Vortex & fermionic zero modes (& Braiding group)

$$\mathcal{L} = \mathcal{L}_b + \mathcal{L}_f$$ abelian-Higgs model

$$\mathcal{L}_b = -\underbrace{|D_\mu\phi|^2}_{(\partial_\mu - i2k A_\mu)\phi} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \underbrace{V(\phi)}_{\text{s.t } \langle\phi\rangle \neq 0}$$

$$\mathcal{L}_\psi = i\bar{\Psi}\gamma^\mu \underbrace{D_\mu\psi}_{=(\partial_\mu - ikA_\mu)\psi} + (i\bar{\Phi}\,\psi^T\gamma^0\psi + c.c.)$$

$$\mathcal{H} = \int d^3x \left[|D_0\phi|^2 + |D_i\phi|^2 + \frac{1}{2}(\vec{E}^2 + B^2) + V(\phi) \right]$$

①

Vortex : (static) field configuration having finite energy (sol. to e.o.m.)

$$|\langle\phi\rangle| = v$$

As $r \to \infty$ (asymptotically),

"vorticity" $\in \mathbb{Z}$ (quantized)

$$\phi \sim v e^{i\nu\varphi}$$

(r, φ) : polar coordinate.

to have finite energy,

$$D_i \phi \sim 0 \quad \text{as} \quad r \to \infty$$

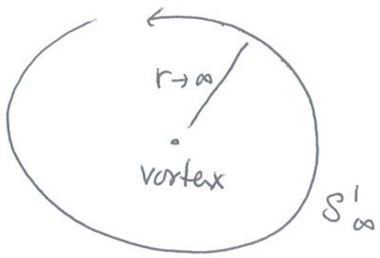
$$\Rightarrow (\partial_i - i2k A_i)(v e^{i\hat{\nu}\varphi})$$

$$= i v e^{i\hat{\nu}\varphi} \hat{\nu} \partial_i \varphi - i(2k) A_i v e^{i\hat{\nu}\varphi} \sim 0$$

$$\therefore \quad A_i \sim \frac{\hat{\nu}}{2k} \partial_i \varphi \quad \text{as} \quad r \to \infty$$

$$A = \frac{\hat{\nu}}{2k} d\varphi$$

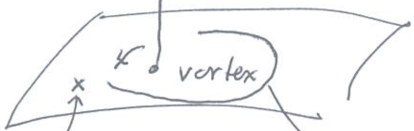
thus $\displaystyle \frac{1}{2\pi} \int_{D_2} dA = \frac{1}{2\pi} \int_{S^1_\infty} A = \frac{\hat{\nu}}{2k}$ .. fractional flux



<u>remark</u> fractional flux is ok?

$\langle \phi \rangle \neq 0 \qquad U(1) \to \underbrace{\mathbb{Z}_{2k}}_{\text{residual gauge sym.}}$

$\frac{\hat{\nu}}{2k} \cdot 2\pi$



this phase is ok because it is the phase of $\mathbb{Z}_{45}$ gauge transf.

charged particle of unit charge under $\mathbb{Z}_{2k}$

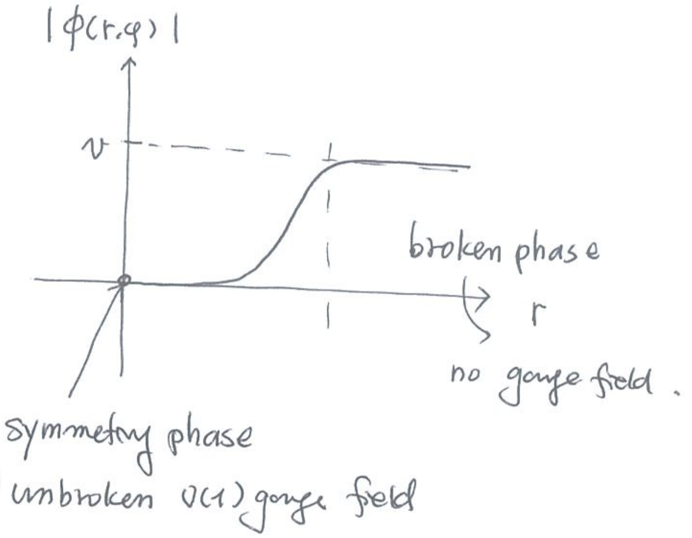
Aharanov - Bohm phase

$e^{i \frac{\hat{\nu}}{2k} 2\pi}$. $e^{i\,2k}$

② $\phi = v e^{i\hat{\nu}\varphi} A(r)$ $\cdots$ (*) $A_0 = 0$.

$$A_i dx^i = \frac{\hat{\nu}}{2k} d\varphi \cdot B(r)$$

one can show that $\exists$ sol. $(A(r), B(r))$ satisfying the field eqn. of motion.



vortex

③ <u>real</u> fermionic zero modes in the background of vortex.

