

Understanding cluster dynamics in percolation and synchronization

- (1) Scaling behaviors of bond percolation
- (2) Spontaneous emergence of synchronization

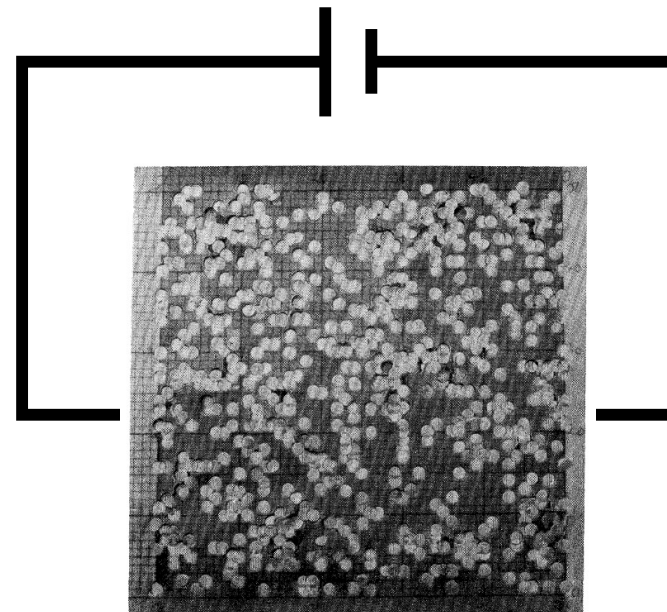
Percolation

Fluid makes a pathway through a porous material between two opposite sides.

Penetration of coffee



Metal insulator transition



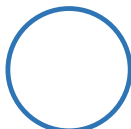
B. J. Last *et al.* Phys. Rev. Lett. (1971)

Bond percolation

 Occupied bond

 Unoccupied bond

 finite cluster

 spanning (infinite) cluster

p : bond occupation probability

$p > p_c$: A spanning cluster exists

$p \leq p_c$: No spanning cluster

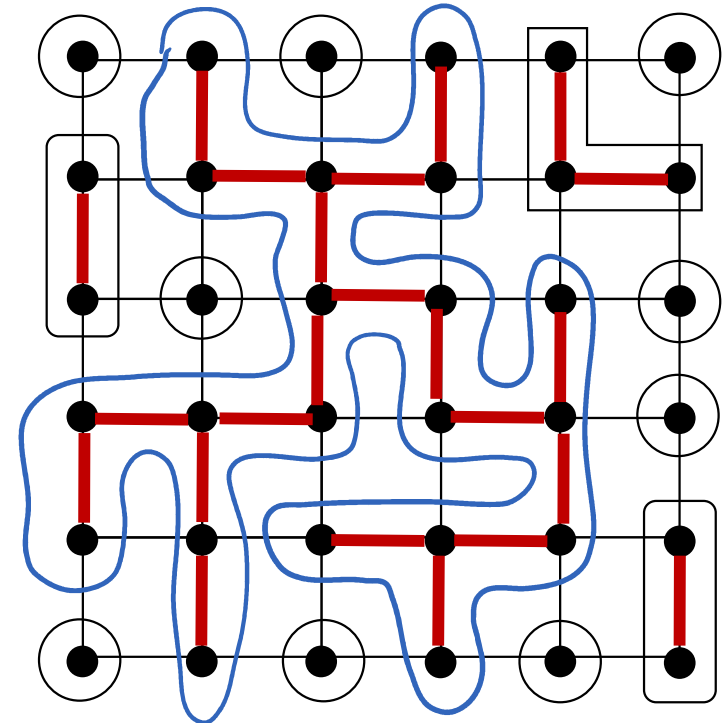
P_∞ : probability that a randomly selected site belongs to the spanning cluster

Cluster size distribution

s	$n_s = \frac{N_s}{N}$
1	9/36
2	2/36
3	1/36

$$P_\infty = 1 - \sum_s' sn_s$$

$$N = L^2 = 36$$



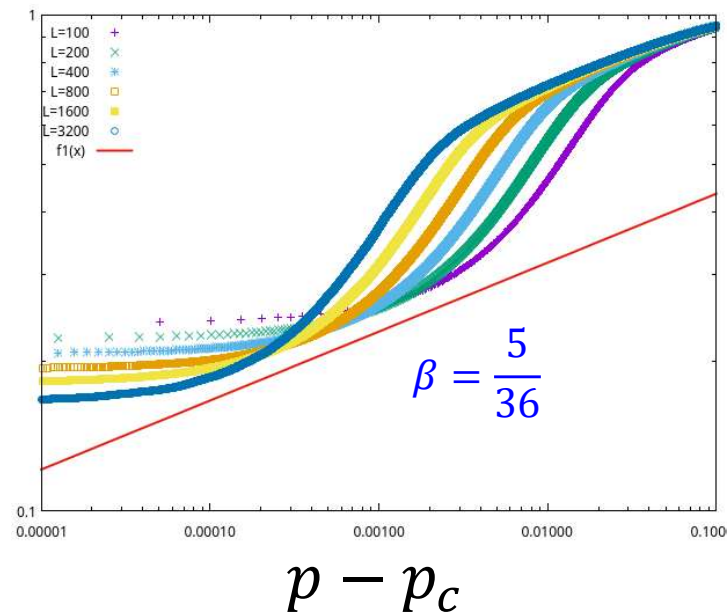
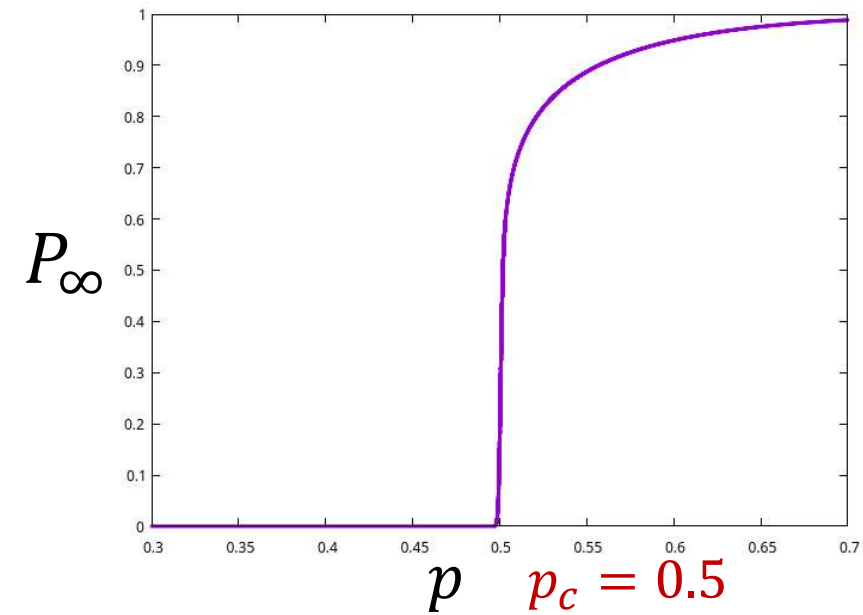
(2d) square lattice

Critical phenomena of bond percolation

(2d) square lattice

$$P_\infty(p) \propto (p - p_c)^\beta \quad \text{as } p \rightarrow p_c^+$$

$$\beta = \frac{5}{36}$$

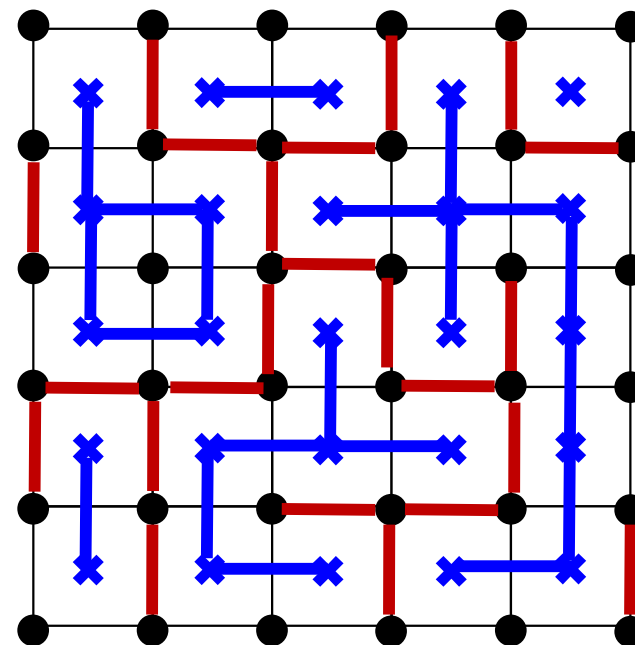


Critical phenomena of bond percolation

Why $p_c = 0.5$?

- (1) Bond is occupied with prob. p \longleftrightarrow Bond is unoccupied with prob. $1 - p$
- (2) Dual graph of unoccupied bonds is also 2d percolation of unoccupied bonds.
- (3) In a given configuration, either the spanning cluster of occupied bonds or unoccupied bonds should exist.
- (4) If $p_c \neq 0.5$, it is a contradiction because both may or may not exist depending on p .

Therefore, $p_c = 0.5$.



