Time Reversal Symmetry

§ classical picture.

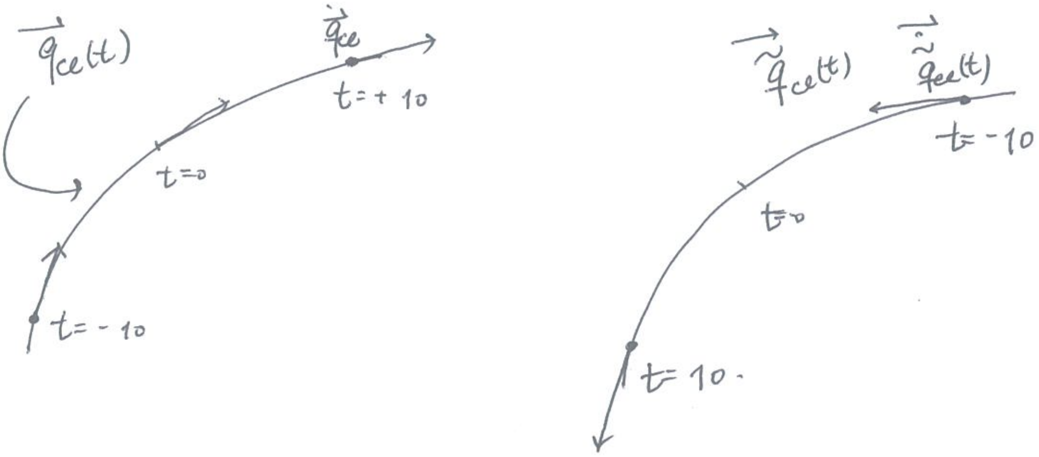
$q_{ce}(t)$: sol to the eqn. of motion

the given system preserves time reversal sym.,

if $q_{ce}(-t) = \tilde{q}_{ce}(t)$ is a sol. to e.o.m.

$\Longleftrightarrow$ e.g. $\ddot{\tilde{q}}_{ce}(t) = -V'(\tilde{q}_{ce}(t))$

$\because$ $\ddot{\tilde{q}}_{ce}(t) \underset{\text{by def}}{=} \ddot{q}_{ce}(-t) \underset{q_{ce}\text{ is a sol. to e.o.m.}}{=} -V'(q_{ce}(-t))$

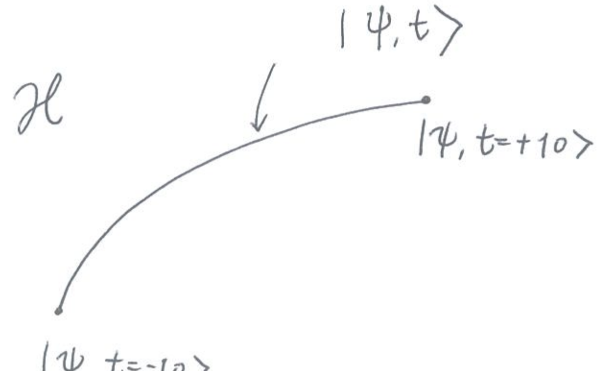


$\dot{\tilde{q}}_{ce}(t) = -\dot{q}_{ce}(-t)$

§ quantum mechanical picture

$|\psi, t\rangle$: time evolution of $|\psi\rangle \in \mathcal{H}$

$$\left(i\hbar \frac{\partial}{\partial t} |\psi, t\rangle = H |\psi, t\rangle \right)$$



if

$$\mathcal{T} i = -i \mathcal{T}$$

$$\mathcal{T} H = H \mathcal{T}$$

$\mathcal{T}$: time reversal op.

$$\Rightarrow \left(-i\hbar \frac{\partial}{\partial t} |\psi, -t\rangle = H |\psi, -t\rangle \right) \equiv i\hbar \frac{\partial}{\partial t} \mathcal{T} |\psi, -t\rangle = H \mathcal{T} |\psi, -t\rangle$$

that is to say, $|\tilde{\psi}, t\rangle = \mathcal{T} |\psi, -t\rangle$ also satisfies

the schrödinger eqn.