~~e.o.m. of ψ;~~

(i)

vortex configuration

eigenfunction

$$i\gamma^{i}(\partial_{i} - ikA_{i})\psi_\lambda - i\phi\psi_\lambda^* = \lambda\psi_\lambda$$

eigenvalue

Schematically,

$$\psi = \sum_\lambda a_\lambda \psi_\lambda$$

complete set.

Grassmannian variable $\Leftarrow$ complex ~~gra~~ coefficient. ~~anti-commuting~~ (~~Grassman~~

insert

$$\Rightarrow \quad \mathcal{L}_f = \int d^3x \; i\bar{\Psi}\gamma^\mu D_\mu \Psi + \left(i\bar{\Phi}\,\psi^T\gamma^0\psi + \text{c.c.}\right)$$

$$= \int dt \sum_{\lambda_0}' \bar{a}_\lambda \dot{a}_\lambda + \lambda \bar{a}_\lambda a_\lambda$$

(ii)

zero mode ? (special role)

$$i\gamma^{0i}(\partial_i - ikA_i)\psi_0 - i\phi\,\psi_0^* = 0$$

for the sake of later convenience, we write

$$\psi_0 = \psi_1 + i\psi_2$$

Dirac

real two-component spinors

$$\phi_{\text{vortex}} = \phi_1 + i\phi_2$$

real scalar configurations

one can rewrite the zero-mode eqn. as follows

~~$(\gamma^i \otimes 1_2)(\partial_i - ik \otimes A_i)$~~

$$\underbrace{\left[(\gamma^i \otimes 1_2)(\partial_i - ik(1_2 \otimes \tilde{\gamma}^0)A_i) + (1_2 \otimes \tilde{\gamma}^i)\phi_i\right]}_{\equiv D}\tilde{\psi} = 0.$$

$\phi_i = (\phi_1, \phi_2)$

where
$$\left\{\begin{array}{l} \gamma^0 = i\tau^2,\ \gamma^1 = \tau^3,\ \gamma^2 = \tau^1 \\ \tilde{\gamma}^0 = i\tau^2,\ \tilde{\gamma}^1 = \tau^3,\ \tilde{\gamma}^2 = \tau^1 \\ \tilde{\psi} = \underbrace{\tilde{\psi}_{\alpha a}}_{\text{real}} \end{array}\right.$$

$\alpha = 1, 2$ ← space-time spinor indices

$a = 1, 2$ ← internal indices

s.t. $\tilde{\psi}_{\alpha 1} = (\psi_1)_\alpha$

$\tilde{\psi}_{\alpha 2} = (\psi_2)_\alpha$

real Majorana spinor

(~"CT transf")

note that, for $M = \gamma^0 \otimes \tilde{\gamma}^0$,

$$MD = -DM$$

$\begin{pmatrix} M^2 = 1 \\ M \doteq \pm 1 \end{pmatrix}$

plays a role as a chiral op. in x dim.

one can thus define

$$\text{Index}[D] = \left\{\begin{array}{c}\#\text{ of} \\ \text{zero mode} \\ \text{with } M \doteq 1\end{array}\right\} - \left\{\begin{array}{c}\#\text{ of} \\ \text{zero mode} \\ \text{with } M \doteq -1\end{array}\right\}$$

(iii) zero mode in the background field with $\hat{\mathbb{V}} = 1$

ansatz : $M = 1$

$$\tilde{\psi} = f(r) \underbrace{\begin{pmatrix} 1 \\ i \end{pmatrix}}_{\gamma_0 \doteq -i} \otimes \underbrace{\begin{pmatrix} 1 \\ -i \end{pmatrix}}_{\tilde{\gamma}_0 \doteq +i} + g(r) \underbrace{\begin{pmatrix} 1 \\ -i \end{pmatrix}}_{\gamma_0 \doteq +i} \otimes \underbrace{\begin{pmatrix} 1 \\ i \end{pmatrix}}_{\tilde{\gamma}_0 \doteq -i}$$

