Standard boundary state of top'l insulator ①

§ ~~Spinors in~~ 1+2 ~~dimensions~~ space-time physics

① Lorentzian gamma matrix & spinor

$$\underbrace{\gamma^0 = i\tau^2, \quad \gamma^1 = \tau^3, \quad \gamma^2 = \tau^1}_{\text{real}}$$

Majorana fermion $\quad \psi^*_\alpha = \psi_\alpha$

Dirac fermion $\quad \psi_\alpha = \psi_{1\alpha} + i\psi_{2\alpha}.$

Majorana fermions.

② time reversal transf. T.

$$T : \psi(t, \vec{x}) \longrightarrow \pm\gamma_0 \psi(-t, \vec{x})$$

sign choice

Check : T :

$\bar{\psi}\gamma^0\psi(t, \vec{x}) \longmapsto (\pm\psi^\dagger(-t,\vec{x})\gamma^{0\dagger})\cancel{\gamma^0\gamma^0}^{-1}(\pm\gamma^0\psi(-t,\vec{x}))$

charge density

$= +\psi^\dagger(-t,\vec{x})\gamma^0\gamma^0\psi(-t,\vec{x})$

$= + \bar{\psi}\gamma^0\psi(-t,\vec{x})$

current

$\bar{\psi}\gamma^i\psi(t,\vec{x}) \longmapsto (\pm\psi^\dagger(-t,\vec{x})\gamma^{0\dagger})\gamma^0\gamma^i(\pm\gamma^0\psi(-t,\vec{x}))$

$= - \bar{\psi}\gamma^i\psi(-t,\vec{x})$

note that

$$T^2 : \psi(t,\vec{x}) \mapsto (\pm)^2(\gamma^0)^2\psi(t,\vec{x}) = (-1)\psi(t,\vec{x})$$

$$\hookrightarrow T^2 = (-1)^F \longmapsto \text{Kramers's doublet.}$$

consistent to

③ ~~T-invariant theory~~ from Lagrangian

(i) ~~kinetic term~~

$$\mathcal{L}_{kin} = i\bar{\psi}\gamma^\mu\partial_\mu\psi = i\psi^\dagger\left(-\partial_t + \gamma^0\gamma^i\partial_i\right)\psi$$

$$\Big\downarrow T$$

$$\partial_t\psi = \psi(t+\varepsilon) - \psi(t)/\varepsilon$$
$$\xmapsto{T} \left(\psi(-t-\varepsilon)-\psi(-t)/\varepsilon\right) \underset{\gamma^0}{=} -\dot{\psi}(-t)$$

$$(-i)\cdot\left(\psi^\dagger(-t,\vec{x})\underbrace{\gamma^{0\dagger}}_{=-\gamma^0}\right)\left(+\gamma^0\dot{\psi}(-t,\vec{x}) + \underbrace{(\gamma^0\gamma^i)^*}_{\text{real}}\partial_i\psi(-t,\vec{x})\right)$$

"anti-linear"

$$= +\gamma^i\partial_i\psi(-t,\vec{x})$$

$$= i\psi^\dagger(-t,\vec{x})\left(-\dot{\psi}(-t,\vec{x}) + \gamma^0\gamma^i\partial_i\psi(-t,\vec{x})\right) = \mathcal{L}_{kin}(\psi(-t,\vec{x}))$$

(ii) minimal coupling

$$\mathcal{L}_{\text{coupling}} = i\bar{\psi}\gamma^\mu(-iA_\mu)\psi = \underbrace{\bar{\psi}\gamma^\mu\psi}_{=j^\mu} A_\mu$$

$$\xrightarrow[T]{} \quad \bar{\psi}\gamma^0\psi(-t,\vec{x})\cdot A_0(-t,\vec{x}) + (-\bar{\psi}\gamma^i\psi(-t,\vec{x}))(-A_i(-t,\vec{x}))$$

$$= \bar{\psi}\gamma^\mu\psi\cdot A_\mu(-t,\vec{x}).$$

(iii) Dirac mass term. (real)

$$\mathcal{L}_m = im\bar{\psi}\psi$$

real. & ~~$U(1)_A$~~ preserves $U(1)_A$ symmetry!

$$\xrightarrow[T]{} \quad \underbrace{(-i)}_{\text{anti-linear}}\cdot m\cdot\left(\pm\psi^\dagger(-t,\vec{x})\gamma^{0\dagger}\right)\gamma^0\left(\pm\gamma^0\psi(-t,\vec{x})\right)$$

$$= +im\ \psi^\dagger\underbrace{\gamma^0\gamma^0\gamma^0}_{=-\gamma^0}\psi(-t,\vec{x}) = -im\bar{\psi}\psi$$

mass sign is flipped under T-transf.

