Short review on $U(1)$ CS theory

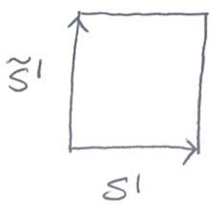
$$\mathcal{L} = \frac{k}{4\pi}\int_M A \wedge dA$$

for any M, $k \in \mathbb{Z}$

for spin mfds M, k needs to be even.

will be discussed later

① Canonical quantization on $T^2 \times R_t = M$ (k: even)



(i) low-energy dynamics :

Q.M. of holonomies $\leftarrow$ classical vacuum configuration

$$A = a(t)\, d\varphi + b(t)\, d\tilde{\varphi}$$

slowly varying over time

$$\int_{S^1} A = 2\pi a$$

$$\int_{\tilde{S}^1} A = 2\pi b$$

$a \cong a+1$
$b \cong b+1$
periodic

$$\tilde{\mathcal{L}} = \frac{k}{2\cancel{4}\pi}(2\pi)^2 \int dt\; b(t)\,\dot{a}(t)$$

(gauge choice $A_0 = 0$)

$2\pi k$.

(ii) quantization

$$[\hat{a}, \hat{b}] = \frac{i}{2\pi k}.$$

position $\quad$ momentum.

Q

a basis of $\mathcal{H}_{T^2}$: $\{ |a\rangle \}$ s.t. $\hat{a}$ 1

$$e^{2\pi i \hat{a}} |a\rangle = e^{2\pi i a} |a\rangle$$

$$e^{2\pi i \hat{b}} |a\rangle = |a + \tfrac{1}{k}\rangle$$

$$\left(\hat{b} = \frac{-i}{2\pi k} \frac{\partial}{\partial a} \right)$$

another basis of $\mathcal{H}_{T^2}$: $\{ |b\rangle \}$ s.t

$$e^{2\pi i \hat{b}} |b\rangle = e^{2\pi i b} |b\rangle$$

...

then, $\quad \langle a | b \rangle \propto e^{2\pi i k b a}$

since a & b are periodic, one demands that

$$\begin{cases} e^{2\pi i k b} = 1 \\ e^{2\pi i k a} = 1 \end{cases}$$

both a & b are quantized

$a, b = \frac{n}{k} \quad n = 0, 1, \ldots, (k-1)$

$$\dim[\mathcal{H}_{T^2}] = k$$

② operator algebra.

$$e^{2\pi i \hat{a}} |a\rangle = e^{2\pi i a} |a\rangle \qquad a = \frac{n}{k} \quad n \in \mathbb{Z}_k.$$

$$e^{2\pi i \hat{b}} |a\rangle = |a + 1/k\rangle$$

note that

$$e^{2\pi i \hat{a}} = e^{i\int_{S^1} A} = W_{n=1}[S^1]$$

$$e^{2\pi i \hat{b}} = e^{i\int_{\tilde{S}^1} A} = W_{n=1}[\tilde{S}^1]$$

$$W_{n}[\mathcal{C}] \equiv e^{in\int_{\mathcal{C}} A}, \quad \text{Wilson line}$$

observables in CS.

(i)

note that $(W_n[\mathcal{C}])^k = 1$

(ii) $$W_{n=1}[S^1] \doteq \begin{bmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{k-1} \end{bmatrix}$$

$\omega^k = 1$

k-th root of unity.

$\{|a\rangle\}$

$$W_{n=1}[\tilde{S}'] \doteq \begin{bmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & 1 & \ddots & \\ & & & \ddots & \\ & & & 1 & 0 \end{bmatrix}$$

$$W_{n=1}[\tilde{S}']\, W_{n=1}[S'] = \omega\, W_{n=1}[S']\, W_{n=1}[\tilde{S}']$$

③ ~~pseudo particle~~ quasi-particle

CS theory describes the low-energy physics of a gapped system.

Wilson line <=> ~~pseudo~~ quasi-particle.

$$\langle W_n[C] \rangle = \int DA \; e^{\frac{i}{4\pi}k\int A\wedge dA \; + \; in\int_C A}$$

$$\| \qquad\qquad\qquad\qquad\qquad\qquad\qquad \longleftarrow$$

$$e^{in\int dt\, A_0(t,\vec{x}_0)} \qquad\qquad\qquad e^{in\int dt\, d^2x\, A_0\, \delta(\vec{x}-\vec{x}_0)}$$

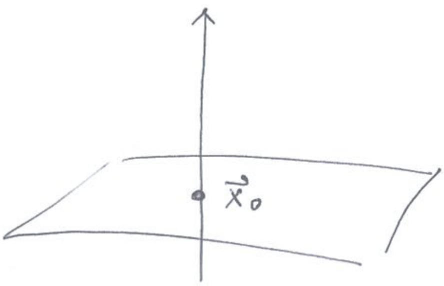
$\therefore$ external charge source at $\vec{x} = \vec{x}_0$

$$\rho = n\,\delta(\vec{x}-\vec{x}_0)$$

$$\frac{\delta}{\delta A_0} : \quad \frac{k}{2\pi} F_{12} = \rho = n\,\delta(\vec{x}-\vec{x}_0)$$