$M \doteq 1$ $\qquad$ $M \doteq 1$

& $\quad f^* = g \quad (\because \text{reality of } \tilde{\psi})$

the equation then becomes

$$-i(\partial_1 + i\partial_2) f - k \underbrace{(A_1 + iA_2)}_{= i\frac{e^{i\varphi}}{2k \cdot r} B(r)} f - ig \underbrace{(\phi_1 + i\phi_2)}_{= ve^{i\varphi} A(r) \ (\hat{\mathbb{V}}=1)} = 0$$

$$+i(\partial_1 - i\partial_2) g - k(A_1 - iA_2) g + if(\phi_1 - i\phi_2) = 0$$

complex conj

$$\Rightarrow \quad \frac{\partial}{\partial r} f(r) = -\left\{ \frac{1}{2r} B(r) f(r) + vA(r) g(r) \right\}$$

normalizable sol. exists only when

$$f(r) = g(r) = f^*(r) \quad \cdots \quad f(r) \text{ is real}$$

$\Rightarrow$ $\cancel{f(r) = \left(\text{real const}\right) \times f}$

$\Rightarrow$

$$f(r) = (\text{real const}) \times \exp\left[-\int_0^r dr' \left[\frac{B(r)}{2r} + v A(r)\right]\right]$$

$$\sim (\text{real const}) \cdot e^{-vr} / \sqrt{r}$$

$$\boxed{\text{thus, } \exists \text{ one } \underline{\underline{\text{real}}} \text{ fermionic zero mode in the background of vortex with } \hat{\nu} = 1}$$

remark ① $\nexists$ zero mode with $M \doteq -1$!

$\Rightarrow$ By applying the <u>Callias index thm</u>, it was shown that, for the vortex background <u>with $\hat{\nu}$</u>.

<u>$\exists$ $\hat{\nu}$ real fermionic zero modes.</u>

~~Furthermore~~

One can also argue that those $\hat{\nu}$ zero modes carry <u>angular momentum</u> $j' = \pm\left(\frac{\hat{\nu}-1}{2}\right), \pm\left(\frac{\hat{\nu}-3}{2}\right), \cdots, -\left(\frac{\hat{\nu}-3}{2}\right), -\left(\frac{\hat{\nu}-1}{2}\right)$

(*)

$$\begin{Bmatrix} \text{fermionic} \\ \text{zero modes} \end{Bmatrix} \Rightarrow \underbrace{\begin{Bmatrix} \text{degenerate} \\ \text{states} \end{Bmatrix}}_{\text{vortex multiplet}}.$$

$$\psi = \underbrace{\sum_i \overset{\text{real}}{a_i}\, \psi_0^i}_{\text{zero modes}} + \underbrace{\sum_\lambda b_\lambda \psi_\lambda}_{\text{non-zero modes}}$$

$$S_Q = \int dt \sum_i{}' a_i \dot{a}_i + \left(\begin{matrix}\text{massive} \\ \text{modes}\end{matrix}\right).$$

Grassmannian

Canonical quantization $\{a_i, a_j\} \propto \delta_{ij}$

representation : gamma matrices
(Clifford algebra)

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij} \leftrightarrow \{a_i, a_j\} \propto \delta_{ij}$$

$\therefore$ for the background with $\hat{\mathbb{V}} = k$ (even), $\exists\ 2^{k/2}$ degenerate states !

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e.g. $\hat{\mathbb{V}} = 2$. $\quad \vec{a} \doteq (\tau^1, \tau^2)$ ← Pauli matrices.

	spin
τ^+	$+1/2$
τ^-	$-1/2$

~~$|j'+1/2\rangle = \tau^+|j'\rangle$, $|j'\rangle$ } two energy degenerate states on which τ^i act~~

$\{ |j'\rangle , \tau^+|j'\rangle \}$: two degenerate energy eigenstates on which $\vec{a} = (\tau^1, \tau^2)$ act.

