KIAS-SNU Winter Camp 2023

Introduction to Many-body Physics

2023. 12. 20 (Wed) 14:00–15:30
22 (Fri) 14:00–15:30
26 (Tue) 11:15–12:45

I. Occupation number representation

II. Linear response theory

III. Path integral method

Project : Many-body approach to superconductivity

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- Many-body problem

— Systems of interacting electrons and ions

$$H = H_{el} + H_{el\text{-}ion} + H_{ion} + \cdots$$

$$H_{el} = -\sum_i \frac{\hbar^2}{2m} \nabla_i^2 + \frac{1}{2}\sum_{i\neq j} \frac{e^2}{|r_i - r_j|}$$

$$H_{el\text{-}ion} = -\sum_{i,I} \frac{Z_I e^2}{|r_i - R_I|}$$

$$H_{ion} = -\sum_I \frac{\hbar^2}{2M_I} \nabla_I^2 + \frac{1}{2}\sum_{I\neq J} \frac{Z_I Z_J e^2}{|R_I - R_J|}$$

→ Statistics, symmetries, effective low energy theories, …

- References

— Fetter and Walecka
— Giuliani and Vignale
— G. Mahan

} operator approach

— Negele and Orland
— P. Coleman
— Altland and Simons

} path-integral approach

I. Occupation number representation

· Second Quantization

① Quantum mechanics of a single particle system

– formulated in terms of $\hat{x}$ and $\hat{p}$

ex) $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$

→ coordinate representation defined in terms of eigenfunctions of $\hat{x}$ (or $\hat{p}$),

$\hat{x}|x\rangle = x|x\rangle$

② Quantum mechanics of a many-particle system

– indistinguishable

ex) bosons and fermions

→ occupation number representation defined in terms of $|n_{\lambda_1}, n_{\lambda_2}, \cdots\rangle$ telling how many particles are in each state $\{\lambda\}$, and the $\hat{a}_\lambda^\dagger, \hat{a}_\lambda$ operators which create and destroy particles in this state.

- Symmetry properties of bosons and fermions

→ Commutation relation for bosons / fermions

$$[\hat{a}_\lambda, \hat{a}_{\lambda'}^\dagger]_\mp = \hat{a}_\lambda \hat{a}_{\lambda'}^\dagger \mp \hat{a}_{\lambda'}^\dagger \hat{a}_\lambda = \delta_{\lambda\lambda'}$$

· Many-body operators

Goal) Express operators of physical interest in terms of creation and annihilation operators.

Note) $\hat{O} = \sum_{m,n} |m\rangle\langle m|\hat{O}|n\rangle\langle n|$

$$\xrightarrow{\text{expect}} \sum_{m,n} \hat{a}_m^\dagger O_{mn} \hat{a}_n \,?$$

① One-body operator

Consider a one-body operator $\hat{F} = \sum_{i=1}^{N} \hat{f}_i$ which consists of N identical operators f_i acting on the ith particle. (i: particle index)

ex) $\hat{\rho}(x) = \sum_i \delta(x - x_i)$ density operator

$\hat{T} = \sum_i \left(-\frac{\hbar^2}{2m}\nabla_i^2\right)$ kinetic energy operator

By inserting the complete relation $\sum_\lambda |\lambda\rangle\langle\lambda| = 1$,

$$\hat{F} = \sum_{i=1}^{N} \left(\sum_{\lambda'} |\lambda'\rangle_i \, {}_i\langle\lambda'|\right) \hat{f}_i \left(\sum_{\lambda} |\lambda\rangle_i \, {}_i\langle\lambda|\right)$$

$$= \sum_{i=1}^{N} \sum_{\lambda'\lambda} |\lambda'\rangle_i \, {}_i\langle\lambda'|f|\lambda\rangle_i \, {}_i\langle\lambda|$$

$$= \sum_{\lambda'\lambda} \langle\lambda'|f|\lambda\rangle \sum_{i=1}^{N} |\lambda'\rangle_i \, {}_i\langle\lambda|$$

Note that $f_{\lambda'\lambda} = \langle\lambda'|f|\lambda\rangle$ is independent of a particle index i because the particles are identical. The operator $\hat{R}_{\lambda'\lambda} = \sum_{i=1}^{N} |\lambda'\rangle_i \, {}_i\langle\lambda|$ searches for each particle in state λ moving it to state λ', which turns out to be

$$\sum_{i=1}^{N} |\lambda'\rangle_i \, {}_i\langle\lambda| n_{\lambda\alpha} n_{\lambda\beta} \cdots\rangle = a_{\lambda'}^\dagger a_\lambda | n_{\lambda\alpha} n_{\lambda\beta} \cdots\rangle.$$

$$\rightarrow \hat{R}_{\lambda'\lambda} = \hat{a}_{\lambda'}^\dagger \hat{a}_\lambda$$

$$\therefore \hat{F} = \sum_{\lambda'\lambda} f_{\lambda'\lambda} \hat{a}_{\lambda'}^\dagger \hat{a}_\lambda$$

Further reading: Marder, App. C.