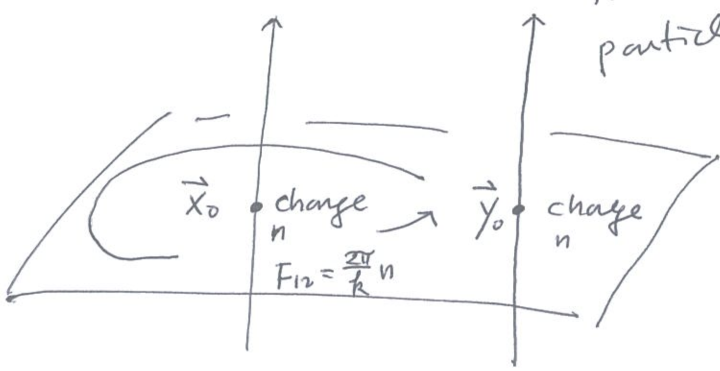
$$F_{12} = \frac{2\pi}{k}\, n\,\delta(\vec{x}-\vec{x}_0)$$

two identical particles



charge n

flux $\frac{2\pi}{k}$



statistical spin

$$e^{in\int_{C} A/2} = e^{\pi i \frac{n^2}{k}}$$

$$\therefore \quad s = \frac{n^2}{2k} \quad \text{mod. } \underline{\text{integer}}$$

Quantum number : spin (statistical)

$$W_n[C] \qquad s = \frac{n^2}{2k} \quad \text{mod integer.}$$

revisit the op.

$$\left(W_{n=1}[C]\right)^k = 1 \quad (?)$$

spin $\frac{k^2}{2k} = \frac{k}{2}$ mod integer

(i) $\therefore$ For $k=$ even, $s=0$. $\quad \left(W_{n=1}[C]\right)^k = \mathbb{1}$

$$W_{n+k}[\mathcal{C}] \cong W_n[\mathcal{C}]$$

$n = 0, \pm 1, \pm 2, \ldots, \pm(\frac{k}{2}-1), \frac{k}{2}.$

$\exists$ finitely many pseudo-particles !

(how many? k).

(ii) what if k = odd?

$(W_{n=1}[\mathcal{C}])^k$ induces no holonomy around it

but carries spin $1/2$

$\uparrow$

signal for a constraint on M.

$\langle W_k[\mathcal{C}] \rangle = \pm 1$

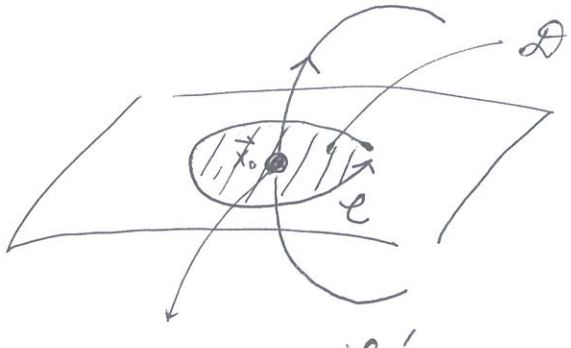
when M is a spin mfd

sign depends on the choice of spin str.

→ We will discuss it later!

(iii)

$$\underbrace{W_n[\mathcal{C}]}_{\text{induces hol.}} W_m[\mathcal{C}'] = e^{2\pi i/k \cdot (nm)} W_m[\mathcal{C}']$$



$$F_{in} = 2\pi/k \cdot m\, \delta(x - \vec{x}_0).$$

$$e^{in\int_{\mathcal{C}} A} \doteq e^{in\int_{\mathcal{D}} F}$$

$$\doteq e^{2\pi i/k \cdot nm.}$$

④ one-form global sym.

~~one-form~~ ~~continuous~~ : $A \to A + \alpha$ ← closed. one-form.

(i) one-form sym. transf.

$$\cancel{\mathcal{L} = \frac{k}{4\pi}\int (A+\alpha)\wedge dA} \qquad \cancel{dA \propto \delta(\mathcal{C})} \qquad \cancel{\oint dA}$$

$$\delta\mathcal{L} = \frac{k}{2\pi}\int \alpha\, dA = k\int_\gamma \alpha$$

$\therefore\ \exists$ discrete $\mathbb{Z}_k$ one-form global sym.

$$A \to A + \alpha/k$$

s.t. $\frac{1}{2\pi}\int_\gamma \alpha \in \mathbb{Z}$ period.

