ARITHMETIC, ALGEBRA, AND THEIR RELATION TO GEOMETRY: A GLOBAL APPROACH

Organized by
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Abstracts

Jessica Carter (Aarhus University)
Philosophical perspectives on the interconnections between arithmetic, algebra and geometry
Abstract: Mathematics is often claimed to be connected in various ways – both horizontally and vertically. This picture has inspired a number of different philosophical positions. Recently J. Cole has formulated a position on the ontology of mathematics. Cole draws on tools from Social Ontology to say that mathematical objects are introduced by collective declarations and that they serve a representational function. J. Ferreirós, on the other hand, formulates a theory of our knowledge of mathematics that is based on the view that mathematics is the outcome of various activities of human agents. Both accounts are based on the idea that mathematics consists of different "levels" or strata and that different levels interact in various ways. In the article 'Mathematical Practice, Fictionalism and Social Ontology' (Carter 2023) I propose a Peircean inspired 'pragmatic' view of mathematics where the reality of abstract mathematical objects depends on whether propositions about them can be reduced to true statements concerning substances at a lower level. This position thus also assumes that mathematics can be organized into different levels. In addition to present a few more details about these philosophical positions, I wish to take the opportunity here to discuss the viability of this picture of mathematics, that is, the image of mathematics as being interconnected in various ways.

Karine Chemla (SPHERE, CNRS & Université Paris Cité)
Polynomials in 13th-century China: A material transformation of diagrammatic work on equations?
Abstract: The mathematical traditions in China that recognize *The Nine Chapters on Mathematical Procedures* (first century CE) as a canon attest to work on algebraic equations conceived as arithmetic operations. These traditions all approach the resolution of these equations by analogy with division, even if this analogy takes different forms in different authors and at different times. On the other hand, the related Chinese sources testify to a radical transformation in the way equations were established between the 1st and 13th centuries. While sources show that, before the 12th century, actors established equations through diagrammatic work, several 13th-century treatises show how this goal could be

achieved using polynomials, the nature of which will be discussed, as well as operations on these polynomials. The historiography of this shift has been poorly understood, probably for two reasons. First, to understand it, we need to take into account material practices prior to the 10th century, which leave only indirect traces in the written record. Second, we need to understand the modalities of their subsequent transformation into (at least partly) paper-based practices. My presentation aims to support the hypothesis that the polynomial algebra evidenced in 13th-century works derives from the diagrammatic work with which ancient actors established equations.

João Cortese (University of São Paulo)
Does a Number have a Form? On Blaise Pascal's Conception of the Relationship between Geometry and Arithmetic

Abstract: Blaise Pascal (1623-1662) was born precisely 400 hundred years ago. One can see in his mathematical works a strong relationship between geometrical forms, numbers, and even "weights".
In his *Treatise on the arithmetical triangle* (written in about 1654 but published only posthumously in 1665), numbers are geometrically disposed according to the famous arithmetical triangle. Pascal is not the first to present this triangle, since it is found in earlier Arabic, Chinese and Sanskrit sources (Edwards 1987; Djebbar 1997; Rashed 1998; Lam Lay Yong 1980). Nevertheless, Pascal's triangle can be considered in its difference from the predecessors (Kyriacopoulos 2000); in particular, his "*usages*" of the triangle allow for new applications, including the use of mathematical induction (which should be related to the Italian mathematician Maurolycus).
In his *Lettres de A. Dettonville* (1658/59), on the other hand, Pascal's method of indivisibles (for calculating areas, volumes and centers of gravity) deals with the model of a "balance" in an Archimedean fashion. However, this ancient approach is "*colored*" "with an enthusiasm for the theory of numbers", as Boyer (1949) would say. The latter affirmation goes in the right direction, but it should be developed in further detail: which features underlie this work, which brings "statics" into geometry, dealing with it "arithmetically" in some sense?
In this communication, I will discuss some aspects of the conception of number that underlie both the *Treatise on the arithmetical triangle* and the *Lettres de A. Dettonville*, relating it to the "geometrical" disposition and "geometrical" context in which they appear.

