

of Ising TFT.

(iv) Why $\dfrac{SU(2)_2 \times U(1)_{-4}}{\mathbb{Z}_2} \cong \dfrac{SU(2)_2}{U(1)_4}$ (Ising model)

see the ref.