New lower bounds on scattering amplitudes

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$$c = \hbar = M_G^2 = 1/(8\pi G) = 1$$

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Introduction

- **What is the ultimate (UV complete) theory ?**
- **What kind of conditions it should satisfy ?**

In order to address these questions,
we investigate S-matrix (scattering amplitude).

Basics of kinematics (scattering amplitude)

S-matrix : $\langle f \,|\, \boldsymbol{S} \,|\, i \rangle_{\text{Heisenberg}} = \langle f; t = \infty \,|\, i; t = -\infty \rangle_{\text{Schrodinger}}$

$(\boldsymbol{S} = 1 + i\boldsymbol{T})$ ⇨ $\langle f | \boldsymbol{T} | i \rangle = (2\pi)^4 \delta^{(4)}(p_i - p_f) \mathcal{M}(i \to f)$

↑ **scattering amplitude**

2 → 2 elastic scattering (p1 + p2 = p3 + p4) :

$$\langle p_3, p_4 | \boldsymbol{T} | p_1, p_2 \rangle = (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) \mathcal{M}(s, t, u)$$

Mandelstam variables : $s = -(p_1 + p_2)^2\,,\ t = -(p_1 - p_3)^2\,,\ u = -(p_1 - p_4)^2\,.$

(CM energy squared) **(momentum transfer squared)**

$$\left(s + t + u = 4m^2, \quad z = \cos\theta = 1 + \frac{2t}{s - 4m^2} \right)$$

All masses are assumed equal.

- **Hard-scattering limit, M(s,z) :**

$$s \to \infty\,, \quad -1 < z = \cos\theta < 1 = \text{fixed}\,.$$

- **Regge limit, M(s,t) :**

$$\left(p_s = |\vec{p}_1| = |\vec{p}_3| = \frac{1}{2}\sqrt{s - 4m^2} \right)$$

$$s \to \infty\,, \quad t = -2p_s^2(1 - \cos\theta) = \text{fixed}\,.$$

(Well-known) upper and lower bounds

- Froissart (upper) bound

$$|\mathcal{M}(s, z = \cos\theta = 1)| < s \ln^2 s, \quad \text{for } s \to \infty$$

- Cerulus-Martin (lower) bound

$$\max_{-a \le \cos\theta \le a} |\mathcal{M}(s, \cos\theta)| \ge \mathcal{N}(s)\, e^{-f(a)\sqrt{s}\,\log(s/s_0)},$$

N(s) : a positive function of s that is subdominant in the s → ∞ limit
f(a) : a positive function of a ∈ (0,1)
s_0 : some energy-squared reference scale

by assuming unitarity, analyticity, polynomial boundedness, a finite mass gap.

$(SS^\dagger = 1)$ **(analytic except at poles and branch cuts)**

(If the bound would be falsified in an experiment, one of the assumptions must be violated !!)

(Jaffe's) classification of localizability

$\rho(-p^2)$ **: Kallen-Lehmann spectral density**

$$\begin{cases} W(z\equiv x-y) = \langle 0|\,\phi(x)\phi(y)\,|0\rangle = \int \frac{d^4p}{(2\pi)^4}\tilde{W}(p)e^{ipz} = \int_0^\infty d\mu\,\rho(\mu)W_{\text{free}}(z;\mu) \\ \tilde{W}(p) = 2\pi\,\Theta(p^0)\rho(-p^2)\,, \quad \rho_{\text{free}}(-p^2) = \delta(p^2+m^2) \end{cases}$$

$$\rho(-p^2) \sim (-p^2)^N \exp\left[c\,(-p^2)^\alpha\right], \quad \begin{cases} 0 \le \alpha < \frac{1}{2} & : \ \text{strictly localizable}\,, \\ \alpha > \frac{1}{2} & : \ \text{non-localizable}\,, \\ \alpha = \frac{1}{2} & : \ \text{quasi-local}\,. \end{cases}$$

Note that W(x-y) with $\alpha \geqq 1/2$ is ill-defined (diverges) even for $x \neq y$.

$$W_{\text{free}}(z;\mu) \equiv \int \frac{d^4p}{(2\pi)^3}\Theta(p^0)\delta(p^2+\mu)e^{ipz} \sim \begin{cases} \frac{(2\sqrt{\mu})^{1/2}}{\left(4\pi\sqrt{z^2}\right)^{3/2}}e^{-\sqrt{\mu z^2}} & \text{for } \ \mu z^2 \gg 1\,, \\ -ie^{-i\pi/4}\frac{(2\sqrt{\mu})^{1/2}}{\left(4\pi\sqrt{-z^2}\right)^{3/2}}e^{-i\sqrt{-\mu z^2}} & \text{for } \ -\mu z^2 \gg 1\,, \end{cases}$$

Examples

$$\rho(-p^2) \sim (-p^2)^N \exp\left[c\,(-p^2)^\alpha\right] \qquad \begin{cases} 0 \leq \alpha < \frac{1}{2} & : \ \text{strictly localizable}\,, \\ \alpha > \frac{1}{2} & : \ \text{non-localizable}\,, \\ \alpha = \frac{1}{2} & : \ \text{quasi-local}\,. \end{cases}$$

- **Standard interacting QFT : α = 0 (polynomial bounded)**

- **Gravity and BH formation : α = (D-2) / (2(D-3)), α > 1/2 for D > 3**

In the usual perturbative QFTs, we can probe arbitrary short-distance scales L ~ E^{-1} with s, -t ~ E^2.