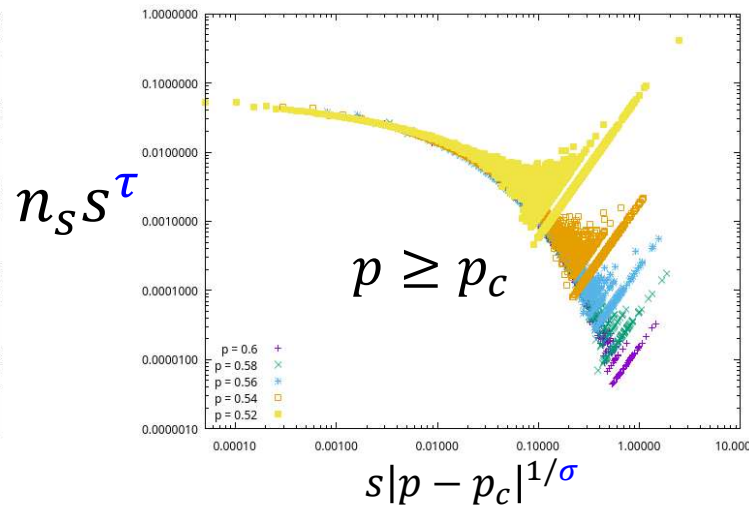
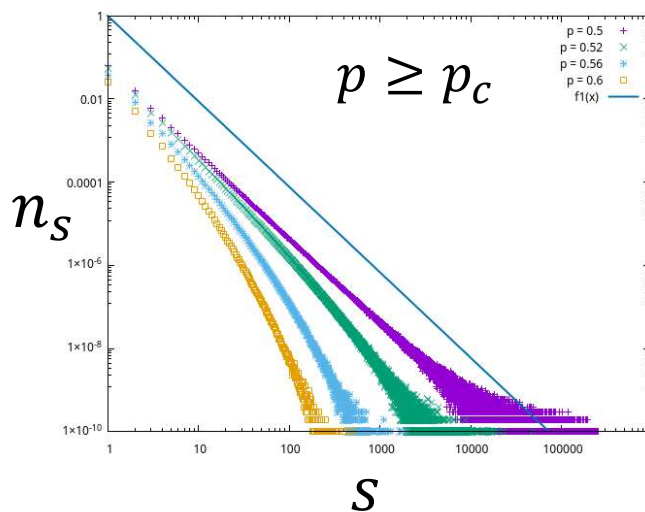
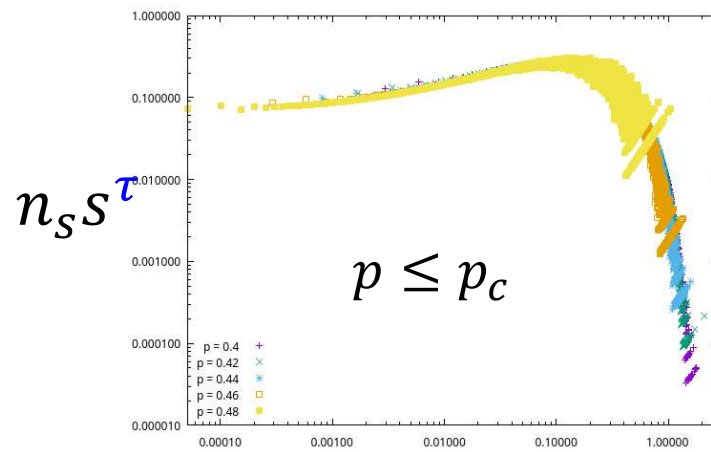
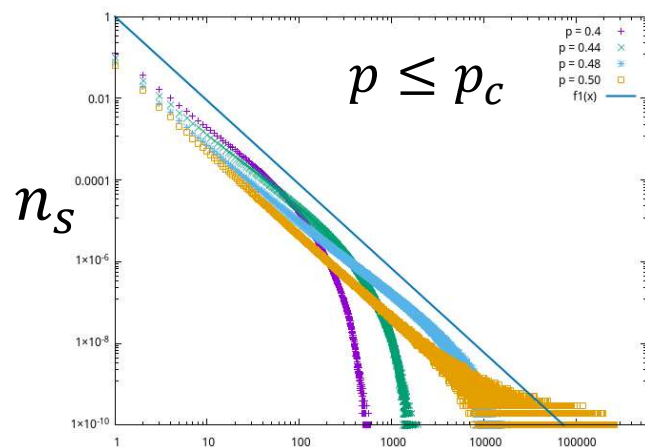
Critical phenomena of bond percolation

(2d) square lattice

$$n_s(p_c) \sim s^{-\tau}$$

$$n_s(p) \sim s^{-\tau} g(s|p - p_c|^{1/\sigma})$$

$$\tau = \frac{187}{91} \quad \sigma = \frac{36}{91}$$



Characteristic cluster size

$$s_\xi = |p - p_c|^{-1/\sigma}$$

Critical phenomena of bond percolation

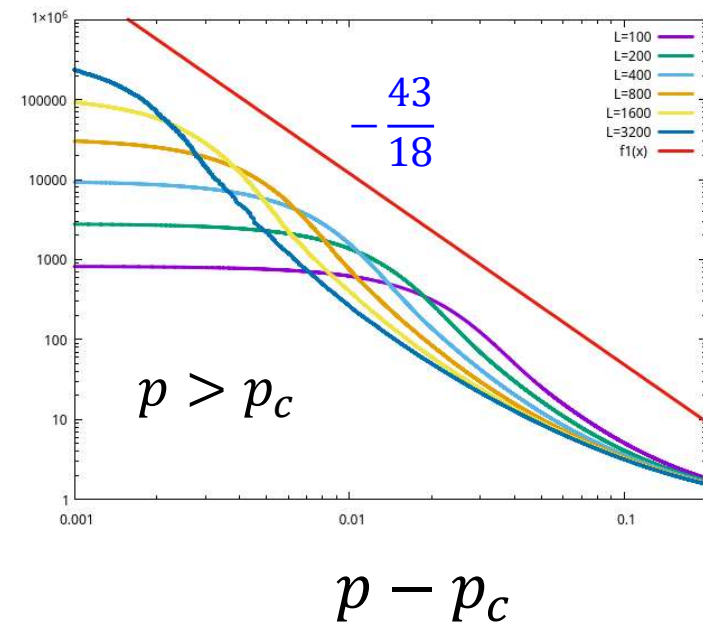
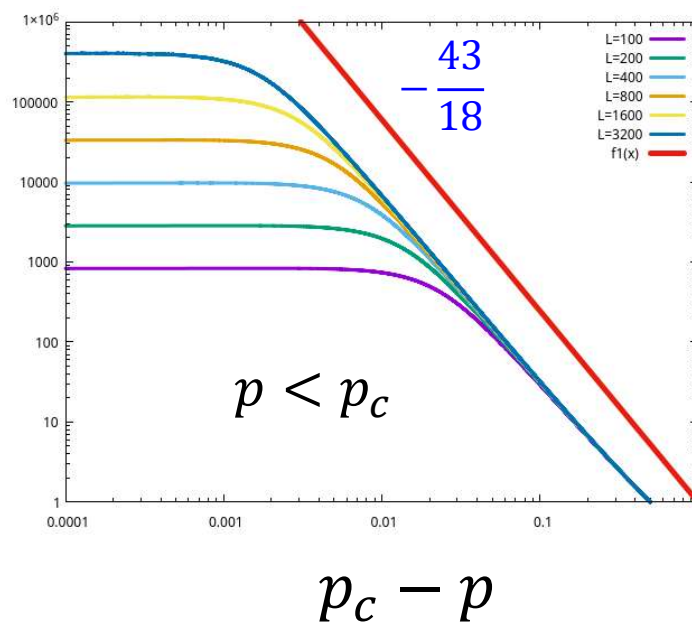
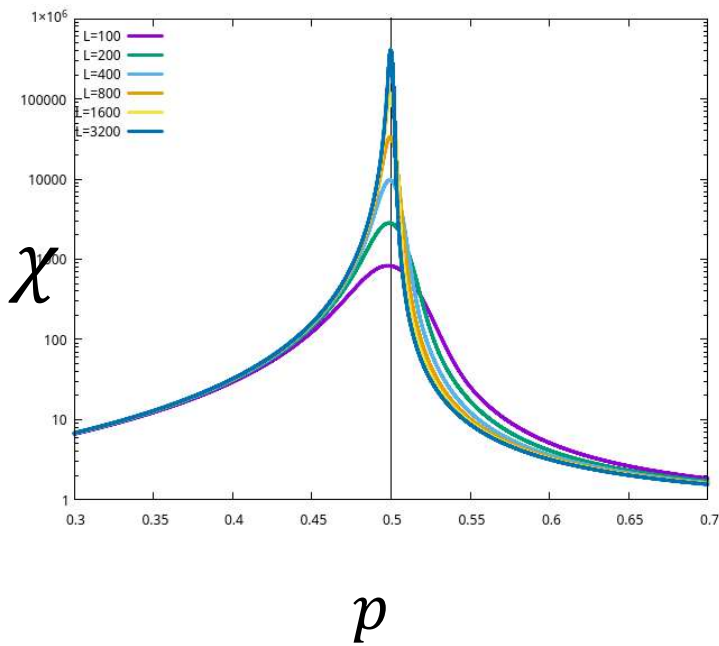
$$\chi = \frac{\sum'_s s^2 n_s}{\sum'_s s n_s}: \text{Average cluster size}$$

Average size of a finite cluster to which a randomly selected node belongs

(2d) square lattice

$$\gamma = \frac{43}{18}$$

$$\chi \propto |p - p_c|^{-\gamma} \quad \text{as } p \rightarrow p_c$$



Origin of critical phenomena of bond percolation: Exact solutions

(1) 1d bond percolation



$$n_s = (1 - p)^2 p^s = (1 - p)^2 \exp(s \ln p) = (1 - p)^2 \exp(s \ln(1 - (1 - p)))$$

$$\propto (1 - p)^2 \exp(-s(1 - p))$$

$$= (1 - p)^2 \exp(-s(p_c - p)^{1/\sigma}) \quad \text{as } p \rightarrow 1^-$$

$$p_c = 1 \quad \sigma = 1$$

Origin of critical phenomena of bond percolation: Exact solutions

(1) 1d bond percolation

$$s_\xi = |p - p_c|^{-1/\sigma} = |p - p_c|^{-1}$$

$$n_{s_\xi} = s_\xi^{-2} \exp(-1)$$

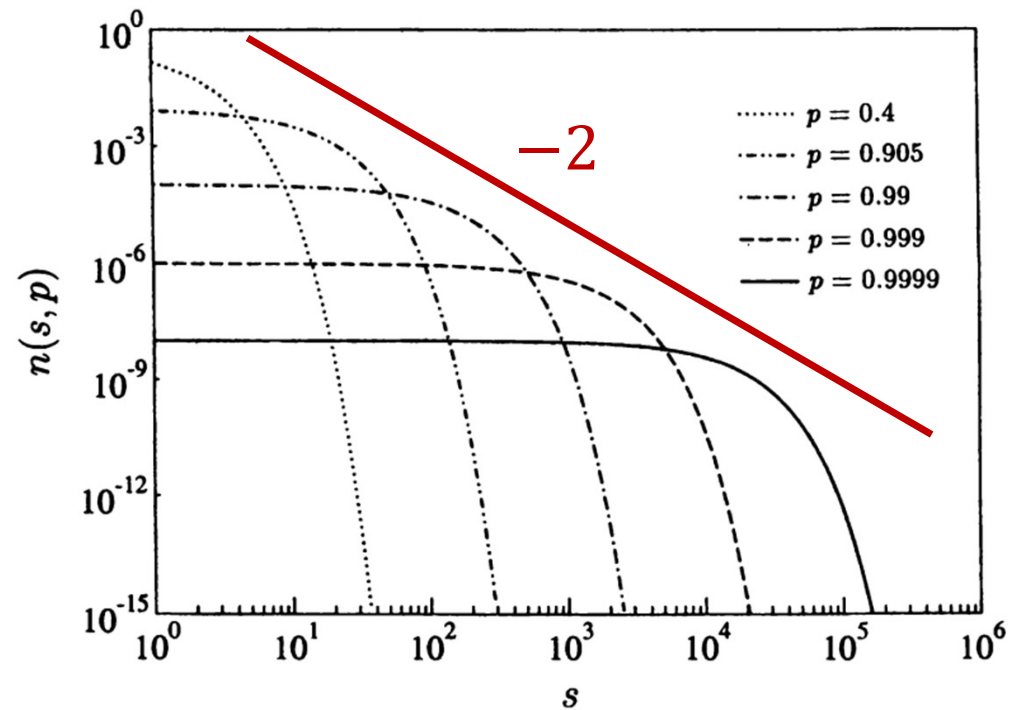
as $p \rightarrow 1^-$

$$= s_\xi^{-\tau} \exp(-1)$$

$$\tau = 2$$

Origin of critical phenomena of bond percolation: Exact solutions

(1) 1d bond percolation



Origin of critical phenomena of bond percolation: Exact solutions

(1) 1d bond percolation

$$P_{\infty} = 1 - \sum_s' sn_s = 1 - \sum_s' s(1-p)^s p^s = 0 \quad \text{for } p < 1$$
$$= 1 \quad \text{for } p = 1$$