$\Rightarrow \mathcal{T}$: anti-linear (& anti-unitary) operator.

note that

$$\boxed{\tilde{\psi}(t, \vec{x}) = \psi^*(-t, \vec{x})}$$

§ anti-linear & anti-unitary

① action on the ket

$$T\left(a|u\rangle + b|v\rangle\right) = a^* T|u\rangle + b^* T|v\rangle$$

② action on the bra

$$\left(\langle v|T\right)|u\rangle \equiv \overline{\langle v|\left(T|u\rangle\right)}$$

③

$$\langle u|\underbrace{T_1 T_2}_{\text{linear op.}}|v\rangle = \left(\langle u|T_1 T_2\right)|v\rangle$$

consistent to the def. of action of linear op on bra/ket

$$= \overline{\left(\langle u|T_1\right)\left(T_2|v\rangle\right)} = \langle u|\left(T_1 T_2|v\rangle\right)$$

④ Hermitian conjugate of ~~anti-linear~~ T

$$\langle v|\left(T^\dagger|u\rangle\right) \equiv \overline{\langle u|\left(T|v\rangle\right)}$$

note that

$$\langle v|\left(T^\dagger|u\rangle\right) = \overline{\left(\langle v|T^\dagger\right)|u\rangle}$$

$$\therefore \quad \left(\langle v|T^\dagger\right)|u\rangle = \langle u|\left(T|v\rangle\right) \; !$$

⑤ anti-unitary op. T

$$T T^\dagger = T^\dagger T = \mathbb{1}$$

T : $|u\rangle \rightarrow T|u\rangle$

$\langle v| \rightarrow \langle v| T^\dagger$

$\mathcal{O} \rightarrow T \mathcal{O} T^\dagger$

$$\langle v'|\mathcal{O}'|u\rangle$$
$$= (\langle v|T^\dagger)\,\mathcal{O}'\,(T|u\rangle) = \overline{\langle v|(T^\dagger \mathcal{O}' T|u\rangle)}$$

$$\overset{\text{anti-unitarity}}{=} \overline{\langle v|\mathcal{O}|u\rangle}$$

⑥ 3 time-reversal sym. ~~Res~~

① $T\hat{x}T^\dagger = \hat{x}$ → "$T|x\rangle = |x\rangle$"

$T\hat{p}T^\dagger = -\hat{p}$

then, the canonical quantization remains unchanged under T transf.

$$[\hat{x}, \hat{p}] = i \rightarrow T[\hat{x}, \hat{p}]T^\dagger = -i$$
$$= [T\hat{x}T^\dagger, T\hat{p}T^\dagger]$$
$$= -[\hat{x}, \hat{p}]$$

· one can also show that

(i) $T\,\hat{x}(t)\,T^{\dagger} = \hat{x}(-t)$

$$T\, e^{iHt}\,\hat{x}\, e^{-iHt}\, T^{\dagger} = e^{-iHt}\,\underbrace{T\hat{x}T^{\dagger}}_{(=\hat{x})}\,e^{+iHt}$$

$$= \hat{x}(-t)$$

Similarly,

(ii) $T\,\hat{p}(t)\,T^{\dagger} = -\,\hat{p}(-t)$

One can also see that

(iii) $T\,\vec{j}\,T^{\dagger} = -\,\vec{j}$ $\longrightarrow$ see the <u>Kramers doublet</u>!

angular momentum.

② $\langle x_f, T/2 \,|\, x_i, -T/2 \rangle$ transition function

$= \langle x_f |\, e^{-iHT}\, | x_i \rangle$

$\hat{x}(t)\,|x_i, t\rangle = x_i\,|x_i, t\rangle$

$|x_i, t\rangle = e^{iHt}\,|x_i\rangle$

$$\langle x_f | e^{-iHT} | x_i \rangle = \langle x_f | \left(e^{-iHT} \mathcal{T} | x_i \rangle \right)$$

$$= \langle x_f | \left(\mathcal{T} \underbrace{e^{iHT} | x_i \rangle}_{= |x_i, T\rangle} \right)$$

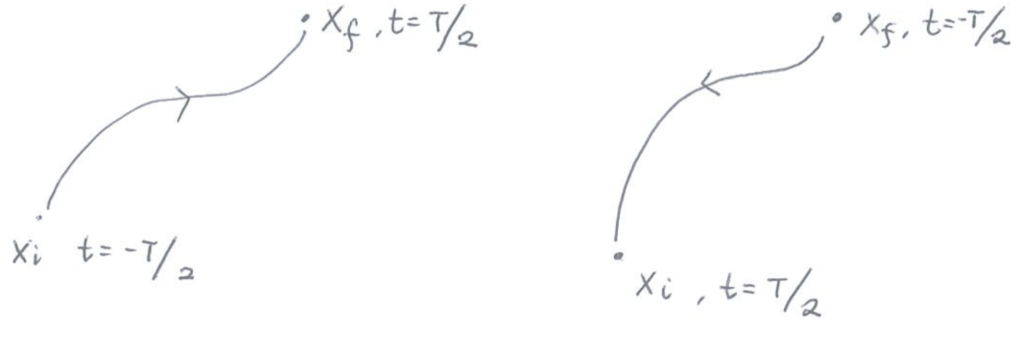
$$\overset{\text{def.}}{=} \overline{\langle x_i, T | \left(\mathcal{T}^{\dagger} | x_f \rangle \right)}$$

$$= \overline{\langle x_i, T/2 | x_f, -T/2 \rangle}$$

$\therefore$ for a time-reversal system,

$$\underline{\langle x_f, T/2 | x_i, -T/2 \rangle = \langle x_f, -T/2 | x_i, T/2 \rangle}$$

! microreversibility !