In GR, there exists a lower limit on the distance scale L that can be probed before BH formation sets in, and this is given by L ≳ 2E / M_p^2.
➔ More energetic probes are affected by a larger uncertainty in resolving distances.

$$\rho(s \sim (-p^2)) \sim e^{S_{\mathrm{BH}}(\sqrt{s})} = e^{c\,(\sqrt{s}/M_p)^{\frac{D-2}{D-3}}}$$

Similar argument

$$\mathcal{M}(s, \cos\theta) \sim e^{-S_{\mathrm{BH}}(\sqrt{s})} = e^{-c\,(\sqrt{s}/M_p)^{\frac{D-2}{D-3}}} \quad \text{for } E \gg M_p$$

New bound

(Well-known) upper and lower bounds

- Froissart (upper) bound

$$|\mathcal{M}(s, z = \cos\theta = 1)| < s \ln^2 s, \quad \text{for } s \to \infty$$

- Cerulus-Martin (lower) bound

$$\max_{-a \leq \cos\theta \leq a} |\mathcal{M}(s, \cos\theta)| \geq \mathcal{N}(s)\, e^{-f(a)\sqrt{s}\,\log(s/s_0)},$$

N(s) : a positive function of s that is subdominant in the s → ∞ limit
f(a) : a positive function of a ∈ (0,1)
s_0 : some energy-squared reference scale

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(If the bound would be falsified in an experiment, one of the assumptions must be violated !!)

Polynomial boundedness ➔ Exponential boundedness

$$|\mathcal{M}(s,z)| \leq A\left(\frac{s}{s_0}\right)^N, \quad \frac{s}{s_0} \gg 1 \quad \Longrightarrow \quad |\mathcal{M}(s,z)| \leq A\left(\frac{s}{s_0}\right)^N e^{\sigma(s/s_0)^{\alpha}}$$

(N, α: positive constant, A: positive parameter relying on z, s0 : some energy-squared reference scale)

Cerulus-Martin (lower) bound is more generalized !!

➔ **New lower bound in the hard-scattering limit:**

$$\max_{-a\leq\cos\theta\leq a} |\mathcal{M}(s,\cos\theta)| \geq \mathcal{N}(s)\, e^{-f(a)\sqrt{s}\,\log(s/s_0)} e^{-g(a)\, s^{\alpha+\frac{1}{2}}},$$

N(s) : a positive function of s that is subdominant in the s ➔ ∞ limit
f(a), g(a) : positive functions of a ∊ (0,1)
s0 : some energy-squared reference scale

- **α=0 consistently recovers the Cerulus-Martin(CM) bound.**
- **Our result admits a violation of the original CM bound even for 0 < α < 1/2.**
  This is interesting since the CM bound has been used as a test of locality in the past.

New lower bound in the Regge limit

Regge limit : $s \to \infty,\ t = -2p_s^2(1-\cos\theta) = \text{fixed}$

$$(p_s = |\vec{p}_1| = |\vec{p}_3| = \frac{1}{2}\sqrt{s-4m^2})$$

$$\Delta \equiv \frac{p_s^2}{4m^2}(1-a) \quad \Longleftrightarrow \quad t = -8m^2\Delta|_{a=\cos\theta}$$

New lower bound in the Regge limit:

$$\max_{8m^2\Delta - 4p_s^2 \le t \le -8m^2\Delta} |\mathcal{M}(s,t)| \ge h(\Delta)\, e^{-\tilde{f}(\Delta)\log(s/s_0) - \tilde{g}(\Delta)s^{\alpha}}$$

h(Δ), f(Δ), g(Δ) : positive functions of Δ, $0 < \Delta < p_s^2/(4m^2)$
s_0 : some energy-squared reference scale

- **α=0 corresponds to the bound for polynomial boundedness**
- **The s-dependence in the lower bound for fixed momentum transfer differs from the one for fixed scattering angle by a factor of sqrt{s} in the exponent. This means that amplitudes in the hard-scattering regime (large angles) can be more suppressed as compared to the ones in the Regge regime (small angles).**

Summary

- **We have generalized the so-called Cerulus-Martin (lower) bound on elastic scattering amplitude in hard-scattering limit by assuming exponential boundedness.**
- **Given a scenario in which the high-energy behavior of an elastic scattering amplitude is known, we can use our new bounds to check whether the starting assumptions are satisfied.**
- **In particular, the degree of (non-)localizability of the underlining UV theory can be constrained.**
- **We have also derived the new (lower) bounds on elastic scattering amplitude in Regge limit by assuming both polynomial and exponential boundedness.**