$$P_{\infty} = (p - p_c)^{\beta} \quad \text{as } p \rightarrow p_c^+ \quad \beta = 0$$

Origin of critical phenomena of bond percolation: Exact solutions

(1) 1d bond percolation

$$\chi = \frac{\sum'_s s^2 n_s}{\sum'_s s n_s} = (1-p)^2 \sum'_s s^2 p^s = \frac{1+p}{1-p}$$

$$\chi \propto (1-p)^{-1} \text{ as } p \rightarrow 1^-$$

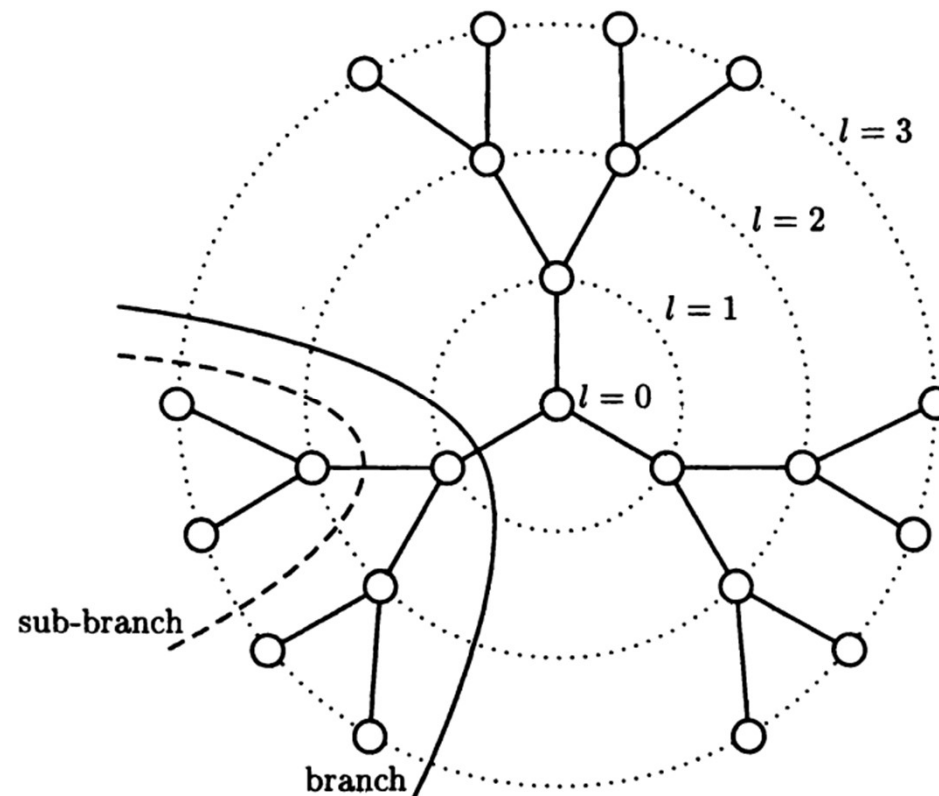
$$\chi \propto (p_c - p)^{-\gamma} \text{ as } p \rightarrow p_c^- \quad \gamma = 1$$

Origin of critical phenomena of bond percolation: Exact solutions

(2) Bond percolation on the Bethe lattice

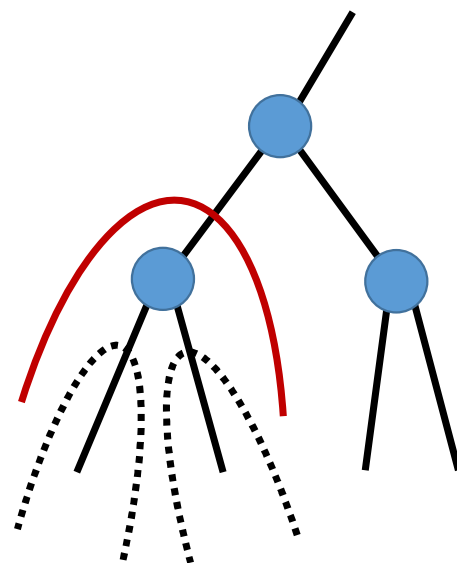
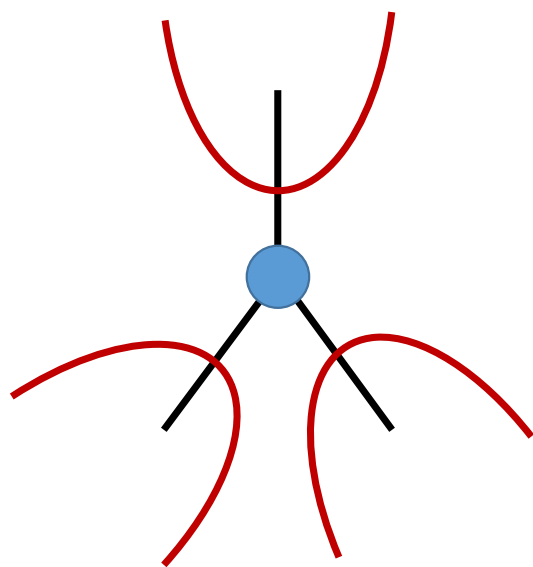
A Cayley tree with coordination number $z = 3$ and $n = 4$ generations is depicted.

When, $n \rightarrow \infty$,
the Cayley tree becomes the Bethe lattice.



Origin of critical phenomena of bond percolation: Exact solutions

(2) Bond percolation on the Bethe lattice



$$p_c = \frac{1}{z-1}$$

$$P_\infty(p) \propto (p - p_c) \text{ as } p \rightarrow p_c^+$$

$$\beta = 1$$

For $z = 3$

$$P_\infty = 1 - \left(\frac{1-p}{p}\right)^3 \text{ for } p \geq \frac{1}{2}$$

$$P_\infty = 1 - (1 - pA)^z$$

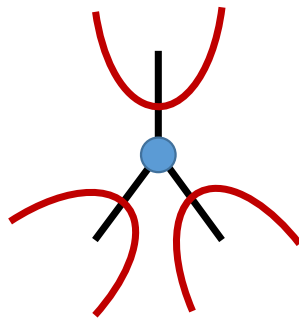
$$A = 1 - (1 - pA)^{z-1}$$

A : probability that a **branch** is connected to the infinite cluster

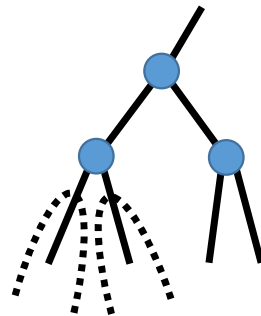
Origin of critical phenomena of bond percolation: Exact solutions

(2) Bond percolation on the Bethe lattice

For $p < p_c$



$$\chi = 1 + zpB$$



$$B = 1 + (z - 1)pB$$

B : average size of the cluster to which seed of each branch belongs

$$\chi = \frac{1 + p}{1 - (z - 1)p} \quad \text{for } p < p_c$$

$$= \frac{1 + p_c}{(z - 1)(p - p_c)} \quad \text{as } p \rightarrow p_c^+$$

$$\chi(p) \propto |p - p_c|^{-\gamma} \quad \text{as } p \rightarrow p_c$$

$$\gamma = 1$$

For $z = 3$

$$\chi = \begin{cases} \frac{1 + p}{1 - 2p} & \text{for } p < \frac{1}{2} \\ \frac{2 - p}{2p - 1} & \text{for } p > \frac{1}{2} \end{cases}$$

Origin of critical phenomena of bond percolation: Exact solutions

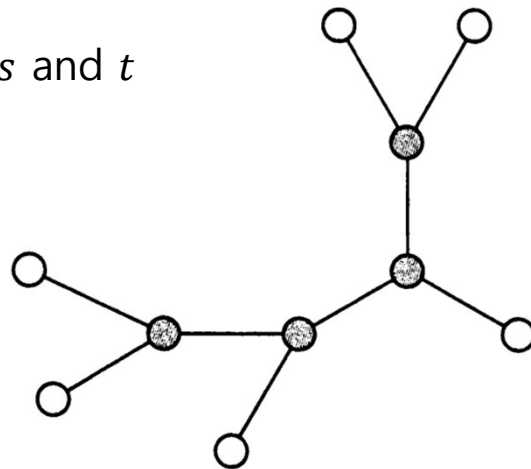
(2) Bond percolation on the Bethe lattice