Veronica Gavagna (Università degli Studi di Firenze)
The relationship between algebra and geometry: from the medieval abachistic tradition to the work of Rafael Bombelli

Abstract: From the middle of the 13th century, al-Khwārizmī's rhetorical algebra began to spread in some regions of Italy, thanks mainly to the dissemination of the vernacularisations of Leonardo Pisano's *Liber Abaci* testified by the extant abacus treatises. The abacus treatises were collections of problems in practical arithmetic, in practical geometry, and more rarely (at least in surviving manuscripts) algebra. In his *Liber Abaci*, Leonardo repurposed al-Khwārizmī's geometric constructions to ensure the generality of algebraic procedures for solving second-degree

equations, and this approach was also transmitted to the later abachist tradition. In this talk I will illustrate some examples of geometry, understood as a demonstrative support for algebraic procedures, drawn from the abachistic tradition; I will also try to outline the evolution of this approach up to the work of Rafael Bombelli - the *Algebra* - in which a new vision of the relationship between algebra and geometry began to emerge; a vision that was not limited to the geometric interpretation of some procedure but began to foreshadow a broader and deeper interaction that found its fulfillment in Viète and Descartes.

Emmylou Haffner (ITEM, ENS Ulm)

Arithmetic in algebra (and conversely?)

Abstract: When defining the concepts of module, ideal and field, Richard Dedekind (1831-1916) also defined ‘arithmetical’ notions to study them — what we could see as arithmetical reinterpretations of relationships such as inclusion. Setting up an ‘arithmetic of modules’ and an ‘arithmetic of ideals’ had two explicit (although admittedly different) aims: to lay clear foundations for the theories, and to be able to compute with the concepts as if they were numbers. In this talk, I will propose to study the prevalence of arithmetic in Dedekind's ‘algebraic’ concepts following a thread started in manuscripts on module theory in the early 1870s up to the fifth version of his algebraic number theory which he never published (ca. 1895-1913, published in 2020 by Katrin Scheel). I will suggest that this ‘arithmetical’ methodology tied intricate links between algebra, arithmetic and logic, using a selection of his unpublished manuscripts and published papers on modules, algebraic numbers, algebraic functions, lattices, and set theory. In doing so, I hope to shed further light on the status of arithmetic and logic in Dedekind's mathematics, but also on his own gradual understanding of (something close to) what we would call algebraic structures.

Michael Harris (Columbia University)

Galois theory by way of geometry

Abstract: Although Galois theory was developed as a theory of symmetries of the roots of polynomials, in the late 20th century the Galois groups of number fields became the central objects of study in their own right in several branches of number theory. What it means to "understand" Galois groups is now bound up with geometry, the meaning of which, in turn, has undergone successive waves of expansion under the influence of Grothendieck and his successors. I will illustrate this with examples from the arithmetic of elliptic curves and from the Langlands program, emphasizing how contemporary number theory blurs the distinction between its objects of study and the means by which they are studied.

Agathe Keller (SPHERE, CNRS & Université Paris Cité)

Bhāskara II on proofs and the interpretation of algebraical operations geometrically.

Abstract: Twice Bhāskara II (b. 1114) evokes in his canonical treatise *Algebra* (*bījagaṇita*) the term proof (*upapatti*), and twice this involves the relations of algebra with geometry. In this presentation, I will look at the mathematical context of these two famous statements of proofs in which are articulated geometrical interpretations of algebraical operations. The first proof

concerns the Pythagorean procedure and the second, the interpretation of multiplicative polynomials, and operations on them as dealing with rectangular areas measured with square units.
The aim of the presentation will be to situate these reasonings in relation to the history of mathematical proofs in Sanskrit mathematical sources, and also to highlight the continuity of some questions found in Sanskrit sources on the geometrical interpretation of arithmetical/algebraical operations and the answer that Bhāskara seems to have given to them here.

Kim Minhyong (University of Edinburgh and KIAS)
Applications of History to Algebra and Geometry?
Abstract: As is well-known, mathematicians tend to focus on their own work in a single-minded fashion. It is quite difficult to get them to devote time and energy to learning other subjects unless the benefit to their mathematics is somehow clear. In reality, the relationship between mathematics and neighbouring areas of inquiry is quite complicated and many boundaries are illusory. For example, it is hard to distinguish the history of mathematics and physics, and, in recent years, it is even common to see reference to 'physical mathematics' as a subject of study. In a similar vein, at an interdisciplinary conference a few years ago, I challenged mathematical biologists to build up a notion of ‘biological mathematics’.
As far as I can tell, the mysterious relationship between algebra and geometry has been a source of tension and fascination among mathematicians for a very long time. Is it conceivable that research in history can help mathematicians to resolve some of the greatest difficulties at the forefront of research? This talk will speculate about this possibility. It will at least try to outline some examples where a *misunderstanding of history* has obstructed progress, both at a collective and an individual level.

