Gottesman-Knill theorem and its quasi-probability description

How powerful is a classical computer?

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Stabilizer sub-theory
Stabilizer state

- A pure N-qubit quantum state:
- $|\psi\rangle = \sum_{i_1,\ldots,i_N=0,1} \alpha_{i_1\ldots i_N} |i_1\ldots i_N\rangle$: $2^N - 1$ real numbers
- A pure quantum state described by its stabilizer group:
  - $Stab(\psi) = \{U \in U(2^N) : U|\psi\rangle = |\psi\rangle\}$
  - $|\psi\rangle \neq |\phi\rangle \rightarrow Stab(\psi) \neq Stab(\phi)$
  - $Stab(\psi)$ too large!

Stabilizer sub-theory

Stabilizer state

- Pauli group: $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
    - $P_1 = \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}$
    - $Z_1 Z_2 \in P_2, -Z_1 Y_2 X_3 \in P_3, iX_1 I_2 Y_3 I_4 \in P_4, -iI_1 Y_2 Y_3 Z_4 I_5 \in P_5$
    - Pauli group elements either commute or anti-commute.
- (Pauli) Stabilizer group of a state $|\psi\rangle$: $S(\psi) = Stab(\psi) \cap P_N$

Stabilizer sub-theory
Stabilizer state

- $|0\rangle$: $S(0) = \{I, Z\} = \langle Z \rangle$, $|0\rangle\langle 0| = \dfrac{1}{2}(I + Z)$
- $|1\rangle$: $S(1) = \{I, -Z\} = \langle -Z \rangle$, $|1\rangle\langle 1| = \dfrac{1}{2}(I - Z)$
- Bell state $|\Phi^+\rangle = \dfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$:
  $S(\Phi^+) = \{I_1 I_2, X_1 X_2, -Y_1 Y_2, Z_1 Z_2\} = \langle Z_1 Z_2, X_1 X_2 \rangle$,
  $|\Phi^+\rangle\langle\Phi^+| = \dfrac{1}{2^2}(I + Z_1 Z_2)(I + X_1 X_2)$

Stabilizer sub-theory

Stabilizer state

- $S(\psi) = \langle U_{g_1}, \ldots, U_{g_N} \rangle$, where $U_{g_i} \in P_N$
- $|\psi\rangle\langle\psi| = \frac{1}{2^N} \prod_{q=1}^{N} (I_q + U_{g_q})$
- A stabilizer group $S(\psi)$ is an abelian group.
- $-I \neq S(\psi)$
- Phases of stabilizer group elements are only ± 1.

Stabilizer sub-theory

Computation: Clifford gate

- Clifford group Cl_N = normalizer of the Pauli group P_n
  - $Cl_N = \{U_g \in U(2^N) : U_g A U_g^\dagger \in P_N \quad \forall A \in P_N\}$
- $S(\psi) = \langle U_{g_1}, \ldots, U_{g_N} \rangle,\ U_g \in Cl_N : U_g |\psi\rangle = U_g \left(U_{g_i} |\psi\rangle \right) = \left(U_g U_{g_i} U_g^\dagger \right) U_g |\psi\rangle,$ $U_g U_{g_i} U_g^\dagger \in P_N$
  - $S(U_g|\psi\rangle) = \langle U_g U_{g_1} U_g^\dagger, \ldots, U_g U_{g_N} U_g^\dagger \rangle$
- Clifford group can be generated by $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$, and CNOT gates.