Number of perimeter sites t of a cluster of size s is $t = 2 + s(z - 2)$

$$n_s = g[s, t](1 - p)^t p^s = g[s, 2 + s(z - 2)](1 - p)^{2 + s(z - 2)} p^s$$

$g[s, t]$: degeneracy factor

The number of different clusters of size s and t



Example of $s = 4$ $t = 6$ cluster

Origin of critical phenomena of bond percolation: Exact solutions

(2) Bond percolation on the Bethe lattice

$$\frac{n_s(p)}{n_s(p_c)} = \left[\frac{1-p}{1-p_c} \right]^{2+s(z-2)} \left(\frac{p}{p_c} \right)^s = \left[\frac{1-p}{1-p_c} \right]^2 \left[\frac{(1-p)^{z-2}p}{(1-p_c)^{z-2}p_c} \right]^s = \left[\frac{1-p}{1-p_c} \right]^2 \exp \left(s \ln \left[\frac{(1-p)^{z-2}p}{(1-p_c)^{z-2}p_c} \right] \right)$$

$$s_\xi = - \frac{1}{\ln \left[\frac{(1-p)^{z-2}p}{(1-p_c)^{z-2}p_c} \right]} \propto |p - p_c|^{-2} = |p - p_c|^{-1/\sigma}$$

$$\sigma = \frac{1}{2}$$

Origin of critical phenomena of bond percolation: Exact solutions

(2) Bond percolation on the Bethe lattice

$$n_s(p) = n_s(p_c) \exp(-s|p - p_c|^{1/\sigma}) \propto s^{-\tau} \exp(-s|p - p_c|^{1/\sigma}) \quad \text{as } p \rightarrow p_c$$

We use the scaling relation $\gamma = \frac{3 - \tau}{\sigma}$ which is derived in the next slide.

$$\tau = \frac{5}{2}$$

Scaling relations between $\beta, \gamma, \tau, \sigma$

We begin with the scaling ansatz $n_s(p) \propto s^{-\tau} g(s|p - p_c|^{1/\sigma})$ for $2 < \tau < 3$, where $g(x)$ is finite for $x \rightarrow 0$, but decreases to 0 rapidly as $x \rightarrow \infty$

$$(1) \quad \beta = \frac{\tau - 2}{\sigma}$$

$$(p - p_c)^\beta \propto P_\infty(p) = 1 - \sum_s' sn_s(p) = \sum_s' sn_s(p_c) - \sum_s' sn_s(p)$$

$$\approx \int_1^\infty s^{1-\tau} [g(0) - g(s(p - p_c)^{1/\sigma})] ds$$

$$\approx \int_{(p-p_c)^{1/\sigma}}^\infty (x(p - p_c)^{-1/\sigma})^{1-\tau} [g(0) - g(x)] (p - p_c)^{-1/\sigma} dx$$

$$= (p - p_c)^{(\tau-2)/\sigma} \int_0^\infty x^{1-\tau} [g(0) - g(x)] dx$$

$p \rightarrow p_c^+$

$$(2) \quad \gamma = \frac{3 - \tau}{\sigma}$$

$$|p - p_c|^{-\gamma} \propto \sum_s' s^2 n_s \approx \int_1^\infty s^{2-\tau} g(s|p - p_c|^{1/\sigma}) ds$$

$$= \int_{|p-p_c|^{1/\sigma}}^\infty (x|p - p_c|^{-1/\sigma})^{2-\tau} g(x) |p - p_c|^{-1/\sigma} dx$$

$$= |p - p_c|^{-(3-\tau)/\sigma} \int_0^\infty (x|p - p_c|^{-1/\sigma})^{2-\tau} g(x) |p - p_c|^{-1/\sigma} dx$$

$p \rightarrow p_c$

Upper critical dimension $d_c = 6$

$d \rightarrow \infty$ (mean-field)

Exponent	1d	2d	3d	4d	5d	6d	Bethe
α	1	$-2/3$	-0.62	-0.72	-0.86	-1	-1
β	0	$5/36$	0.41	0.64	0.84	1	1
γ	1	$43/18$	1.80	1.44	1.18	1	1
ν	1	$4/3$	0.88	0.68	0.57	1/2	1/2
σ	1	$36/91$	0.45	0.48	0.49	1/2	1/2
τ	2	187/91	2.18	2.31	2.41	5/2	5/2
$D(p = p_c)$	1	91/48	2.53	3.06	3.54	4	4

(Phase) Synchronization

Synchronization of N oscillators

$$\mathbf{x}_1(t) = \mathbf{x}_2(t) = \cdots = \mathbf{x}_N(t)$$

$\mathbf{x}_i(t)$: state of oscillator i at time t



$$N = 8$$

$$\mathbf{x}_1(t) = \mathbf{x}_2(t) = \cdots = \mathbf{x}_8(t)$$

Synchronization examples



Synchronized fireflies

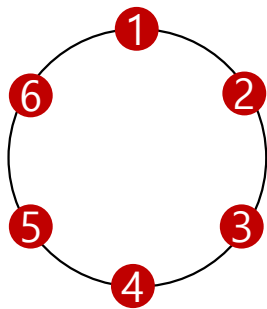
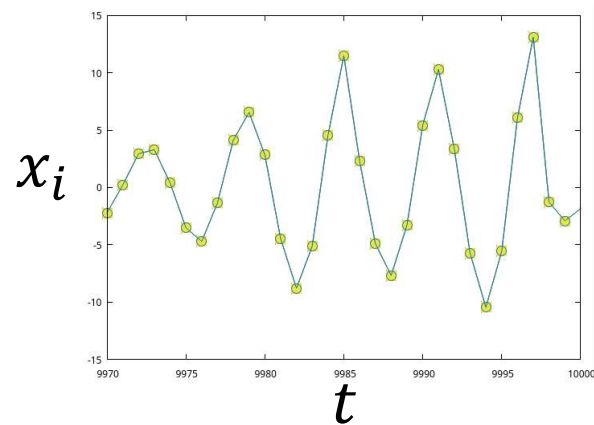


Synchronized Swimming

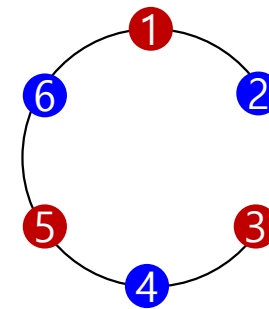
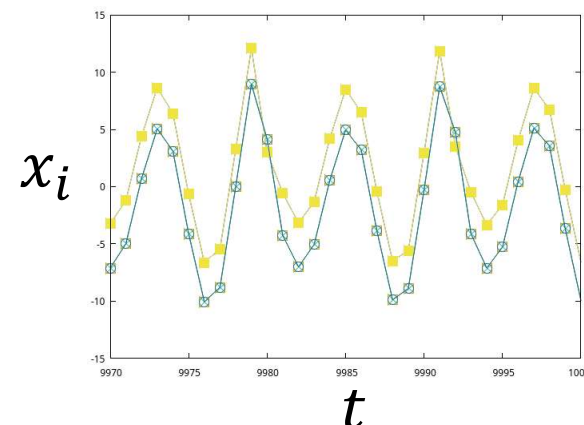
Global synchronization vs. cluster synchronization

On a given network of N oscillators

Global synchronization



Cluster synchronization



Global synchronization is a cluster synchronization of the single giant cluster

Oscillator

Each oscillator has n -dimensional vector $\mathbf{x}(t)$ as its phase, where $\mathbf{x}(t)$ evolves following $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$:

Example

(1) Rössler oscillator

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} -y - z \\ x + ay \\ b + z(x - c) \end{pmatrix}$$

(2) Lorenz oscillator

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \sigma(y - x) \\ x(\rho - z) - y \\ xy - \beta z \end{pmatrix}$$

(3) Kuramoto oscillator

$$x = \phi \in [0, 2\pi)$$

$$F = \omega$$

Oscillator

Trajectory

Rössler oscillator

$$\frac{dx}{dt} = -y - z$$

$$\frac{dy}{dt} = x + ay$$

$$\frac{dz}{dt} = b + z(x - c)$$

Chaotic oscillator

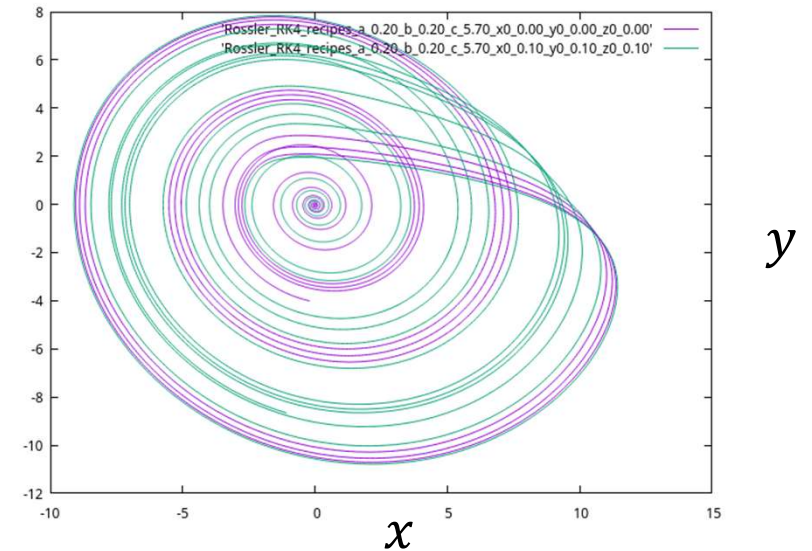
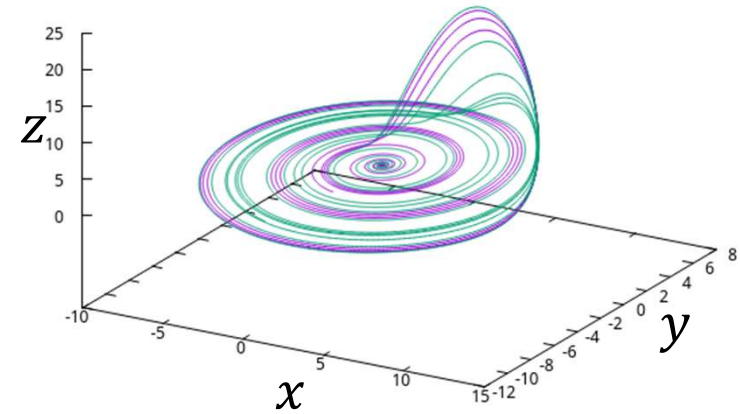
—————