(iv) Complex mass term.

complex mass .

$$\tilde{\mathcal{L}}_m = m\, \psi^T \gamma^0 \psi - m^* \psi^\dagger \gamma^0 \psi^*$$

$$\propto i m_2 \left(\psi_1^T \gamma^0 \psi_1 - \psi_2^T \gamma^0 \psi_2\right) + i m_1 \left(\psi_2^T \gamma^0 \psi_1 + \psi_1^T \gamma^0 \psi_2\right).$$

$m = m_1 + i m_2$

Let $m_2 = 0$. $\quad i m_1 \psi_2^T \gamma^0 \psi_1$

If $T: \begin{cases} \psi_1 \to (+)\gamma^0 \psi_1 \\ \psi_2 \to (-)\gamma^0 \psi_2 \end{cases}$, $\quad i m_1 \psi_2^T \gamma^0 \psi_1 (t, \vec{x})$

$$\to (-i) m_1 \left(-\psi_2^T \gamma^{0T}\right) \gamma^0 \left(+\gamma^0 \psi_1\right)(-t, \vec{x})$$

$$= + i m_1 \psi_2^T \underbrace{\gamma^{0T} \gamma^0 \gamma^0}_{= \gamma^0} \psi_1 (-t, \vec{x})$$

$\therefore \tilde{\mathcal{L}}_m = i m_1 \psi_2^T \gamma^0 \psi_1$ preserves the T-symmetry,

but breaks the $U(1)_A$-symmetry.

~~(*) T-invariant theory & Real partition function.~~

def of ν_T

$$T : \underset{\text{Dirac}}{\psi(t, \vec{x})} \longrightarrow \gamma_0 \psi(-t, \vec{x})$$

Majorana fermions.

Since, $\psi(t, \vec{x}) = \psi_1(t, \vec{x}) + i\,\psi_2(t, \vec{x})$, one can see

that

$$T : \psi_1(t, \vec{x}) \longmapsto \oplus\, \gamma_0\, \psi_1(-t, \vec{x})$$

(anti-linear op.)

$$\psi_2(t, \vec{x}) \longmapsto \ominus\, \gamma_0\, \psi_2(-t, \vec{x})$$

$$\underline{\nu_T \equiv n_+ - n_-}$$

# of fermion with $-$ sign

of (massless) Majorana fermions transforming under T with $+$ sign

~~for a standard~~

$i\psi_1^T \gamma^0 \psi_2$

As shown ~~in the previous~~ above, T-invariant mass term exists only when ψ_1 & ψ_2 transf. under T with opposite signs.

$\Rightarrow$ ν_T is an invariant $\in \mathbb{Z}_{16}$. interaction.

For a standard boundary state of a top'l insulator,

$n_+ = n_- = 1$ $\quad \therefore \quad \nu_T = 0$.

③ charge conjugation $\mathcal{C}$

$\mathcal{C} : \psi \longmapsto \psi^\dagger$ or $\psi_1 \to \psi_1$

$\psi_2 \to -\psi_2$

Hence, $\mathcal{C}T$; $\psi_1(t, \vec{x}) \Rightarrow \gamma_0 \psi_1(-t, \vec{x})$

$\psi_2(t, \vec{x}) \Rightarrow \gamma_0 \psi_2(-t, \vec{x})$

Similarly $\nu_{CT} = \tilde{n}_+ - \tilde{n}_-$ is an invariant

under CT-~~symmetric~~ conserving interactions mod 16

For the standard boundary state of TI,

$\nu_{CT} = 2$

④ ~~Odd~~ ~~Odd~~ Chern-Simons action ?

Let's consider a theory of massive Dirac fermion

preserving $U(1)_A$ symmetry

$$\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu^A \psi + im\bar{\psi}\psi$$

$$Z[A] = e^{i\Gamma[A]} = \int D\psi D\bar{\psi}\, e^{i\int d^4x\, \mathcal{L}(\psi,\bar{\psi})}$$

local terms
w.r.t. A
(gauge-inv.)

def.
$$\stackrel{\mathrm{def.}}{\equiv} \det[i\gamma^\mu D_\mu + im] \Big/ \underbrace{\det[i\gamma^\mu D_\mu + iM]}$$

Pauli-Villars regulator
that preserves $U(1)_A$.