" same probability. "

Path-integral rep.

$$(\text{LHS}) = \int_{\substack{\vec{x}(t=-T/2)=x_i \\ \vec{x}(t=T/2)=x_f}} D\vec{x}(t)\; e^{i\int_{-T/2}^{T/2} dt\, \mathcal{L}(\vec{x}(t),\, \dot{\vec{x}}(t))}$$

$$(\text{RHS}) = \int_{\substack{\vec{x}(t=-T/2)=x_f \\ \vec{x}(t=T/2)=x_i}} D\vec{x}(t)\; e^{-i\int_{-T/2}^{T/2} dt\, \mathcal{L}(\vec{x}(t),\, \dot{\vec{x}}(t))}$$

~~for T-inv. system~~ $\cancel{= \mathcal{L}(\vec{x}(-t), -\dot{\vec{x}}(t))}$

$\vec{y}(t) \equiv \vec{x}(-t)$

for a T-inv. system

$$\left(= \int_{\substack{\vec{y}(t=T/2)=x_f \\ \vec{y}(t=-T/2)=x_i}} D\vec{y}(t)\; e^{-i\int_{-T/2}^{T/2} dt\, \mathcal{L}(\vec{y}(t),\, \dot{\vec{y}}(t))} \right)$$

$\Rightarrow$ this implies that $\overline{(\text{LHS})} = (\text{RHS})$

{ In other words, the partition function have to be real! }

§ Kramer's singlet / doublet.

① note that $\quad T\hat{\vec{j}}T^{\dagger} = -\hat{\vec{j}}$ ← angular momentum.

$$T\hat{j}^a\hat{j}^a T^{\dagger} = \hat{j}^a\hat{j}^a$$

$$\therefore \quad T|j,m\rangle = c(j,m)\,|j,-m\rangle$$

Since $\quad |j,m\rangle = \alpha(j,m)\,(J^-)^{j-m}\,|j,j\rangle$

$$\cancel{= \beta(j,m)\,(J^+)^{j+m}\,|j,-j\rangle},$$

$$T|j,m\rangle = \alpha(j,m)\,(-J^+)^{j-m}\,\underbrace{|j,-j\rangle}_{c(j,j)}$$

$$= \alpha(j,m)\,\beta(j,m)\,c(j,j)\,(-1)^{j-m}\,|j,-m\rangle$$

$$\alpha(j,m) = \left[\prod_{a=m+1}^{a=j}\sqrt{(l+a)(l-a+1)}\right]^{-1}$$

$$\therefore \quad \alpha(j,m)\beta(j,m) = 1$$

$$\beta(j,m) = \left[\prod_{b=-j}^{b=m-1}\sqrt{(l-b)(l+b+1)}\right]$$

$$T|j,m\rangle = \overset{\text{phase}}{c(j,j)}\,(-1)^{j-m}\,|j,-m\rangle$$

$$T^2|j,m\rangle = |c(j,j)|^2(-1)^{2j}\,|j,m\rangle$$

$$\Rightarrow \boxed{T^2 = (-1)^{2j}} \quad \begin{cases} +1 & \text{for integer spin (boson)} \\ -1 & \text{for half-integer spin (fermion).} \end{cases}$$

② ~~Since When $[T, H] = 0$ energy eigenstate & $T^2 = -1$~~

② When $T^2 = (-1)$ & $[T, H] = 0$,

$$H|n\rangle = \overset{\text{real}}{E_n}|n\rangle$$

$$H(T|n\rangle) = E_n(T|n\rangle)$$

$T|n\rangle = \alpha|n\rangle$

$(-1)|n\rangle = T^2|n\rangle = \alpha^* T|n\rangle$

$= |\alpha|^2|n\rangle$

inconsistent!!

$|n\rangle$ & $T|n\rangle$ are linearly independent! $\Leftarrow$ Kramer's doublet

~~$|n\rangle = \alpha T|n\rangle$ $\quad T|n\rangle = \alpha^* T^2|n\rangle = -\alpha^*|n\rangle$~~