$$(x(0), y(0), z(0)) = (0, 0, 0)$$

—————

$$(x(0), y(0), z(0)) = (0.1, 0.1, 0.1)$$

$$a = 0.2, b = 0.2, c = 5.7$$



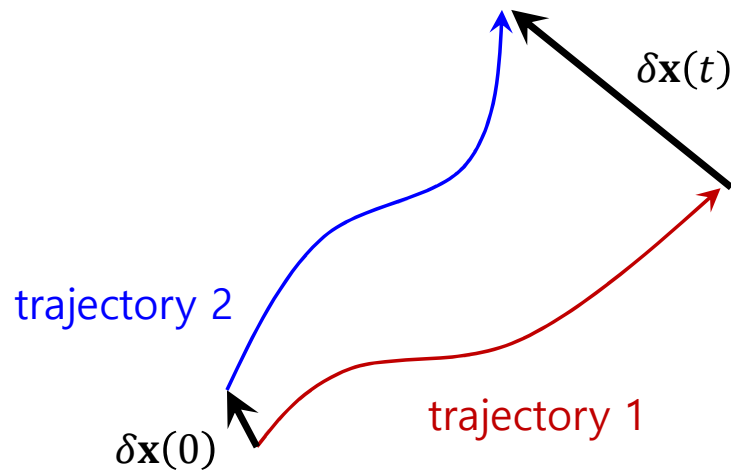
Oscillator

(Maximal) Lyapunov exponent Λ of a trajectory

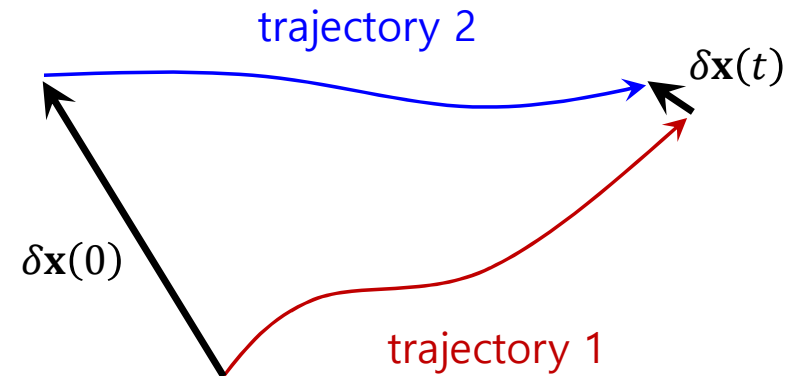
$$\|\delta\mathbf{x}(t)\| \approx e^{\Lambda t} \|\delta\mathbf{x}(0)\| \quad \text{as } t \rightarrow \infty$$

Butterfly effect

$\Lambda > 0$



$\Lambda < 0$



Oscillator

(Maximal) Lyapunov exponent Λ of a trajectory

$$\|\delta\mathbf{x}(t)\| \approx e^{\Lambda t} \|\delta\mathbf{x}(0)\| \quad \text{as } t \rightarrow \infty$$

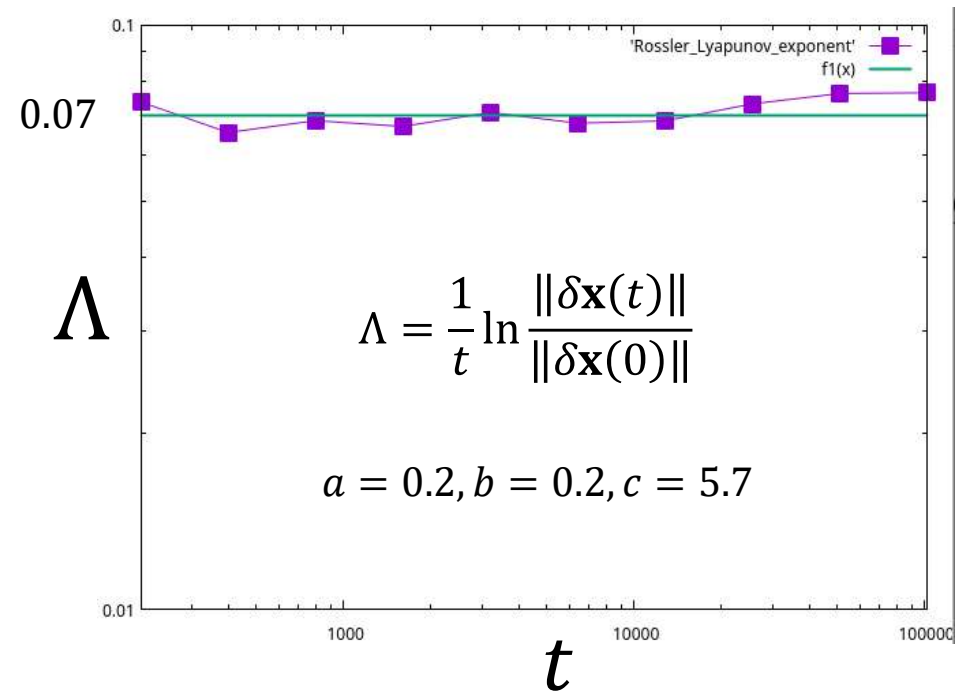
(Example) Rössler oscillator

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$$

$$\delta\dot{\mathbf{x}} = D\mathbf{F}(\mathbf{x})\delta\mathbf{x}$$

$$= \begin{pmatrix} \frac{\partial F_x}{\partial x} & \frac{\partial F_x}{\partial y} & \frac{\partial F_x}{\partial z} \\ \frac{\partial F_y}{\partial x} & \frac{\partial F_y}{\partial y} & \frac{\partial F_y}{\partial z} \\ \frac{\partial F_z}{\partial x} & \frac{\partial F_z}{\partial y} & \frac{\partial F_z}{\partial z} \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x - c \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}$$

$\Lambda > 0 \rightarrow$ Chaotic behavior of the trajectory



Synchronization of coupled chaotic oscillators

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PHYSICAL REVIEW LETTERS

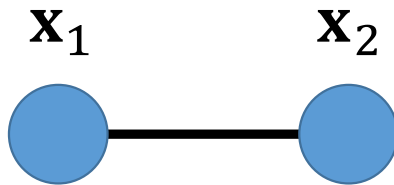
19 FEBRUARY 1990

Synchronization in Chaotic Systems

Louis M. Pecora and Thomas L. Carroll

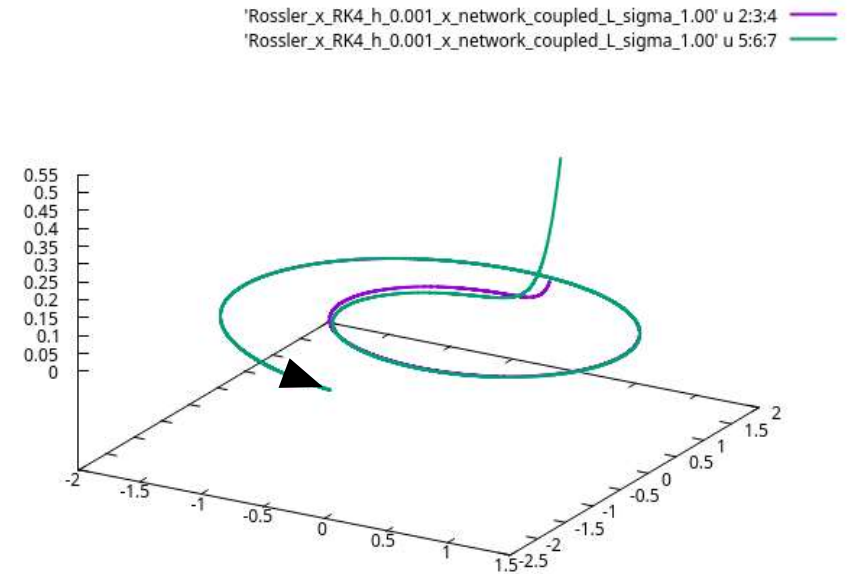
Code 6341, Naval Research Laboratory, Washington, D.C. 20375

(Received 20 December 1989)



$$\dot{\mathbf{x}}_1 = \mathbf{F}(\mathbf{x}_1) - [x_2 - x_1]$$

$$\dot{\mathbf{x}}_2 = \mathbf{F}(\mathbf{x}_2) - [x_1 - x_2]$$



Governing equation of coupled oscillators on a network

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) - \sigma \sum_j L_{ij} \mathbf{H}(\mathbf{x}_j)$$

$$\text{Laplacian matrix } L_{ij} = \begin{cases} \text{deg}(i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and they are connected.} \\ 0 & \text{if } i \neq j \text{ and they are disconnected.} \end{cases}$$

(Note)

Kuramoto oscillators of the same natural frequency follow the above governing equation near the synchronization manifold ($\phi_i - s = \delta\phi_i \ll 1$ for all i) approximately.

$$\dot{\phi}_i = \omega + K \sum_{j \in \Lambda_i} \sin(\phi_j - \phi_i) = \omega + K \sum_{j \in \Lambda_i} \sin(\delta\phi_j - \delta\phi_i) \approx \omega - K \sum_j L_{ij} \delta\phi_j$$

Properties of the Laplacian matrix

In this presentation, we only consider an **undirected** and **connected** network.

1. L is a symmetric matrix, and it is diagonalizable with a set of orthonormal eigenvectors.

↔ There exists a set of orthonormal eigenvectors $\{\mathbf{u}_\kappa\}_{1 \leq \kappa \leq N}$ satisfying $L\mathbf{u}_\kappa = \lambda_\kappa \mathbf{u}_\kappa$

2. L is positive-semidefinite, that is $\lambda_\kappa \geq 0$ for all κ .

3. The minimum eigenvalue $\lambda_1 = 0$ is non-degenerate and corresponding eigenvector $\mathbf{u}_1 = \frac{1}{\sqrt{N}}(1, \dots, 1)^\top$. Therefore, $0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_N$.

4. Row sum of L is zero, that is $\sum_j L_{ij} = 0$ for all i .

Mechanism for global synchronization

(1) Existence of synchronous trajectory

Synchronous trajectory follows $\dot{\mathbf{s}} = \mathbf{F}(\mathbf{s})$, which is the dynamical equation of a single oscillator.

(Proof)

Let $\mathbf{x}_i = \mathbf{s}$ for all i and substitute into the governing equation. Then, we obtain

$$\dot{\mathbf{x}}_i = \dot{\mathbf{s}} = \mathbf{F}(\mathbf{s}) - \sigma \sum_j L_{ij} H(\mathbf{s}) = \mathbf{F}(\mathbf{s}) \quad \text{for all } i$$

Row sum of L is zero.

Mechanism for global synchronization

(2) Stability of synchronous trajectory

Master stability function of an oscillator can determine the stability of global synchronization of any networks composed of those oscillators.