Eunsoo Lee (Seoul National University, South Korea)
Naming Mathematical Curves and Scientia Penitus Nova
Abstract: This paper examines the evolution of the naming of mathematical curves from ancient Greece to the 17th century. By analyzing the process of curve naming, the paper aims to explore how the mindset of mathematicians in understanding curves has evolved over time. The various methods of naming curves based on their appearance, essential properties, points loci, generation methods, and algebraic equations illustrate the gradual acceptance of curves as geometrical objects, tools, and solutions to mathematical problems. Ultimately, the paper will revisit Descartes’ classification of curves, and thus, his *scientia penitus nova*, through the lens of this historical understanding of curve naming.

Antoni Malet (Institut d’Història de la Ciència (UAB)/Laboratoire SPHERE (UMR 7219, CNRS-Université Paris Cité))
Between arithmetic, algebra and geometry: the arithmetization of geometrical magnitude in early modern Europe
Abstract: My presentation will focus on the conceptual shift that transformed the well-established Euclidean categories of number and the geometrical magnitudes in early modern mathematics (c1550 - c1700). By way of introduction we will

shortly consider how the early Renaissance algebraic and arithmetical practices/ideas (heavily indebted to medieval Arabic mathematics) clashed with the letter and the spirit of Euclid's *Elements*, which at the time provided the basis for mathematical education. Next, we will turn to analyse the increased social role of metrological practices, and its reflection in early modern practical geometry (as a discipline) and practical geometries (a mathematical book genre that gained enormous popularity). We will discuss some new approaches to measuring that were given currency in practical geometries and which contributed to materialize both a new notion of number (versus the Euclidean notion, restricted to natural numbers), and a new arithmetical understanding of geometrical magnitude. As a case study we will pay particular attention to the arithmetic and practical geometry of Simon Stevin (1548 - 1620). A creative mathematician in his own right, Stevin pioneered the introduction of the theory and practice of decimal fractions among mathematical practitioners. Stevin 's *Arithmetic* (1585) as well as his *Practice of geometry* (1605) provide vantage points to look into the mathematical arguments that made possible to supersede the venerable Euclidean notions. Finally, we shall consider an important debate that was caused, so to speak, by the changing nature of geometrical magnitude. Once the arithmetization of magnitudes gained ground, the arguments for abandoning the Euclidean notions of ratio and proportionality multiplied. The main arguments crossed in this debate provide substantial evidence that mathematical innovations must sometimes pay the price of mathematical inconsistency.

Nicolas Michel (Wuppertal Universität)

Algebra as resource, algebra as model. Some reflections from the history of enumerative geometry.

Abstract: Hermann Schubert's calculus, a symbolic and computational tool for the enumeration of geometrical figures satisfying given conditions, was the culmination of decades of collective work on such problems, by mathematicians distributed all across Europe. Upon its publication in the 1870s, the mysterious efficiency of its symbolic apparatus and the sheer size of the numbers it produced -- such as that of 5,819,539,783,680 (spatial) cubics touching twelve given quadrics -- drew both admiration and suspicion.

Many early enthusiasts, like Arthur Cayley or Charles Sanders Peirce, sought to interpret it as an application of (Boole's) "algebraic logic" to geometry. At the same time, leading algebraic geometers such as Georges-Henri Halphen or Eduard Study also castigated it as an intuitive and therefore unreliable method, lacking proper foundations in rigorous algebraic techniques. Schubert himself, attempting to defend the validity of the principles underlying his calculus, reframed them as geometrical applications of the Fundamental Theorem of Algebra.

In sum, years after Klein had proclaimed the end of the divide between synthetic and analytic geometry (or, at least, of its relevance to modern mathematics), the rise of enumerative methods seemed yet to be rife with lingering traces of this methodological and epistemological opposition. In this talk, I shall explore the images of algebra which structured various approaches to enumerative problems in geometry, and use this historical episode to think anew the manifold relation between geometry and algebra in the 19th century. Rather than an opposition between geometries with and without algebra, I shall argue, one may perhaps

consider an opposition between two uses of algebra within geometry -- as an epistemic resource, or as a formal model.

Reviel Netz (Stanford University)

Why Euclid Can't Count

Abstract: Why do the canonical Greek geometrical texts contain so few numbers? The talk will present evidence for the avoidance of numbers in many Greek mathematical texts, noting several contrasts: (1) the evidence from papyri, (2) Imperial-era authors, (3) Late Ancient commentaries. The contrasts all cohere around the practices of mathematical education, and the question arises what mathematical education could have been like at the early era of the formation of the Greek mathematical genre. I conclude that the likelihood is that, even as early as the fourth century BCE, the Greek mathematical genre was formed in contrast with contemporary mathematical educational practice. The reasons for this must remain speculative, and the talk will conclude by offering some speculations concerning the sociology of early Greek mathematics.