Stabilizer sub-theory

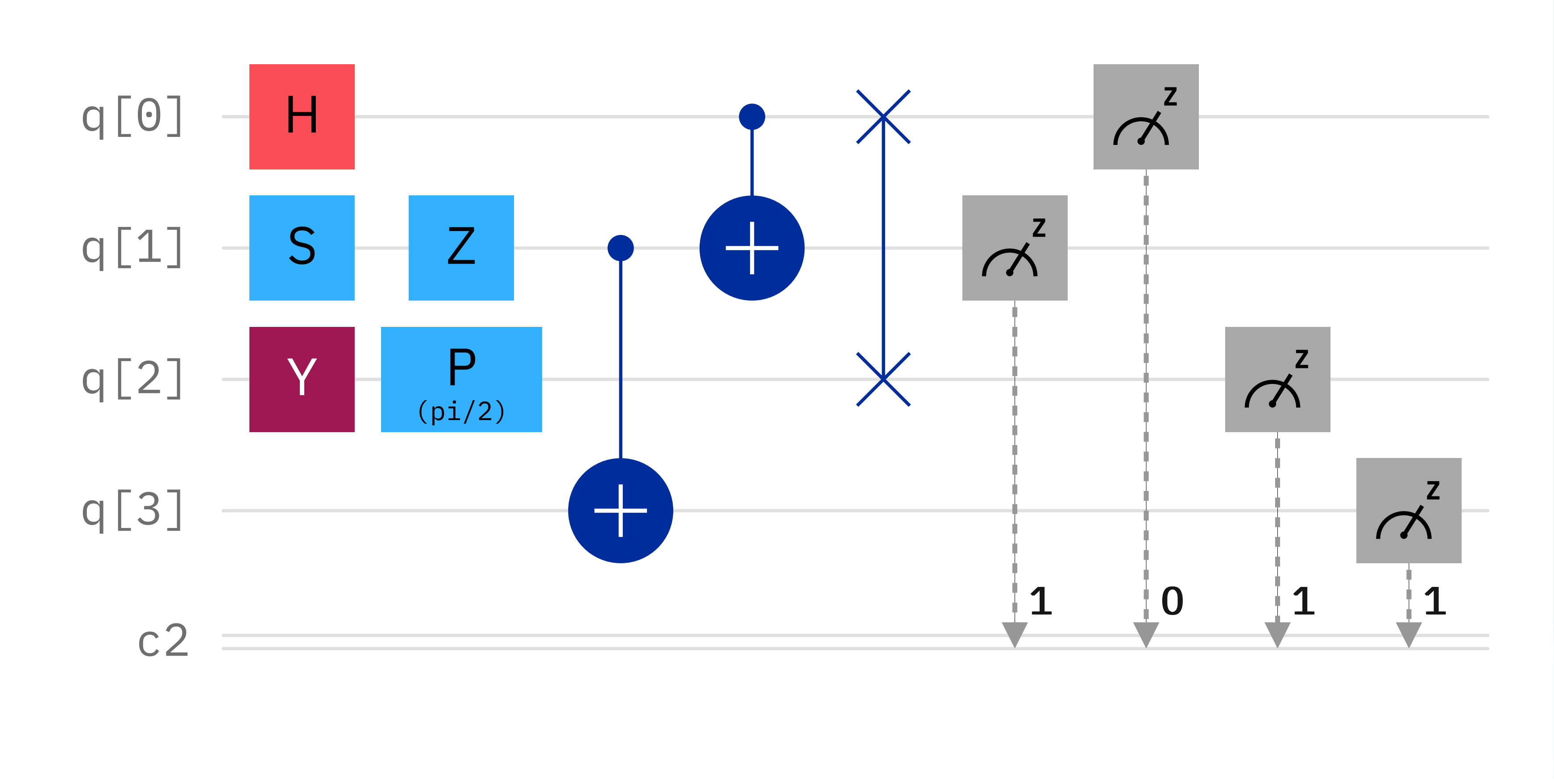
Measurement

- A stabilizer state $|\psi\rangle$ with its stabilizer group $S(\psi) = \langle g_1, \ldots, g_N \rangle$
- Measurement of $g \in P_N$ (with phase $+1$):
    - Check if g commutes with g_i for all $i = 1,\ldots,N$.
        - If so, the outcome is 1 for $g \in S(\psi)$ or the outcome is -1 for $-g \in S(\psi)$.
            - The state does not change.
        - If not, choose a anti-commuting generator g_1, then update other anti-commuting generator by multiplying g_1: $(\{g, g_1\} = \{g, g_2\} = 0 \to [g, g_1 g_2] = 0\}$
            - Sample an outcome ± 1 with probability $1/2$:
            - If 1, replace g_1 to g. If -1, replace g_1 to $-g$.

