Gauging a symmetry

1. **Gauged 2d harmonic oscillator - quantized**

   A two dimensional isotropic harmonic oscillator is described by the Lagrangian,

   $$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}m\omega^2(x^2 + y^2)\,.$$

   We can gauge the rotation symmetry on the (x, y) plane. The goal of this problem is to construct a *quantum* theory after the gauging. We can take two approaches.

   (a) Solve the constraint in the classical theory first, and then quantize it.

   (b) Quantize the original theory, and impose the constraints in the quantum theory. (In the quantum theory, we cannot impose $\phi = 0$ and $\chi = 0$ simultaneously.)

   (c) Compare (a) and (b). Do we obtain the same result? Should we? Try to argue, in general, whether the two approaches should give the same result or not.

   (d) How can we incorporate the constraints in the path integral quantization?

2. **Gauging a non-abelian symmetry**

   The lectures focused on the abelian symmetry algebra, $\{\phi_A, \phi_B\} = 0$. A non-abelian generalization is given by

   $$\{\phi_A, \phi_B\} = f_{AB}{}^C\phi_C\,.$$

   The coefficients $f_{AB}{}^C$ could be functions of dynamical variables, but for simplicity let us assume that they are constants. A prototypical example is the rotation symmetry of a particle in a central potential in $\mathbb{R}^3$ whose Lagrangian is

   $$L = \frac{1}{2}m\dot{\vec{r}}^2 - V(r)\,, \quad r = \sqrt{\vec{r}^2}\,.$$

   The non-abelian symmetry is the SO(3) rotation symmetry generated by $\phi_i = \epsilon_{ijk}x_j p_k$.

   (a) Repeat all the steps of gauging the symmetry in the Lagrangian formulation. What are the most notable differences compared to the abelian case?

   (b) Repeat all the steps of gauging the symmetry in the Hamiltonian formulation. What are the most notable differences compared to the abelian case?

   (c) Give another (elementary and realistic) example of a non-abelian symmetry. Solve its equations of motion before and after the gauging.