Young Sook OH (independent scholar)

Arithmetic Operations and Geometric Justification in Joseon Society

Abstract: One of the most important features of mathematics in the late Joseon Dynasty was the use of counting rods as the primary computational tool. The calculation using counting rods had flourished since ancient times in East Asia. Whilst in other parts of East Asia, it was rapidly replaced by the calculation using abacus or one using brush and paper, in Joseon society, it remained as the main computational tool until the late 19th century. The transition from the calculation using counting rods to one using brush and paper was a long process that occurred throughout two whole centuries, specifically from the 17th century; it was also luckily documented in mathematical texts, providing great evidence for future historians. Hence, using these texts, I will explore the role of mathematical tools and geometric justification in this transition, focusing on the examples of arithmetic operations, root extractions, and higher order equations. Whilst the simple arithmetic operations were able to be easily substituted, those that had complex structures, reliant on the algorithm using counting rods, required other rationales. One significant example of these rationales was the geometric justification of the operation. Through the close examination and comparison of how the geometric justification work with respect to the computational tools, this talk will aim to shed new light on the relationship between geometry and arithmetic operations in ancient East Asian mathematics.

PAN Shuyuan (Institute for the History of Natural Sciences, Chinese Academy of Sciences, Beijing, China)

How was Chinese Resources borrowed When the Frist Chinese Euclid was Established in the early 17th Century

Abstract: The year of 1607 saw the first Chinese translation of the *Elements* published by the Italian Jesuit Matteo Ricci in collaboration with the Chinese literatus Xu Guangqi. The translation was and is still regarded as the most representative mathematical text of the so-called "Western learning" (*xixue*), namely, the knowledge introduced from Europe into China in early modern times. It is noteworthy that Chinese materials were used in the translation, which mainly based on Christoph Clavius's Euclid (1574) . On the one hand, translators inevitably and deliberately employed Chinese terms to convey mathematical concepts and ideas from Europe; on the other hand, quotations from an ancient Chinese classic *Chuang-Tzu* 莊子, with Ricci and Xu's reflections, were inserted into the explanation of a postulate, and the criticisms of the famous problem 'angulus contactus'. Interestingly, those treatments were related, closely or indirectly, to the distinction and connection between numbers, magnitudes, and general quantities. In this talk, we will discuss how Ricci and Xu understood quantitative concepts when they borrowed Chinese resources to make the translation.

Eleonora Sammarchi (ETH Zürich)
Uses of geometry in relation to algebra and arithmetic. Some case studies taken from Arabic algebraic texts (9th-12th c.).

Abstract: If algebraic texts are an emblematic context in which one can identify the various arithmetical approaches that characterized mathematics in the Islamicate world, they also provide the historian with the opportunity to study specific uses and applications of geometry. In particular, it is interesting to analyze the way in which scholarly geometry (especially Euclidean) changed as a result of the introduction of algebra. The question thus becomes what kind of geometry do Arabic algebraists use? In this talk, I will present some examples of how geometry has been applied in relation to algebra and to arithmetic by taking into account different traditions of algebraic texts written in Arabic between the 9th and the 12th century. I will focus on the analysis of the difference between "purely" geometrical reasonings and "algebrized" geometrical reasonings, where geometrical magnitudes are multiplied, added or subtracted just as numbers would be.

Ivahn Smadja (CAPHI, University of Nantes)
A Prussian Brahmagupta: British Indology, Higher Mathematics and the Dragon's Seed of Hegelianism

Abstract: In a letter, dated June 14, 1846, to his former student Leopold Kronecker, German mathematician Ernst Eduard Kummer (1810-1893) discussed aspects of his recent major mathematical breakthrough, *viz*. his famous theory of ideal complex numbers. He also incidentally mentioned another work in progress, which he presented as "a fairly nice thing." In the midst of an intensely creative period, he launched into closely reading ancient Sanskrit mathematical sources, delving into the work of the British Indologist, Henry Thomas Colebrooke (1765-1837). During these decisive months, he had fallen under the spell of an age-old enigma, a problem about cyclic rational quadrilaterals, upon which he stumbled, while studying French geometer Michel Chasles's *Aperçu historique* (1837). In Colebrooke's translation, Chasles had first singled out a collection of verses by Brahmagupta, presumably containing what he called "Brahmaguta's geometry," a consistent and completely general theory of rational cyclic quadrilaterals. A major difficulty, however, lay in the fact that these statements were largely under-