(ii) Under $\mathbb{Z}_k$ one-form sym. transf

$$U_{g=e^{2\pi i/k}} W_n[\mathcal{C}] \longrightarrow$$

$$W_n[\mathcal{C}] = e^{in\int_{\mathcal{C}} A} \longrightarrow e^{in\int_{\mathcal{C}} A+\alpha/k}$$

$$= e^{in/k \int_{\mathcal{C}}\alpha} \cdot W_n[\mathcal{C}] \qquad \int_{\mathcal{C}}\alpha \to 2\pi$$

$$= e^{2\pi i\, n/k}\, W_n[\mathcal{C}]$$

changed op.

generator of $\mathbb{Z}_k$.

$$U_{g=e^{2\pi i/k}}[\mathcal{C}'] = ?$$

$\Rightarrow$ one can easily see that the $\mathbb{Z}_k$ generator can be identified as the <u>Wilson line</u> $W_{n=1}[\mathcal{C}']$!

indeed $\left(U_{g=2\pi i/k}[\mathcal{C}']\right)^k = W_k[\mathcal{C}'] = 1$.

• <u>remark</u> for a Maxwell theory

$$U_{g=2\pi i/k}[\Sigma_{(2)}] = e^{i\int_{\Sigma_{(2)}} *F}.$$

(iii) anomaly ?

to see the so-called 't Hooft anomaly,

we would like to gauge $\mathbb{Z}_k$ and see

if there exist any obstructions !

gauging : sum over all possible insertions of group elements $U_g[\mathcal{C}]$

<u>remark</u>

$$U_{g=e^{2\pi i/k}}[\mathcal{C}]\; U_{g=e^{2\pi i/k}}[\mathcal{C}'] = e^{2\pi i \left(\frac{1}{k}\right)} U_{g=e^{2\pi i/k}}[\mathcal{C}']$$

(25)

statistical spin of f

it is problematic because the generator

also transfs. under $\mathbb{Z}_k$ → ~~sign of~~ signal of anomaly typical insertion.

$$\langle \mathcal{U}_{g=2\pi i/k}[\mathcal{C}] \rangle = \int \mathcal{D}A \; e^{\frac{i}{4\pi} k \int A \wedge dA + \cancel{2\pi i/k}\, i \int A \wedge \delta(C)}$$

poincare dual of $\mathcal{C}$

$$\frac{k}{4\pi} \int A \wedge dA + \int A \wedge \delta(\mathcal{C})$$

$d\tilde{\delta} \cancel{=} \delta(\mathcal{C})$

$$= \frac{k}{4\pi} \int \left(A + 2\pi \tilde{\delta}/k \right) \wedge \left(dA + 2\pi/k\, \delta(\mathcal{C}) \right)$$

$\overset{?}{=}$

$dA' = dA + 2\pi/k\, \delta(\mathcal{C})$ is not allowed due to the wrong Dirac quantization. $\frac{1}{2\pi} \int dA' \notin \mathbb{Z}$

Instead, $dA' = k\, dA + \underline{\underline{2\pi\, \delta(\mathcal{C})}}$ is allowed!

$$\downarrow$$

$$\frac{1}{2\pi} \int dA' \in \mathbb{Z}$$

$\Rightarrow$ thus "$\int \mathcal{D}A'\, e^{i \frac{1}{4\pi} \left(\frac{1}{k}\right) \int A' \wedge dA'}$"

wrong coefficient!

$\therefore$ $\mathbb{Z}_k$ one-form global sym. is anomalous.

(iv) $\underset{\underset{\mathbb{Z}}{\cap}}{k = 4p}$ $\mathbb{Z}_2 \subset \mathbb{Z}_k$ is non-anomalous.

$\mathbb{Z}_2$ generator : $U_{g=-1}[\mathcal{C}] = W_{2p}[\mathcal{C}]$

ⓐ (W) $U_{g=-1}[\mathcal{C}'] \, U_{g=-1}[\mathcal{C}] = e^{2\pi i \left(2 \cdot \frac{(2p)^2}{2(4p)}\right)} U_{g=-1}[\mathcal{C}]$

$\left(2 \cdot \frac{(2p)^2}{2(4p)}\right) = p$

neutral under $\mathbb{Z}_2$ transf.

ⓑ ~~Well~~ After $\mathbb{Z}_2$ gauging, the well-defined gauge field is $A' = 2A$.

$$\mathcal{L}[A'] = \frac{1}{4\pi} p \int 2A \wedge d(2A)$$

$$= \frac{1}{4\pi} p \int_M A' \wedge dA'$$

p = even M ← any mfd

p = odd M ← spin mfd

allowed observables

p = even

$$W'_n[\mathcal{C}] = e^{in\int_{\mathcal{C}} A'} = \underline{e^{2in\int_{\mathcal{C}} A}}$$

$n = 0, \pm 1, \ldots \pm (\frac{p}{2} - 1), \frac{p}{2}.$

$W_n'[\mathcal{C}] = W_{2n}[\mathcal{C}]$ ~~is~~ are $\mathbb{Z}_2$ invariants

$\because$ $\mathcal{U}_{g=\underset{-1}{\cancel{2\pi i/k}}}[\mathcal{C}'] \, W_{2m}[\mathcal{C}]$

$$= e^{2\pi i \underset{\substack{\| \\ n}}{\frac{(2p)(2m)}{4p}}} W_{2m}[\mathcal{C}]$$

$W_{2m+1}[\mathcal{C}]$ are odd under $\mathbb{Z}_2$

$\hookrightarrow$ projected out once we gauge $\mathbb{Z}_2$!