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9 MARCH 1998

Master Stability Functions for Synchronized Coupled Systems

Louis M. Pecora and Thomas L. Carroll

Code 6343, Naval Research Laboratory, Washington, D.C. 20375

(Received 7 July 1997)

We show that many coupled oscillator array configurations considered in the literature can be put into a simple form so that determining the stability of the synchronous state can be done by a master stability function, which can be tailored to one's choice of stability requirement. This solves, once and for all, the problem of synchronous stability for any linear coupling of that oscillator. [S0031-9007(98)05387-3]

Phys. Rev. Lett. **80** 2109 (1998)

Mechanism for global synchronization

(2) Stability of synchronous trajectory

※ Derivation of the master stability function

Substitute $\mathbf{x}_i = \mathbf{s} + \delta\mathbf{x}_i$ into the governing equation. Then, we obtain

Row sum of L is zero.

$$\dot{\mathbf{s}} + \delta\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{s} + \delta\mathbf{x}_i) - \sigma \sum_j L_{ij} \mathbf{H}(\mathbf{s} + \delta\mathbf{x}_j) \approx \mathbf{F}(\mathbf{s}) + D\mathbf{F}(\mathbf{s})\delta\mathbf{x}_i - \sigma \sum_j L_{ij} \mathbf{H}(\mathbf{s}) - \sigma \sum_j L_{ij} D\mathbf{H}(\mathbf{s})\delta\mathbf{x}_j$$

, where $D\mathbf{F}(\mathbf{s})$ and $D\mathbf{H}(\mathbf{s})$ are Jacobian matrices of \mathbf{F} and \mathbf{H} , respectively, at $\mathbf{x} = \mathbf{s}$.

$$\begin{array}{l} \xrightarrow{\text{blue}} \delta\dot{\mathbf{x}}_i = D\mathbf{F}(\mathbf{s})\delta\mathbf{x}_i - \sigma \sum_j L_{ij} D\mathbf{H}(\mathbf{s})\delta\mathbf{x}_j \quad \text{for } 1 \leq i \leq N \\ \nearrow \\ \dot{\mathbf{s}} = \mathbf{F}(\mathbf{s}) \end{array}$$

Mechanism for global synchronization

(2) Stability of synchronous trajectory

※ Derivation of the master stability function

We use a new Nn -dimensional vector $\boldsymbol{\eta} = (\mathbf{U} \otimes \mathbf{I}_n) \delta \mathbf{X}$, where $\mathbf{U}^\top = (\mathbf{u}_1, \dots, \mathbf{u}_N)$ for an orthonormal set of eigenvectors $\{\mathbf{u}_\kappa\}_{1 \leq \kappa \leq N}$ satisfying $\mathbf{L}\mathbf{u}_\kappa = \lambda_\kappa \mathbf{u}_\kappa$. For convenience, we let $\mathbf{u}_1 = \frac{1}{\sqrt{N}}(1, \dots, 1)^\top$ with $\lambda_1 = 0$. Therefore, $\mathbf{U}\mathbf{L}\mathbf{U}^\top = \text{diag}(\lambda_1, \dots, \lambda_N)$ with $\lambda_1 = 0$.

Then, the result on the previous slide is written using the new coordinates $\boldsymbol{\eta}$ such that

$$\dot{\boldsymbol{\eta}} = (\mathbf{U} \otimes \mathbf{I}_n) \delta \dot{\mathbf{X}} = (\mathbf{U} \otimes \mathbf{I}_n) [\mathbf{I}_N \otimes \mathbf{D}\mathbf{F}(\mathbf{s}) - \sigma \mathbf{L} \otimes \mathbf{D}\mathbf{H}(\mathbf{s})] (\mathbf{U} \otimes \mathbf{I}_n)^\top \boldsymbol{\eta} = [\mathbf{U} \otimes \mathbf{D}\mathbf{F}(\mathbf{s}) - \sigma \mathbf{U}\mathbf{L} \otimes \mathbf{D}\mathbf{H}(\mathbf{s})] (\mathbf{U} \otimes \mathbf{I}_n)^\top \boldsymbol{\eta}$$

$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{A}\mathbf{C}) \otimes (\mathbf{B}\mathbf{D})$

$$= [\mathbf{I}_N \otimes \mathbf{D}\mathbf{F}(\mathbf{s}) - \sigma \mathbf{U}\mathbf{L}\mathbf{U}^\top \otimes \mathbf{D}\mathbf{H}(\mathbf{s})] \boldsymbol{\eta} = [\mathbf{I}_N \otimes \mathbf{D}\mathbf{F}(\mathbf{s}) - \sigma \text{diag}(\lambda_1, \dots, \lambda_N) \otimes \mathbf{D}\mathbf{H}(\mathbf{s})] \boldsymbol{\eta}$$

$$(\mathbf{U} \otimes \mathbf{I}_n)^\top = \mathbf{U}^\top \otimes \mathbf{I}_n$$



For $\boldsymbol{\eta}_\kappa = (\mathbf{u}_\kappa^\top \otimes \mathbf{I}_n) \delta \mathbf{X}$ $\dot{\boldsymbol{\eta}}_\kappa = [\mathbf{D}\mathbf{F}(\mathbf{s}) - \sigma \lambda_\kappa \mathbf{D}\mathbf{H}(\mathbf{s})] \boldsymbol{\eta}_\kappa$ for $1 \leq \kappa \leq N$

Mechanism for global synchronization

$$\|\delta\mathbf{X}_\perp\| = \|\delta\mathbf{X} - [(\mathbf{u}_1^\top \otimes I_n)\delta\mathbf{X}](\mathbf{u}_1 \otimes I_n)\| \rightarrow \mathbf{0} \text{ as } t \rightarrow \infty$$

(2) Stability of synchronous trajectory

※ Derivation of the master stability function

$$\|\boldsymbol{\eta}\|^2 = \boldsymbol{\eta}^\top \boldsymbol{\eta} = \delta\mathbf{X}^\top (\mathbf{U}^\top \otimes I_n) (\mathbf{U} \otimes I_n) \delta\mathbf{X} = \delta\mathbf{X}^\top (\mathbf{U}^\top \mathbf{U} \otimes I_n) \delta\mathbf{X} = \|\delta\mathbf{X}\|^2$$

$\mathbf{U}^\top \mathbf{U} = I_N$

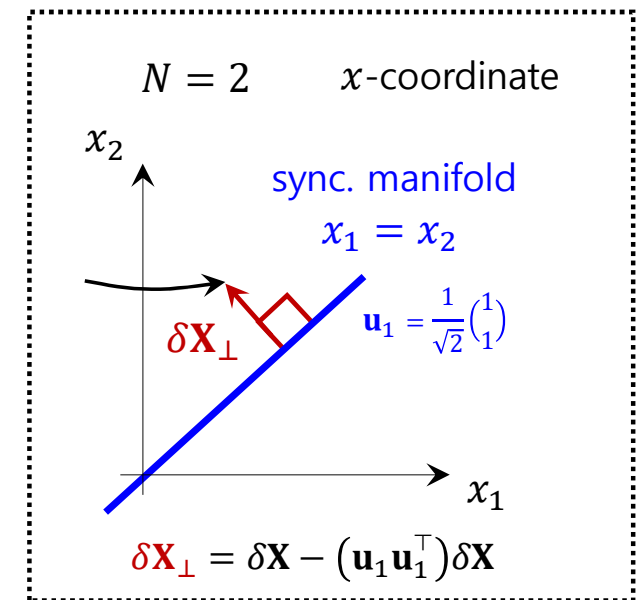
Component of $\delta\mathbf{X}$ perpendicular to the synchronization manifold

$$\delta\mathbf{X}_\perp = \delta\mathbf{X} - \begin{pmatrix} \frac{1}{N} \sum x_i \\ \vdots \\ \frac{1}{N} \sum x_i \end{pmatrix} = \delta\mathbf{X} - (\mathbf{u}_1 \mathbf{u}_1^\top \otimes I_n) \delta\mathbf{X} = \delta\mathbf{X} - (\mathbf{u}_1 \otimes I_n) \boldsymbol{\eta}_1$$

$$\|\delta\mathbf{X}_\perp\|^2 = \delta\mathbf{X}_\perp^\top \delta\mathbf{X}_\perp = \|\delta\mathbf{X}\|^2 - \|\boldsymbol{\eta}_1\|^2 = \|\boldsymbol{\eta}\|^2 - \|\boldsymbol{\eta}_1\|^2 = \sum_{k=2}^N \|\boldsymbol{\eta}_k\|^2$$

$$\boldsymbol{\eta}_1 = (\mathbf{u}_1^\top \otimes I_n) \delta\mathbf{X}$$

(Example)



Mechanism for global synchronization

(2) Stability of synchronous trajectory

※ Derivation of the master stability function

In summary, synchronous trajectory ($\mathbf{x}_i = \mathbf{s}$ for all i) of the governing equation $\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) - \sigma \sum_j L_{ij} \mathbf{H}(\mathbf{x}_j)$ follows $\dot{\mathbf{s}} = \mathbf{F}(\mathbf{s})$ and stability of the global synchronization is determined using $\dot{\boldsymbol{\eta}}_\kappa = [D\mathbf{F}(\mathbf{s}) - \sigma \lambda_\kappa D\mathbf{H}(\mathbf{s})]\boldsymbol{\eta}_\kappa$.

We assume long time behavior of $\|\boldsymbol{\eta}_\kappa\|$ as

$$\|\boldsymbol{\eta}_\kappa\| \approx e^{\Lambda_\kappa t} \|\boldsymbol{\eta}_\kappa(0)\| \quad \text{as } t \rightarrow \infty \quad \text{for } 2 \leq \kappa \leq N$$

$$\|\delta\mathbf{X}_\perp\|^2 = \sum_{\kappa=2}^N \|\boldsymbol{\eta}_\kappa\|^2 \approx \sum_{\kappa=2}^N e^{2\Lambda_\kappa t} \|\boldsymbol{\eta}_\kappa(0)\|^2 \quad \text{as } t \rightarrow \infty$$

When $\Lambda_\kappa < 0$ for all $2 \leq \kappa \leq N$, $\|\delta\mathbf{X}_\perp\|^2 \rightarrow 0$ as $t \rightarrow \infty$ \longrightarrow Global synchronization is stable.

Otherwise, $\|\delta\mathbf{X}_\perp\|^2 \rightarrow \infty$ as $t \rightarrow \infty$ \longrightarrow Global synchronization is unstable.