Stabilizer sub-theory

- A stabilizer state can be described by N matrices ($M_{2^N}(\mathbb{C})$).
- There are a finite number $O\left(2^{N^2}\right)$ of stabilizer state: a tiny portion in the whole Hilbert space
- Stabilizer states include typical (entangled) states such as Bell states, GHZ states, cluster states.
- Stabilizer formalism is utilized in the quantum error correction.
- Stabilizer circuits can be efficiently simulated classically.

Gottesman-Knill theorem



- An N-qubit stabilizer circuit composed of
  - all qubits in $|0\rangle$ state initially,
  - Clifford gates: $Cl_N = \{U_g : U_g A U_g^\dagger \in P_N \quad \forall A \in P_N\}$
  - Pauli measurements
- can be efficiently simulated by classical computers.

Gottesman-Knill theorem

Stabilizer state

- Computing variable: $C(n,m) = (-i)^{nm} Z^n X^m = \omega^{-\frac{1}{2}nm} Z^n X^m$ $(n, m = 0,1; \omega = e^{\pi i} = -1)$
  - $C(0,0) = I, C(1,0) = Z, C(0,1) = X, C(1,1) = -iZX = Y$
- $C(n_1, \ldots, n_N, m_1, \ldots, m_N) = \bigotimes_{q=1}^{N} (-i)^{n_q m_q} Z_q^{n_q} X_q^{m_q} = C(k)$,
  - $k \in K = \{(n_1, \ldots, n_N, m_1, \ldots, m_N) : n_q, m_q \in \mathbb{Z}_2 \quad \forall q = 1, \ldots, N\}$
  - $C(1,1,0,0) = Z_1 Z_2$, $C(0,0,1,1) = X_1 X_2$, $C(1,0,1,0) = Y_1 I_2$, $C(1,1,1,0) = Y_1 Z_2$
- $C(k)^\dagger = C(k)$, Tr $C(k)C(k') = \frac{1}{2^N}\delta_{k,k'}$: CVs consist of a basis of $M(2^N, \mathbb{C})$

Gottesman-Knill theorem

Stabilizer state

- A stabilizer group element $(-1)^r C(k)$ is represented by $2N+1$ bits $(k; r)$
  - $X_1 \sim (0,1;0)$, $-Z_1X_2 \sim (1,0,0,1;1)$, $-Z_1X_2Y_3 \sim (1,0,1,0,1,1;1)$
- A stabilizer group for an N-qubit state has N generators: $S(\psi) = \langle g_1, \ldots, g_N \rangle$
- $N(2N+1)$ bits describe a stabilizer state (Tableau representation):
- $$\begin{bmatrix} n_{11} & \cdots & n_{1N} & m_{11} & \cdots & m_{1N} & | r_1 \\ & \vdots & & & \vdots & & \\ n_{N1} & \cdots & n_{NN} & m_{N1} & \cdots & m_{NN} & | r_N \end{bmatrix}$$

Gottesman-Knill theorem

Clifford gate

- $U_g C(k) U_g^\dagger = \omega^{\phi_g(k)} C(\psi_g(k))$ for $U_g \in Cl_N = \langle H, S, CNOT \rangle$
- $\psi_g(k + k') = \psi_g(k) + \psi_g(k')$ linear map
- $\phi_g : K \to \{0,1\}$ nonlinear map
- Any Clifford gate can be decomposed into 11 layers of H, S, $CNOT$.
- For N qubits, m gates can be calculated in $poly(N)m$.

Gottesman-Knill theorem

Measurement

- A stabilizer state $|\psi\rangle$ with its stabilizer group $S(\psi) = \langle g_1, \ldots, g_N \rangle$
- Measurement of $g \in P_N$ (with phase $+1$):
  - Compute $k_g \Omega k_{g_i}^T$ for all $i = 1, \ldots, N$, where $\Omega = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$:
    - If the value is all 0, check whether $g \in S(\psi)$ for the outcome 1 or $-g \in S(\psi)$ for the outcome -1: Gaussian elimination
      - The state does not change.
    - If not, choose a generator g_1 such that $k_g \Omega k_{g_1}^T \neq 0$, then update anti-commuting generators ($k_g \Omega k_{g_j}^T \neq 0$) by $k_{g_1} \oplus k_{g_j}$.
    - Sample an outcome ± 1 with probability $1/2$. If 1, replace k_{g_1} to k_g. If -1, replace k_{g_1} to k_{-g}.
- A Pauli measurement can be simulated in $poly(N)$.