(to define QFT ,
we need to specify ~~the choice~~
~~of regularization scheme~~
how to regularize it!)

$$i\Gamma[A] = \mathrm{Tr}\log[i\gamma^\mu D_\mu + im] - \mathrm{Tr}\log[i\gamma^\mu D_\mu + iM]$$

quotation mark

$$i\Gamma[A] \text{ “}=\text{” } i \cdot \frac{1}{4\pi} \cdot \frac{1}{2} \cdot \mathrm{sgn}(m)\, \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$$

$$- i \frac{1}{4\pi} \cdot \frac{1}{2}\, \mathrm{sgn}(M)\, \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$$

$$+ (\cdots)$$

actual computation

$$\frac{\delta^2}{\delta A_\mu \delta A_\nu} \Gamma[A] \propto \varepsilon^{\mu\nu\rho} p_\rho$$

If we set $M < 0$, one can show that

~~$i\Gamma[A] = i \frac{1}{4\pi}$~~

$$i\Gamma[A] = \begin{cases} i \cdot \frac{1}{4\pi} \cdot 1 \cdot \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho & (m > 0) \quad m \to \infty \\ 0 & (m < 0) \quad |m| \to \infty \end{cases}$$

$$\boxed{\mathcal{L}_{CS} = \frac{1}{4\pi} \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho}$$

Chern-Simon coupling.

↑

topological field theory

describing the gapped phase

Dirac mass

$$\begin{cases} m > 0 & : \text{non-trivial gapped phase} \\ m < 0 & : \text{trivial gapped phase.} \end{cases}$$

○ What if we start with a massless Dirac fermion?

$M<0$

$$i\Gamma[A] = i\,\frac{1}{4\pi}\left(\frac{1}{2}\right)\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho \quad ??$$

wrong coeff.
(ill-defined)

§ 3 dim'l Euclidean physics

In order to make the analysis well-controlled, we consider the theory in the <u>Euclidean space</u>

① gamma matrices & spinor.

$$\underbrace{\gamma^1 = \tau^3 , \quad \gamma^2 = \tau^1 , \quad \gamma^3 = \tau^2}_{\text{Hermitian}}$$

$\mathcal{C}^{\alpha\beta} = i\tau^2$

$\mathcal{C}_{\alpha\beta} = -i\tau^2$

$\mathcal{C}\gamma^\mu \mathcal{C}^{-1} = -\gamma^{\mu *} = -\gamma^{\mu T}$

<u>pseudo-real</u> spinor : $\psi^{*\alpha} \overset{(?)}{=} \mathcal{C}^{\alpha\beta}\psi_\beta$

spinor index

~~reality~~ condition : ⊕

$$(\psi^*)^* = (\mathcal{C}\psi)^* = \mathcal{C}\mathcal{C}^*\psi$$

$\mathcal{C}\mathcal{C}^* = -1$

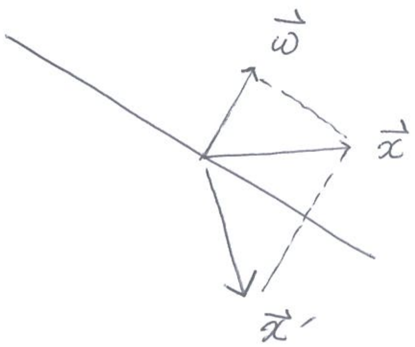
$$= (-1)\psi$$

∴ inconsistent.

② T-transf in ~~Lorentzian~~ Minkowski spacetime $\simeq$ Reflection transf. in Euclidean space.

reflection symmetry

ω^μ : unit vector



$$x^\mu \longmapsto \underbrace{x^\mu - 2\omega^\mu(\vec{\omega}\cdot\vec{x})}_{= x'^\mu}$$

the reflection acts on spinor as

$$R : \psi(x^\mu) \longmapsto \gamma^\mu \omega_\mu \psi(x'^\mu)$$

<u>check</u> :

$$R : \psi^T \mathcal{C} \gamma^\mu \psi \to \psi^T \underbrace{(\omega\cdot\gamma)^T} \mathcal{C} \gamma^\mu (\omega\cdot\gamma)\psi(x'^\mu)$$

$$= -\mathcal{C}\,\omega_\nu \gamma^\nu \gamma^\mu$$

$$= \psi^T \mathcal{C}\left[\gamma^\mu \cdot \cancel{\omega^2}^{\,1} - 2(\omega\cdot\gamma)\omega^\mu\right]\psi$$

$$= \psi^T \mathcal{C} \gamma^\mu \psi(x'^\mu) - 2\omega^\mu \psi^T \mathcal{C}(\omega\cdot\gamma)\psi($$