Mechanism for global synchronization

(2) Stability of synchronous trajectory

※ Derivation of the master stability function

Based on long time behavior $\|\boldsymbol{\eta}_\kappa\| \approx e^{\Lambda_\kappa t} \|\boldsymbol{\eta}_\kappa(0)\|$ with $\dot{\boldsymbol{\eta}}_\kappa = [D\mathbf{F}(\mathbf{s}) - \sigma\lambda_\kappa D\mathbf{H}(\mathbf{s})]\boldsymbol{\eta}_\kappa$ and $\dot{\mathbf{s}} = \mathbf{F}(\mathbf{s})$,

We find that Λ_κ is a function of $\sigma\lambda_\kappa$ for a given oscillator ($\mathbf{F}(\mathbf{x})$ and $\mathbf{H}(\mathbf{x})$).

Therefore, if we calculate Λ_\perp of $\|\boldsymbol{\eta}_\kappa\| \approx e^{\Lambda_\perp t} \|\boldsymbol{\eta}_\kappa(0)\|$ for general values of α using $\dot{\boldsymbol{\eta}}_\kappa = [D\mathbf{F}(\mathbf{s}) - \alpha D\mathbf{H}(\mathbf{s})]\boldsymbol{\eta}_\kappa$ with $\dot{\mathbf{s}} = \mathbf{F}(\mathbf{s})$ in advance,

we can determine stability of the global synchronization of $\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) - \sigma \sum_j L_{ij} \mathbf{H}(\mathbf{x}_j)$ by checking whether $\sigma\lambda_\kappa$ for $2 \leq \kappa \leq N$ belong to the range of α where $\Lambda_\perp(\alpha) < 0$.

$\Lambda_\perp(\alpha)$: Master stability function

Mechanism for global synchronization

(2) Stability of synchronous trajectory

※ Usefulness of the master stability function

- (1) Calculation of the master stability function only requires dimensions of phase space of a single oscillator.
- (2) Determination using the master stability function only requires eigenvalues of a given Laplacian matrix.

Therefore, if we calculate Λ_{\perp} of $\|\boldsymbol{\eta}_{\kappa}\| \approx e^{\Lambda_{\perp} t} \|\boldsymbol{\eta}_{\kappa}(0)\|$ for general values of α using $\dot{\boldsymbol{\eta}}_{\kappa} = [D\mathbf{F}(\mathbf{s}) - \alpha D\mathbf{H}(\mathbf{s})]\boldsymbol{\eta}_{\kappa}$ with $\dot{\mathbf{s}} = \mathbf{F}(\mathbf{s})$ in advance,

we can determine stability of the global synchronization of $\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) - \sigma \sum_j L_{ij} \mathbf{H}(\mathbf{x}_j)$ by checking whether $\sigma \lambda_{\kappa}$ for $2 \leq \kappa \leq N$ belong to the range of α where $\Lambda_{\perp}(\alpha) < 0$.

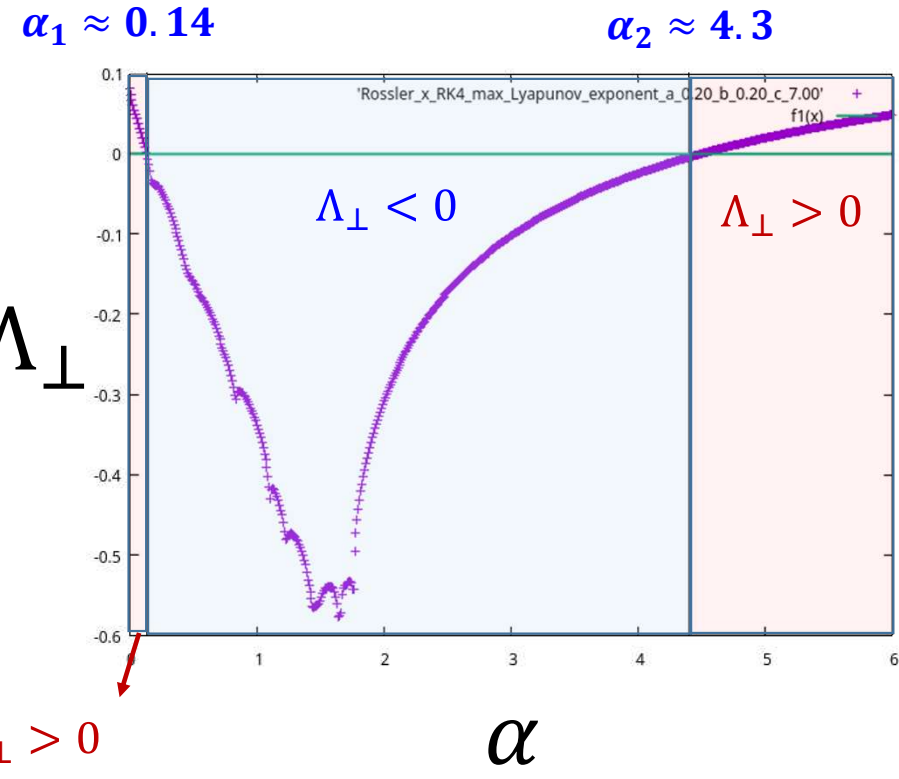
$\Lambda_{\perp}(\alpha)$: Master stability function

Example: x -coupled Rössler oscillators with $a = 0.2, b = 0.2, c = 7$ on a ring of length $N = 6$

Master stability function for x -coupled Rössler oscillators with $a = 0.2, b = 0.2, c = 7$ is obtained in the range $\alpha \in [0, 6]$.

$$\dot{\boldsymbol{\eta}} = [D\mathbf{F}(\mathbf{s}) - \alpha D\mathbf{H}(\mathbf{s})]\boldsymbol{\eta} \quad \text{with} \quad \dot{\mathbf{s}} = \mathbf{F}(\mathbf{s}) \quad \text{where} \quad \mathbf{F}(\mathbf{x}) = \begin{pmatrix} -y - z \\ x + 0.2y \\ 0.2 + z(x - 7) \end{pmatrix} \quad \mathbf{H}(\mathbf{x}) = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$$

$$\text{Then, } D\mathbf{F}(\mathbf{s}) - \alpha D\mathbf{H}(\mathbf{s}) = \begin{pmatrix} -\alpha & -1 & -1 \\ 1 & 0.2 & 0 \\ z & 0 & x - 7 \end{pmatrix}$$



Application to a given network of that oscillator

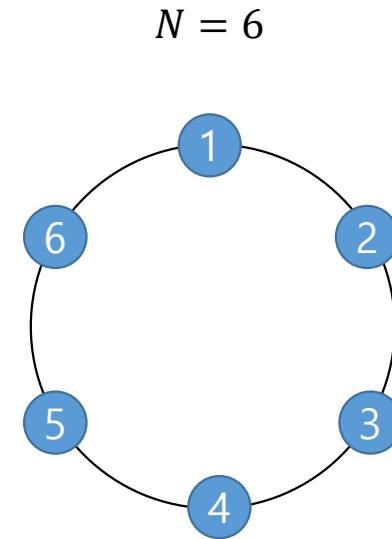
When $\sigma \lambda_k \in (\alpha_1, \alpha_2)$ for all nonzero eigenvalues λ_k of a given network, the global synchronization in the network is stable.

↔ When $\frac{\alpha_1}{\lambda_2} < \sigma < \frac{\alpha_2}{\lambda_N}$, the global synchronization is stable.

Example: x -coupled Rössler oscillators with $a = 0.2, b = 0.2, c = 7$ on a ring of length $N = 6$

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) - \sigma \sum_j L_{ij} \mathbf{x}_j$$

$$L = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

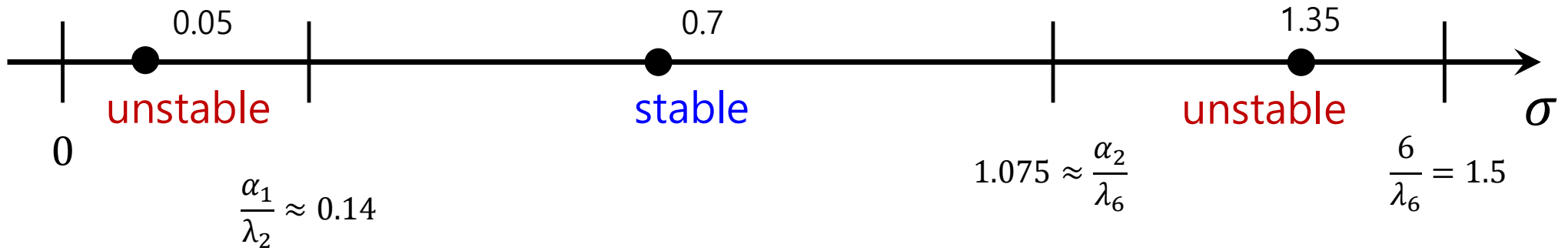
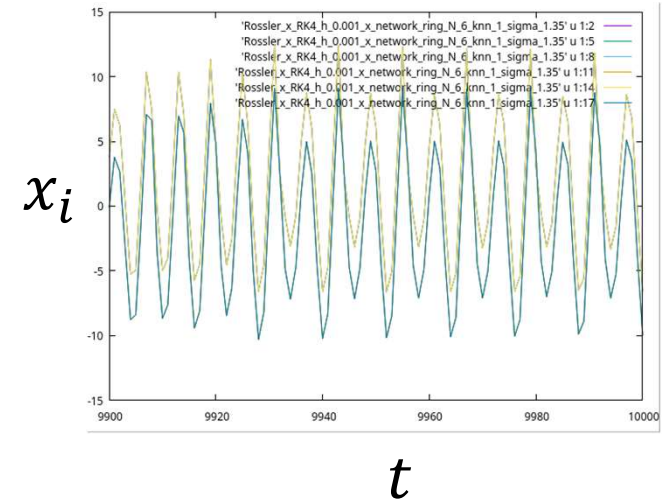
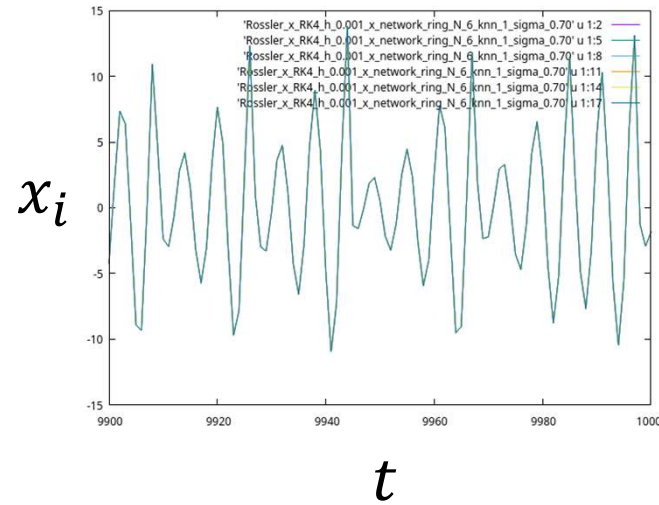
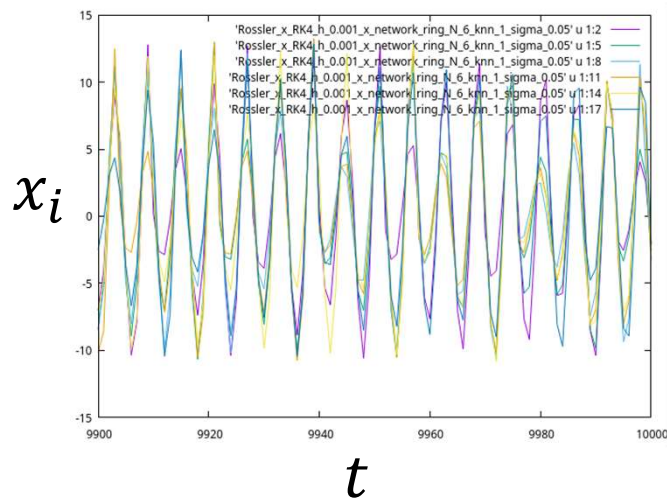


$$\lambda_1 = 0 \quad \lambda_2 = 1 \quad \lambda_3 = 1 \quad \lambda_4 = 3 \quad \lambda_5 = 3 \quad \lambda_6 = 4$$