Gottesman-Knill theorem

- Stabilizer sub-theory
  - $S(\psi) = \langle g_1, \ldots, g_N \rangle$: $N(2N+1)$ bits
  - $\begin{bmatrix} n_{11} & \cdots & n_{1N} & m_{11} & \cdots & m_{1N} & | r_1 \\ & \vdots & & & \vdots & & \\ n_{N1} & \cdots & n_{NN} & m_{N1} & \cdots & m_{NN} & | r_N \end{bmatrix}$
  - m Clifford gates: $O(poly(N)m)$
  - Pauli measurement: $O(poly(N))$
- All the processes can be efficiently simulated with classical computers.

Quasi-probability description

- Can we describe a stabilizer circuit classically?
- $\bigotimes_{q=1}^{N} |0\rangle_q \xrightarrow{U_g} U_g\left(\bigotimes_{q=1}^{N} |0\rangle_q\right) \xrightarrow{M_Z} p_Z(s)$
- $p_Z(s) = \sum_{r'} \xi_Z(s\,|\,r')\left(\sum_{r} \Gamma_g(r'\,|\,r) W_0(r)\right)$ is possible?

Quasi-probability description

- $K_0 = \{(n_1, \ldots, n_N, 0, \ldots, 0) : n_q \in \mathbb{Z}_2 \quad \forall q = 1, \ldots, N\}$

  $\rho_0 = \otimes_{q=1}^{N} |0\rangle\langle 0|_q = \otimes_{q=1}^{N} \frac{1}{2}(I_q + Z_q)$

- $\quad = \frac{1}{2^N} \sum_{n_1=0,1} \cdots \sum_{n_N=0,1} Z_1^{n_1} \cdots Z_N^{n_N} = \frac{1}{2^N} \sum_{k_0 \in K_0} C(k_0)$
  - (* $C(n, m) = (-i)^{nm} Z^n X^m$)
- $\rho_g = U_g \rho_0 U_g^\dagger = \frac{1}{2^N} \sum_{k_0 \in K_0} U_g C(k_0) U_g^\dagger = \frac{1}{2^N} \sum_{k_0 \in K_0} \omega^{\phi_g(k_0)} C(\psi_g(k_0))$

Quasi-probability description

- $C(n,m) = (-i)^{nm} Z^n X^m = \omega^{-\frac{1}{2}nm} Z^n X^m$ corresponds to the Heisenberg-Weyl displacement operator:
  - $D(x,p) \propto e^{ipX} e^{-ixP}$ in quantum optics
  - $Z|0\rangle = |0\rangle, Z|1\rangle = -|1\rangle,$
  - $X|0\rangle = |1\rangle, X|1\rangle = |0\rangle.$
- We construct a quasi-probability distribution for N qubits.

Quasi-probability description

- Characteristic function of a matrix A:
  - $\chi_A(k) = \text{Tr}\ \left\{C(k)^\dagger A\right\}$
- Characteristic function of a stabilizer state ρ_g:
- $\chi_g(k) = \chi_{\rho_g}(k) = \text{Tr}\ \left\{C(k)^\dagger \rho_g\right\} = \sum_{k_0 \in K_0} \omega^{\phi_g(k_0)} \delta_{k,\psi_g(k_0)}$
  - (Quantum optics: $\chi_s(\alpha) = \text{Tr}\ D_s(\alpha)\rho$)

Quasi-probability description

- Quasi-probability distribution of ρ_g:

$$W_g(r|\Theta_g) = \frac{1}{2^{2N}} \sum_{k\in K} \omega^{\Theta_g(k)} \omega^{r\cdot k} \chi_g(k)$$

- $$= \frac{1}{2^{2N}} \sum_{k_0\in K_0} \omega^{\Theta_g\circ\psi_g(k_0)+r\cdot\psi_g(k_0)+\phi_g(k_0)}$$

- What choice of Θ_g makes $W_g(r|\Theta_g) \geq 0$ for all $r \in K$?
- Proof technique: $\sum_{k\in K} \omega^{r\cdot k} = \sum_{k_1=0,1} \cdots \sum_{k_{2N}=0,1} \prod_j \omega^{r_j k_j} = \prod_j (1+\omega^{r_j}) \geq 0$