Let us choose $\underline{\omega^\mu = (1, 0, 0)}$ for convenience.

$$R : \psi(x^\mu) \longrightarrow \gamma^1 \psi(-x, \vec{y})$$

(3) R anticommutes with the Dirac op.

(i) kinectic term

$a=2,3.$

$$i\psi^T \mathcal{C}\gamma^\mu \partial_\mu \psi \longmapsto i\psi^T \underbrace{\gamma^{1T}\mathcal{C}}_{-\mathcal{C}\gamma^1} \left(\gamma^1\gamma^1(-\partial_1\psi) + \gamma^a\partial_a\psi\right)(-x,\vec{y})$$

$$= i\psi^T\mathcal{C}\gamma^1\partial_1\psi(-x,\vec{y}) + i\psi^T\mathcal{C}\gamma^a\partial_a\psi(-x,\vec{y})$$

$$= i\psi^T\mathcal{C}\gamma^\mu\partial'_\mu\psi(x'^\mu)$$

(ii) minimal coupling

$$\psi^T\mathcal{C}\gamma^\mu\psi \cdot A_\mu(x) \longmapsto (-\psi^T\mathcal{C}\gamma^1\psi)\cdot(-A_1)(-\vec{x},\vec{y})$$

$$+ (\psi^T\mathcal{C}\gamma^a\psi)\cdot(A_a)(-\vec{x},\vec{y})$$

$$= \psi^T\mathcal{C}\gamma^\mu\psi \cdot A_\mu(x'^\mu)$$

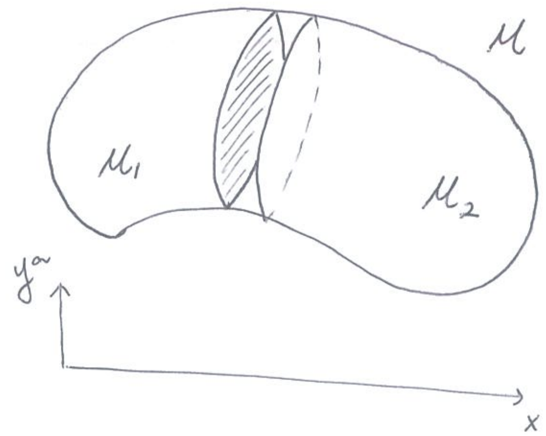
$$\therefore \quad \mathcal{L}_{kin} \longmapsto \mathcal{L}_{kin}[\psi(x'), A(x')]$$

(4) <u>REAL</u> Euclidean partition function

$$Z = \int D\psi(x)\, e^{-\int_M d^3x\, \mathcal{L}_E[\psi(x^\mu)]}$$

Once you specify the field config. on ∂M_2, one can a c-number. $\langle \psi(x) = \tilde{\psi}(x)\big|_{\partial M_1} \, in \rangle$

$$= \langle out \mid in \rangle$$



(i) $$| in \rangle = \int D\psi(x)\, e^{-\int_{M_1} d^3x\, \mathcal{L}_E[\psi(x^\mu)]}$$

on ∂M_1

(ii) $$\langle out | = \int D\psi(x)\, e^{-\int_{M_2} d^3x\, \mathcal{L}_E[\psi(x^\mu)]}$$

on ∂M_2

$$= \int D\psi'(x)\, e^{-\int_M d^3x\, \mathcal{L}_E[\psi'(x)]}$$

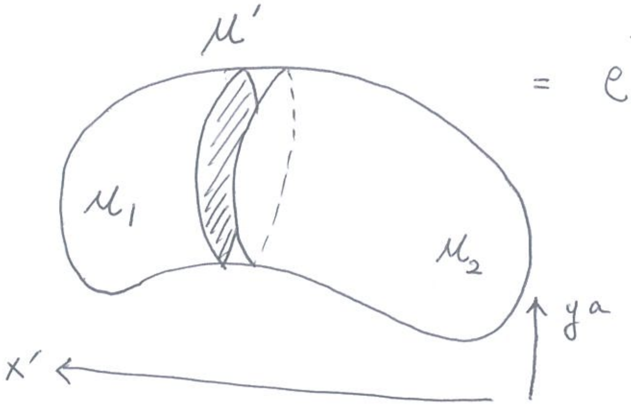
$$\psi'(x) = \gamma' \psi(-x, \vec{y})$$

$$= \int D\psi(-x, \vec{y})\, e^{-\int_M d^3x\, \mathcal{L}_E[\psi(-x, \vec{y})]}$$