For a homogeneous system with states $\lambda = \{k, \sigma\}$
in the plane-wave basis $\langle x | k \rangle = \frac{1}{\sqrt{V}} e^{ik\cdot x}$, $\leftarrow V$: volume

$$\langle k', \sigma' | f_1 | k, \sigma \rangle_1$$ evaluated for a particle, say $i=1$

$$\int dx_1 |x_1\rangle\langle x_1| = 1$$

$$= \frac{1}{V}\int dx_1\, e^{-i(k'-k)\cdot x_1} f_{\sigma'\sigma}(x_1) = \frac{1}{V} f_{\sigma'\sigma}(k'-k)$$

$$\therefore \hat{F} = \frac{1}{V}\sum_{k,k'}\sum_{\sigma,\sigma'} f_{\sigma'\sigma}(k'-k)\, \hat{a}^{\dagger}_{k'\sigma'} \hat{a}_{k\sigma}$$

$$\overset{q=k'-k}{=} \frac{1}{V}\sum_{k,q}\sum_{\sigma,\sigma'} f_{\sigma'\sigma}(q)\, \hat{a}^{\dagger}_{k+q,\sigma'} \hat{a}_{k,\sigma}$$

Ex) Density operator $\hat{\rho}(x) = \sum_i \delta(x - x_i)$

$$\langle k', \sigma' | \delta(x - x_1) | k, \sigma \rangle_1 = \frac{1}{V} e^{-i(k'-k)\cdot x}\, \delta_{\sigma'\sigma}$$

$$\therefore \hat{\rho}(x) = \frac{1}{V}\sum_{k',k}\sum_{\sigma',\sigma} e^{-i(k'-k)\cdot x}\, \delta_{\sigma'\sigma}\, \hat{a}^{\dagger}_{k'\sigma'} \hat{a}_{k\sigma}$$

$$\overset{q=k-k'}{=} \frac{1}{V}\sum_{q} e^{iq\cdot x} \sum_{k} \hat{a}^{\dagger}_{k-q,\sigma} \hat{a}_{k\sigma}$$

$\left(\frac{1}{V}\sum_q \to \int \frac{d^dq}{(2\pi)^d}\right)$

$$= \int \frac{d^dq}{(2\pi)^d} e^{iq\cdot x}\, \hat{\rho}(q)$$

where $\hat{\rho}(q) = \sum_{k,\sigma} \hat{a}^{\dagger}_{k-q,\sigma} \hat{a}_{k\sigma}$

② Two-body operator

$$\hat{O} = \frac{1}{2}\sum_{i\neq j} \hat{U}_{ij} \leftarrow \left(\begin{array}{l} \sum_{\lambda_1,\lambda_2} |\lambda_1\lambda_2\rangle_{ij}\,{}_{ij}\langle\lambda_1\lambda_2| = 1 \\ |\lambda_1\lambda_2\rangle_{ij} = |\lambda_1\rangle_i|\lambda_2\rangle_j, \quad {}_{ij}\langle\lambda_1\lambda_2| = {}_j\langle\lambda_2|\,{}_i\langle\lambda_1| \end{array}\right.$$

$$= \frac{1}{2}\sum_{i\neq j}\sum_{\{\lambda\}} |\lambda_1'\lambda_2'\rangle_{ij}\,{}_{ij}\langle\lambda_1'\lambda_2'|U|\lambda_1\lambda_2\rangle_{ij}\,{}_{ij}\langle\lambda_1\lambda_2|$$

$$= \frac{1}{2}\sum_{\{\lambda\}} \langle\lambda_1'\lambda_2'|U|\lambda_1\lambda_2\rangle \sum_{i\neq j} |\lambda_1'\lambda_2'\rangle_{ij}\,{}_{ij}\langle\lambda_1\lambda_2|$$

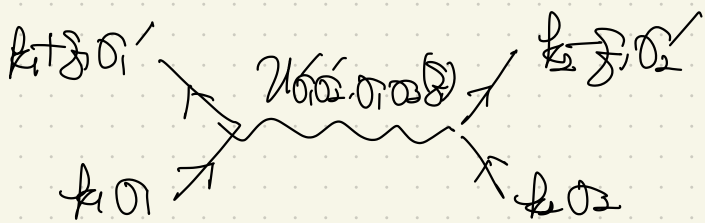
independent of particle indices

searches for two particles in state (λ_1,λ_2) moving from to (λ_1',λ_2')

$$= \frac{1}{2}\sum_{\{\lambda\}} U_{\lambda_1'\lambda_2',\lambda_1\lambda_2}\, \hat{a}^\dagger_{\lambda_1'}\hat{a}^\dagger_{\lambda_2'}\hat{a}_{\lambda_2}\hat{a}_{\lambda_1}$$

For a homogeneous system,

$$\hat{O} = \frac{1}{2V}\sum U_{\sigma_1'\sigma_2',\sigma_1\sigma_2}(q)\, \hat{a}^\dagger_{k_1+q,\sigma_1'}\hat{a}^\dagger_{k_2-q,\sigma_2'}\hat{a}_{k_2,\sigma_2}\hat{a}_{k_1,\sigma_1}$$



Ex.) Coulomb interaction with $U(x_i-x_j) = \frac{e^2}{|x_i-x_j|}$

$$\hat{O} = \frac{1}{2V}\sum_{k_1,k_2,q}\sum_{\sigma_1,\sigma_2} U(q)\, \hat{a}^\dagger_{k_1+q,\sigma_1}\hat{a}^\dagger_{k_2-q,\sigma_2}\hat{a}_{k_2,\sigma_2}\hat{a}_{k_1,\sigma_1}$$

where $U(q) = \frac{4\pi e^2}{q^2}$ in 3D and $U(q) = \frac{2\pi e^2}{q}$ in 2D.