$|j'\rangle$: angular mom. j'

$\tau^+|j'\rangle = |j'+\frac{1}{2}\rangle$

CT-symmetric. i.e,

$$+j' = -(j' + \tfrac{1}{2}) \Rightarrow j' = -1/4$$

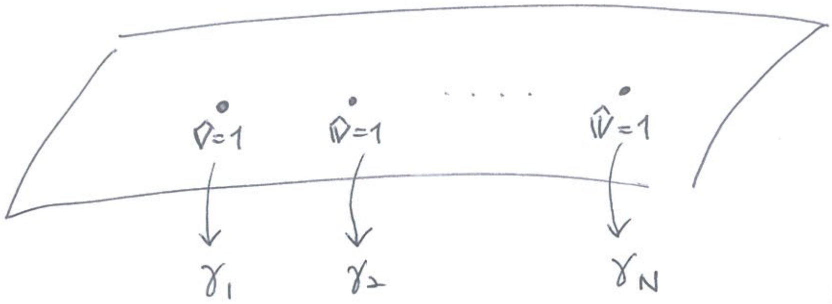
vortex multiplet $\{ |j' = -\frac{1}{4}\rangle , |j' = +\frac{1}{4}\rangle \}$

(5) Non-abelian statistics & Braiding group.

(i) $\hat{\mathbb{V}} = 1 \longrightarrow \exists$ single real fermion zero mode

$$\langle \gamma \rangle = \pm 1 \quad \text{no degenerate state}$$

(ii) Let's consider a system of (widely) separated $\underset{\text{(even)}}{N}$ $\hat{\mathbb{V}} = 1$ vortices.



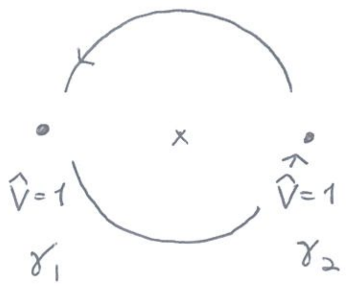
each vortex is ~~asso~~ associated with fermion zero-mode

$$\{\gamma_a, \gamma_b\} = 2\delta_{ab}$$

$\therefore$ the system of N $\hat{\mathbb{V}}=1$ vortices has $2^{N/2}$ degenerate states

$\Rightarrow$ each vortex carries $\sqrt{2}$ quantum states

signal of non-abelian statistics.



adiabatic move of N vortices leads to the Berry phase (connection) in the space of $2^{N/2}$ degenerate (ground) states ; monodromy Braiding group.

⑥ Braiding group.

How to find out the rep. of Braiding group on the space of $2^{N/2}$ states?

- two elementary facts.

(i) each vortex has its own zero-mode, uniqued determined up to sign, i.e, $\pm \gamma_i$ $(i=1,2,\cdots,N)$

→ braid group : permutation and sign changes of γ_i

(ii) the action of braid group must commutes with $(-1)^F$

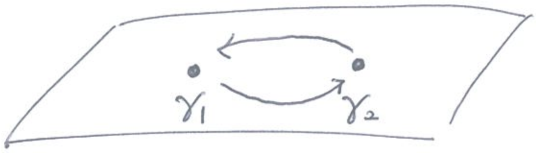
$\{(-1)^F, \gamma_i\} = 0$. fermion # op.

$(-1)^F \doteq \gamma_1 \cdots \gamma_N$ in the space of $2^{N/2}$ states.

B_{12} : $(\gamma_1, \gamma_2) \longrightarrow (\gamma_2, -\gamma_1)$

minus sign otherwise

$[B_{12}, (-1)^F] \neq 0$.



counter-clockwise

note that $B_{12}^2 = -1$ braiding ≠ permutation

B_{12}^{-1}

clockwise

$B_{12}^{-1} = -B_{12}$

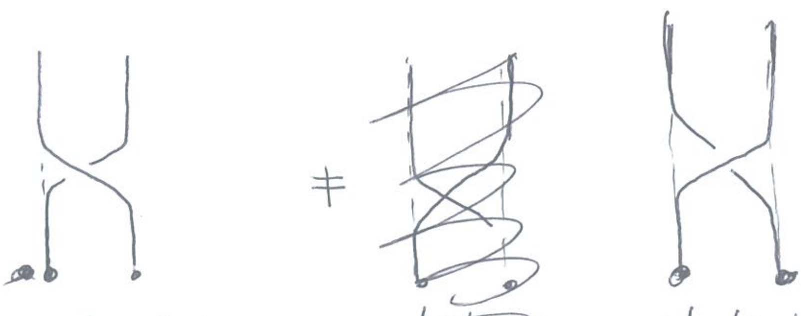
For a system of 2 $\hat{V}=1$ vortices, $\exists$ two degenerate states