R-sym. theory

$$\because \mathcal{L}_E[\gamma' \psi(-x, \vec{y})] = \mathcal{L}_E[\psi(-x, \vec{y})]$$

$$= e^{-\int_{M'} d(\overset{x'}{\overset{\shortparallel}{(-x)}}) \int d^2y\, \mathcal{L}_E[\psi(\overset{x'}{\overset{\shortparallel}{-x}}, \vec{y})]}$$



$$= \langle in \mid out \rangle$$

one can see that, for a theory preserving R,

$$\langle \text{out} | \text{in} \rangle = \langle \text{in} | \text{out} \rangle$$

... ~~reflection positivity~~.

$\therefore$ one can naively expect that the partition function of a massless Dirac fermion on $\mathcal{M}$ is real.

~~But, this is not the case due to quantum corre~~

quantum mechanically, $\exists$ R-anomaly.

④ R-anomaly.

$$Z_\psi = \lim_{M\to\infty} \prod_a \underbrace{\frac{\lambda_a}{\lambda_a + iM}}_{\text{Pauli-Villars regulator}} \quad \text{where} \quad \underbrace{i\gamma^\mu D_\mu}_{\text{hermitian}} \psi_a = \overset{\text{real}}{\lambda_a} \psi_a$$

U(1)$_A$ symmetry is preserved ~~conserved~~
but R-symmetry is broken.

$$\sim \prod_a (-i\lambda_a)/M \sim |Z_\psi| e^{-\frac{\pi}{2} i \sum_a \mathrm{sgn}(\lambda_a)}$$

$M>0$

eta-invariant -

$$Z_\psi = |Z_\psi| \, e^{-\frac{\pi}{2} i \eta(g, A)}$$

... $U(1)_A$ gauge-invariant!

well-defined, unlike CS with ill-defined level 1/2

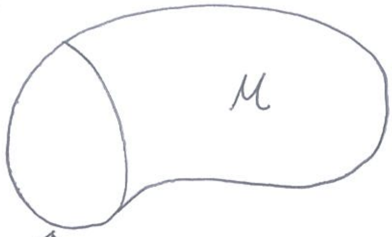
non-trivial phase : R-symmetry is broken.

note if we define the theory with a regulator preserving R but breaking $U(1)_A$, one can obtain the real partition function but not-inv. under $U(1)_A$.

$\nexists$ regulator preserving both R & $U(1)_A$!!

(5) Q Chern-Simons v.s. eta-invariant

$$e^{\frac{i}{4\pi}\int A dA} \quad \text{v.s.} \quad e^{-\pi i \eta(g, A)}$$



$N = \partial M$

(i) $$2\pi \,\mathrm{Index}\left(i\Gamma^\mu D_\mu\right)_M$$

$$= \underbrace{2\pi \int_{M} \hat{A}(R)}_{\text{gravitational CS coupling } \Omega(g_N)} + \frac{1}{4\pi}\int_N A\,dA - \pi\eta(N, A)$$

thus, $$e^{-\pi i \eta(N, A)} = e^{\frac{i}{4\pi}\int A\,dA + i\Omega(g)}$$

But, $$e^{-\frac{\pi}{2} i \eta(N, A)} = \underset{\text{ambiguity}}{(\pm 1)}\, e^{\frac{i}{4\pi}\left(\frac{1}{2}\right)\int_N A\,dA + \frac{i}{2}\Omega(g)}$$

(ii) But, one can also see that

from the APS index thm,

$$\frac{\delta \eta(N, A)}{\delta A} = \frac{\delta \mathcal{L}_{CS}(A)}{\delta A}$$ ~~under δA~~

$\Rightarrow$ ~~local physics~~ the same e.o.m !! the same local physics

$\Rightarrow$ global properties (may) differ.