When $0.14 \approx \frac{\alpha_1}{\lambda_2} < \sigma < \frac{\alpha_2}{\lambda_6} \approx 1.075$, the global synchronization is stable and observable beginning with random initial phases.

When $0 < \sigma < \frac{\alpha_1}{\lambda_2} \approx 0.14$ and $1.075 \approx \frac{\alpha_2}{\lambda_6} < \sigma < \frac{6}{\lambda_6} = 1.5$, the global synchronization is unstable and it is not observable beginning with random initial phases.

Example: x -coupled Rössler oscillators with $a = 0.2, b = 0.2, c = 7$ on a ring of length $N = 6$



Synchronizability

Phys. Rev. Lett. **91**, 014101 (2003)

Nat. Phys. **9**, 89–92 (2013)

If the master stability function $\Lambda_{\perp}(\alpha) < 0$ in a range $\alpha \in (\alpha_1, \alpha_2)$, global synchronization is stable only when the coupling strength $\sigma \in \left(\frac{\alpha_1}{\lambda_2}, \frac{\alpha_2}{\lambda_N}\right)$. This means that range of the coupling strength in which the global synchronization is stable exists only when $\frac{\alpha_1}{\lambda_2} < \frac{\alpha_2}{\lambda_N}$ or $\frac{\lambda_N}{\lambda_2} < \frac{\alpha_2}{\alpha_1}$.

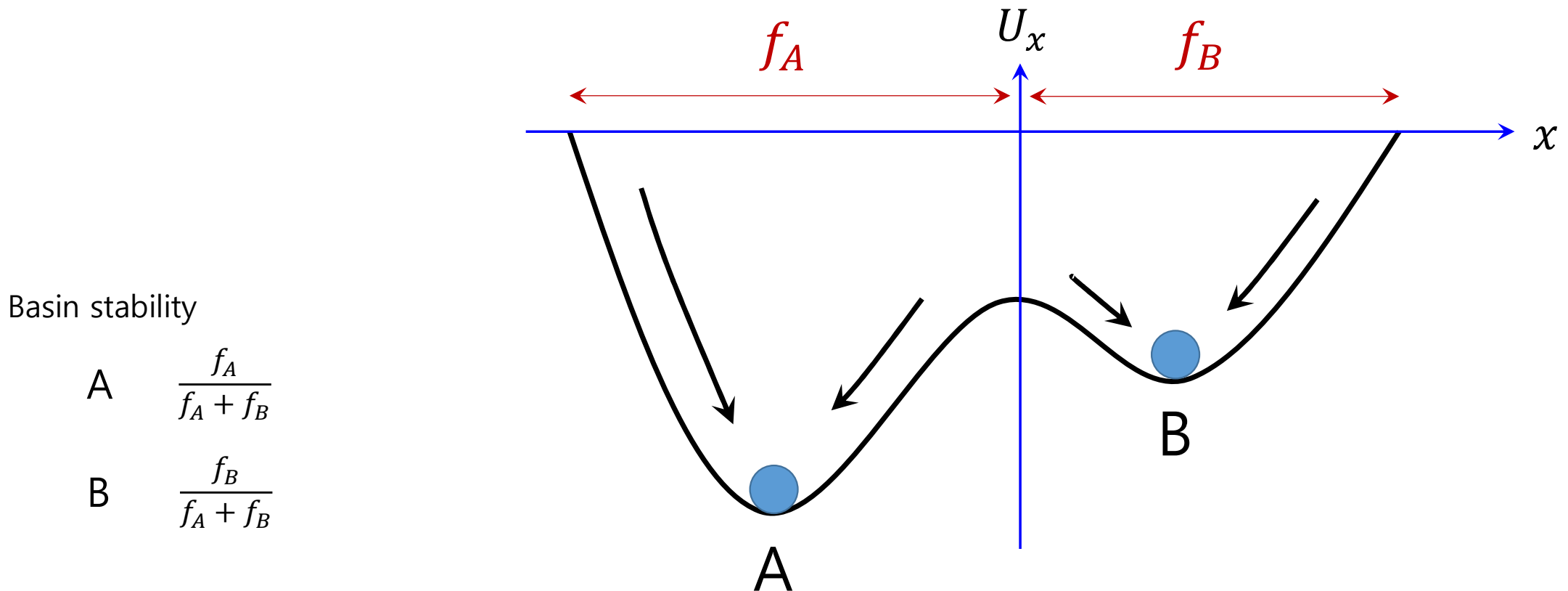
※ Note that we only consider $\alpha > 0$ for convenience.

For a given oscillator, this condition is easier to satisfy as $\frac{\lambda_N}{\lambda_2}$ decreases. Therefore, the $\frac{\lambda_N}{\lambda_2}$ is called “synchronizability” of the given network structure (Laplacian matrix).

Basin stability

Nat. Phys. 9, 89–92 (2013)

If there are multiple stable states (multistability), phase space is divided into regions that are attracted to each stable state. Then, fraction of each region of whole phase space becomes the basin stability of corresponding stable state.



Application of theoretical framework to Kuramoto model of identical oscillators

$$\phi_i \in [0, 2\pi)$$

$$\dot{\phi}_i = \omega + K \sum_{j=1}^N A_{ij} \sin(\phi_j - \phi_i)$$

For a rotating frame $\phi_i \rightarrow \phi_i + \omega t$

$$\dot{\phi}_i = K \sum_{j=1}^N A_{ij} \sin(\phi_j - \phi_i)$$

Adjacency matrix A_{ij} $\left\{ \begin{array}{ll} 0 & \text{if } i = j \\ 1 & \text{if } i \neq j \text{ and they are connected.} \\ 0 & \text{if } i \neq j \text{ and they are disconnected.} \end{array} \right.$

Application of theoretical framework to Kuramoto model of identical oscillators

Global synchronization

(1) Trajectory

Substitute $\phi_i = s$ into the governing equation

$$\dot{s} = 0 \quad (\text{phase locked state})$$

(2) Stability

Substitute $\phi_i = s + \delta\phi_i$ into the governing equation and expand up to linear order of $\delta\phi_i$

$$\delta\dot{\Phi} = -KL\delta\Phi \quad \text{where} \quad \delta\Phi = \begin{pmatrix} \delta\phi_1 \\ \vdots \\ \delta\phi_N \end{pmatrix} \xrightarrow{\eta_\kappa = \mathbf{u}_\kappa^\top \delta\Phi} \dot{\eta}_\kappa = -K\lambda_\kappa \eta_\kappa \quad \text{for } 1 \leq \kappa \leq N$$
$$\Lambda_\kappa = -K\lambda_\kappa < 0 \quad \text{for all } 2 \leq \kappa \leq N \quad \text{when } K > 0.$$

→ Global synchronization of identical oscillators is stable when $K > 0$.

Application of theoretical framework to Kuramoto model of identical oscillators (problem2 explanation)

Cluster synchronization

q -twisted states on a ring

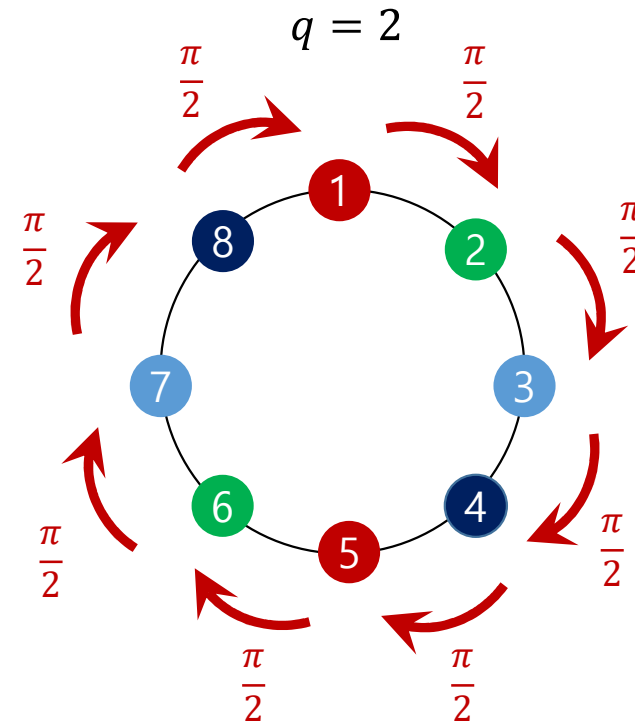
$$\phi_i^* = \left[\frac{2\pi q}{N} i \right] \bmod 2\pi$$

(1) Trajectory

Show that ϕ_i^* is a phase locked state ($\dot{\phi}_i^* = 0$).

(2) Stability

Derive dynamical equation of $\delta\Phi$ for $\delta\phi_i = \phi_i - \phi_i^*$ up to linear order of $\delta\phi_i$. Then, find the condition for which $\|\delta\Phi_{\perp}\| \rightarrow 0$ as $t \rightarrow \infty$, where $\delta\Phi_{\perp}$ is component of $\delta\Phi$ perpendicular to ϕ_i^* manifold.



Symmetry-induced cluster synchronizations (Optional)

TBA