Quasi-probability description

Phase decomposition

- $U_g C(k_0) U_g^\dagger = \omega^{\phi_g(k_0)} C(\psi_g(k_0))$
- $\psi_g(k+k') = \psi_g(k) + \psi_g(k')$
- $\phi_g(k+k') = \phi_g(k) + \phi_g(k') + \frac{1}{2}\psi_g(k)^T \Omega \psi_g(k') - \frac{1}{2} k^T \Omega k'$, where $\Omega = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$
- $\phi_g\left(k = \sum_{j=1}^{J} k_j\right) = \sum_{j=1}^{J} \phi(k_j) + \theta_g(k) - \theta(k)$
- $\theta\left(k = \sum_{j=1}^{J} k_j\right) = \frac{1}{2} \sum_{j_1 > j_2}^{J} k_{j_1}^T \Omega k_{j_2}$, $\theta_g\left(k = \sum_{j=1}^{J} k_j\right) = \theta\left(\psi_g(k) = \sum_{j=1}^{J} \psi_g(k_j)\right) \circ \psi_g^{-1}$

Quasi-probability description

- Quasi-probability distribution:
- $$W_g(r\,|\,\Theta_g) = \frac{1}{2^{2N}} \sum_{k_0 \in K_0} \omega^{\Theta_g \circ \psi_g(k_0) + r \cdot \psi_g(k_0) + \phi_g(k_0)}$$

$$k_0 = (n_1, \ldots, n_N, 0, \ldots, 0)$$
$$= (n_1, 0, \ldots, 0) + (0, n_2, 0, \ldots, 0) + \cdots + (0, \ldots, n_N, 0, \ldots, 0)$$

- $$= \sum_{q=1}^{N} k_0^{(q)}$$

Quasi-probability description

- Quasi-probability distribution of a N-qubit stabilizer state is non-negative with $\Theta_g = (\theta_g - \theta) \circ \psi_g^{-1}$; $\psi_g(0) = 0$, $\phi_g(0) = 0$

$$W_g(r \,|\, \Theta_g) = \frac{1}{2^{2N}} \sum_{k_0 \in K_0} \omega^{\Theta_g \circ \psi_g(k_0) + r \cdot \psi_g(k_0) + \phi_g(k_0)}$$

$$= \frac{1}{2^{2N}} \prod_{q=1}^{N} \sum_{k_0^{(q)}=0,1} \omega^{r \cdot \psi_g(k_0^{(q)}) + \phi_g(k_0^{(q)})}$$

- $$= \frac{1}{2^{2N}} \prod_{q=1}^{N} (1 + \omega^{\square})$$

$$\geq 0$$

Quasi-probability description

Transformation matrix

- $W_g(r'|\Theta_g) = \sum_r \Gamma_g(r'|r)W_0(r|\Theta_0)$
- Using $\Theta_g = (\theta_g - \theta) \circ \psi_g^{-1}$, we can show $\Gamma_g(r'|r) \geq 0$ for all r, r'.

Quasi-probability description

Measurement

- $p_Z(s) = \sum_r \xi_Z(s \mid r) W_g(r \mid \Theta_g)$

  $$\xi_Z(s \mid r) = \frac{1}{2^N} \sum_{k_0 \in K_0} \omega^{\Theta_g(k_0) + s \cdot k_0 + r' \cdot k_0}$$

- $$= \frac{1}{2^N} \sum_{k_0 \in K_0} \omega^{(\theta_g - \theta) \circ \psi_g^{-1}(k_0) + s \cdot k_0 + r' \cdot k_0}$$

- $\xi_Z(s \mid r) \geq 0$ is not assured.

Conclusion

- Stabilizer state $|\psi\rangle$: (Pauli) Stabilizer group $S(\psi)$
- Stabilizer circuit = stabilizer state + Clifford gates + Pauli measurement
- G-K theorem: Stabilizer circuits can be efficiently simulated by classical computers
- Classical probability description of N-qubit stabilizer circuits is not successful with the Heisenberg-Weyl displacement operator with additional phases.
  - Classical probability description of N-qudit stabilizer circuits with odd dimension is possible.