specified, dealing with properties of quadrilaterals whose validity conditions were not fully spelled out.
Kummer then took up this interpretive puzzle and rebutted Chasles's claim about generality on higher mathematical grounds. In his view, a completely general theory of cyclic rational quadrilaterals would require much more powerful methods than Brahmapupta's. In order to show that these new methods, which he set out to create, were distinct from Brahmagupta's more elementary ones, Kummer went back to Colebrooke's text. He reinterpreted and recombined Brahmagupta's statements within a new algebraic framework implying different levels of generality. He showed how some reconstructed formulas, obtained in the first place at the lower level corresponding to elementary methods '*à la Brahmagupta*', could be regained, at a higher level of generality, as a particular case of much more general formulas, allowing for an enhanced understanding of the whole problem. Kummer thus made it explicit why, in his view, Brahmagupta could not achieve the intended generality, for want of an access to the higher level of generality which only his own reworking of the problem would provide.
In the light of this case study, this paper will focus on the ways in which the relationship between geometry, algebra and arithmetic may be approached, in connection with the generality of methods. It will also analyse how Kummer's reading of Colebrooke was shaped by a combination of intermeshing historical factors, specific to the Berlin context, blending British Indology, German philology, higher mathematics in full bloom and ingrained Hegelian convictions.

Don Zagier (Max-Planck Institut für Mathematik, Bonn, and International Centre for Theoretical Physics, Trieste)

Abstract: ***tba***

Félix Fanglei Zheng (independent scholar)

Jordanus Nemorarius's De numeris datis*: an algebra in the form of arithmetical problems demonstrated with Greek geometrical analysis-synthesis structure of propositions*

Abstract: Jordanus Nemorarius's *De numeris datis* is an interesting case for studying the relationship between arithmetic, algebra, and geometry in the history of mathematics. This 13th-century work was praised by its modern critical editor and English translator as the first advanced algebra by contrasting it with Viète's equation method. However, this work had often been excluded from most stories of the development of algebra, which focused mainly on the increasing capability of solving difficult equations or the inventions of abstract algebraic signs. The work's absence in many early histories of Algebra may be caused by its arithmetical form, which many researchers might consider a setback in the development of algebra. This presentation will first show how Jordanus invents this unique work — transforming algebraic problems into arithmetical problems of finding numbers, and demonstrating the solutions in the framework of Greek geometrical analysis-synthesis. I will then try to reveal more obvious but formal similarities to Viète's work than what Hughes' contrast showed.

Zhou Xiaohan Célestin (The Institute for the History of Natural Sciences, Chinese Academy of Sciences)

Yang Hui's geometric reasoning using duan *(pieces [of diagram]) to account for mathematical methods to solve algebraic equations*

Abstract: From the 11th century on, the word *duan* (段), which did not bear any mathematical significance before that time, began to be largely used in mathematical works. On the one hand, this word was used as a grammar word whose combination with a numerical value designates the number of pieces of plane or solid figures in the statement of a mathematical problem or the commentary to the procedure. On the other hand, in the context of mathematical reasoning, the word *duan per se* turned out to be a general designation for plane rectangular figures. For the latter usage, extant mathematical works of the 13th century show that the term *duan* was regularly combined with the verbs such as "change/transform (*bian* 變)", "deduce (*yan* 演)" or with the noun "strip (*tiao* 条)" to form new technical expressions. Along with the emergence of this word in the history of mathematics, diagrams (*tu* 圖) working as visual aids began to be engraved in the printed mathematical works. These expressions and the diagrams are in large part related to mathematical problems that are perceived by modern observers as being solved by algebraic equations.

This talk will focus on one of the most representative authors of mathematical works of this period, Yang Hui (fl. 1261 CE). I will examine how the geometric reasoning using pieces of diagram ground the mathematical methods for solving equations. In his *Quick Methods for Multiplication and Division for the Surface of the Fields and Analogous Problems* (田畝比類乘除捷法 1275 CE), Yang Hui quoted his precursor Liu Yi's (ca. 11th century) "deduction with pieces of diagram." Indeed, Liu Yi had successfully proved the correctness of the detailed procedures of solving quadratic equations (Chemla, 2019). In his commentaries on the mathematical classic *The Nine Chapters*, which dates back to the first century CE, through transforming diagrams which has been cut into pieces (*duan*) into new forms, Yang established the quadratic equations to be solved. Through detailed textual analysis, this talk will also address the uses of and the possible differences between the above expressions including this key term *duan*.

K. Chemla, 2019, "The Proof Is in the Diagram: Liu Yi and the Graphical Writing of Algebraic Equations in Eleventh-Century China", Endeavour, 42 (2018) 60–77