on which $\gamma_1 \doteq \tau^1$ & $\gamma_2 \doteq \tau^2$

then $B_{12}\, \gamma^i\, B_{12}^{-1} = (\tau^2, -\tau^1)$

$$\rightarrow \quad B_{12} \doteq \pm \begin{pmatrix} e^{\frac{\pi}{4}i} & 0 \\ 0 & e^{-\frac{\pi}{4}i} \end{pmatrix}$$ on the space of 2 degenerate states.

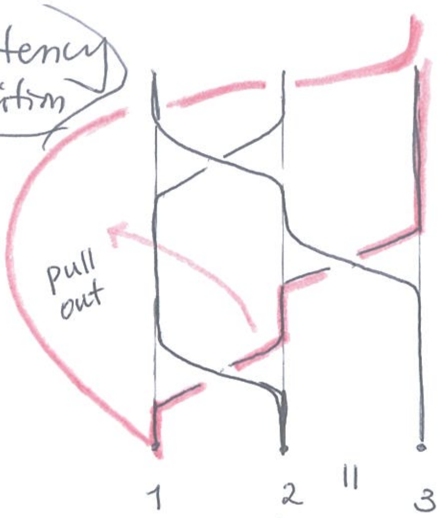
(iii) One can also describe the move for braiding group as



counter clockwise ~~clockwise~~ clockwise.

$\boxed{Q}$: does the operation $B_{i,i+1} : (\gamma_i, \gamma_{i+1}) \longmapsto (+\gamma_{i+1}, -\gamma_i)$ give the rep. of the consistent braiding group?

Consistency Condition



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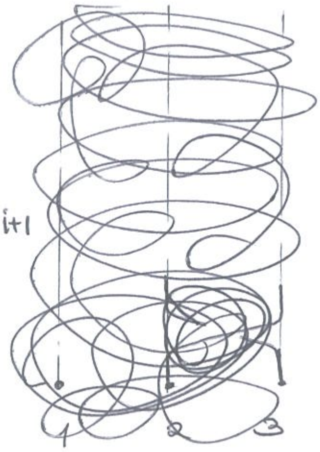
$B_{i,i+1} B_{i+1,i+2} B_{i,i+1}$ =



$B_{i+1,i+2} B_{i,i+1} B_{i+1,i+2}$

One can show that ~~the al~~

$$B_{i,i+1} B_{i+1,i+2} B_{i,i+1} : (\gamma_i, \gamma_{i+1}, \gamma_{i+2}) \longrightarrow (\gamma_{i+2}, -\gamma_{i+1}, \gamma_i)$$

and

$$B_{i+1,i+2} B_{i,i+1} B_{i+1,i+2} : (\gamma_i, \gamma_{i+1}, \gamma_{i+2}) \longrightarrow (\gamma_{i+2}, -\gamma_{i+1}, \gamma_i)$$

$$\therefore \quad B_{i,i+1} : (\gamma_i, \gamma_{i+1}) \rightarrow (\gamma_{i+1}, -\gamma_i)$$

does the represent the Braiding operation.

remark

fusion.

$$\left\{ \begin{array}{l} \text{two } \hat{\nu} = 1 \text{ vortices} \\ \text{has two degenerate} \\ \text{states} \end{array} \right\} = \left\{ \begin{array}{l} \text{one } \hat{\nu} = 2 \text{ vortex} \\ \text{has two degenerate} \\ \text{states} \end{array} \right\}$$

each zero mode carries spin = 0.

two degenerate state carry $\pm \frac{1}{4}$ angular mom.

But braiding matrix

$$B_{12} \doteq \pm \begin{pmatrix} e^{\frac{\pi}{4} i} & 0 \\ 0 & e^{-\frac{\pi}{4} i} \end{pmatrix}$$

consistent with

~ " rotation by π angle "