Two point functions in two-dimensional inhomogeneous field theories

O-Kab Kwon

(Sungkyunkwan University)

In collaboration with
Chanju Kim, Yoonbai Kim, Hanwool Song, Driba D. Tolla, Jeongwon Ho, Sang-A Park, Sang-Heon Yi

High1 Workshop on Particle, String and Cosmology
Jan.21(Sun) – Jan.27(Sat), 2024 High1 Resort

Supersymmetric Inhomogeneous Field Theories and Curved Background

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Contents

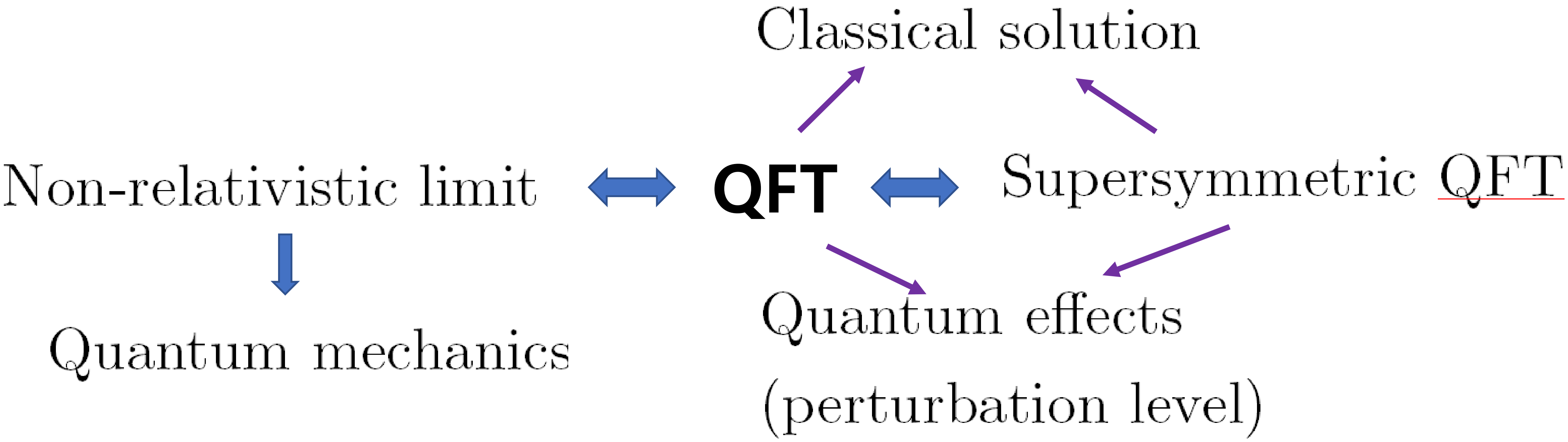
- **What is theoretical origins of the inhomogeneous couplings?**
- **Inhomogeneous field theory (IFT) = Field theory on curved space (FTCS)**
- **Quantization and Supersymmetrization of IFT**
- **A new supersymmetric background**
- **Discussion**

QFT vs Inhomogeneous field theory (IFT)

- (canonical, renormalizable, local) **QFT**

$$S = \int d^d x \, \mathcal{L}(\phi_a, \partial_\mu \phi_a; m, g_n)$$

Constant → Poincare symmetric



QFT vs Inhomogeneous field theory (IFT)

- (canonical, renormalizable, local) **QFT**

$$S = \int d^d x\, \mathcal{L}(\phi_a, \partial_\mu \phi_a; m, g_n)$$

Constant ➔ Poincare symmetric

- Classical and **quantum approaches**
- Preferred vacuum (Minkowski spacetime: global vacuum)
- Canonical quantization: $[\phi_a(t,\vec{x}), \pi_a(t,\vec{y})] = i\delta^{d-1}(\vec{x}-\vec{y})$ $\quad \pi_a(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a}$

$$\phi(t,\vec{x}) = \int \frac{d^{d-1}p}{(2\pi)^{p-1}} \frac{1}{\sqrt{2w_p}} \left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x}\right) \qquad a_p, \qquad a_p^\dagger$$

QFT vs Inhomogeneous field theory (IFT)

- (canonical, renormalizable, local) **QFT**

$$S = \int d^d x \, \mathcal{L}(\phi_a, \partial_\mu \phi_a; m, g_n)$$

Constant ➔ Poincare symmetric

- One particle state:

  select the prepared vacuum: Minkowski vacuum $|0\rangle$

$$a_p^\dagger |0\rangle = \# |p\rangle \xrightarrow{\text{Poincare transformation}} |\Lambda P\rangle = U(\Lambda)|p\rangle$$

Unitary transform

Construct the Hilbert space from vacuum

Poincare should be preserved at quantum level as well

QFT vs Inhomogeneous field theory (IFT)

- (canonical, renormalizable, local) **QFT**

$$S = \int d^d x \, \mathcal{L}(\phi_a, \partial_\mu \phi_a; m, g_n)$$

Constant ➔ Poincare symmetric

- One particle state:

select the preferred vacuum: Minkowski vacuum $|0\rangle$

Unitary transform

$$a_p^\dagger |0\rangle = \# |p\rangle \xrightarrow{\text{Poincare transformation}} |\Lambda P\rangle = U(\Lambda)|p\rangle$$

➔ In the canonical quantization of a field theory, preferred vacuum, Poincare symmetry are necessary!

➔ If not, no particle notion and cannot construct global Hilbert space

QFT vs Inhomogeneous field theory (IFT)

- (canonical, renormalizable, local) Inhomogeneous **QFT (IFT)**

$$S = \int d^d x \, \mathcal{L}(\phi_a, \partial_\mu \phi_a; m(x), g_n(x))$$

Arbitrary functions ➔ Poincare symmetry is broken

Classical solutions (condensed matter, cosmology, nuclear physics)

Non-relativistic limit ⇔ **IFT** ⇔ Supersymmetric IFT (SIFT)

↓

Quantum mechanics with space-dependent mass

Canonical quantization is not possible without the Poincare symmetry
➔ **algebraic QFT (our proposal)**

QFT vs Inhomogeneous field theory (IFT)

- Merits of SQFT:

1. 2^{nd}-order diff. eq. → 1^{st} –order diff. eq.
   → **analytic solutions (classical level)**

2. Exact results in quantum level (ex. Index calculations)

2. **BPS solutions (classical level)**: protected from quantum corrections (Hierarchy problem)

3. superparticles?? Applications to **condensed matter physics (classical level)**
(ex. models of top. Superconductor)

Origin of the inhomogeneous parameters

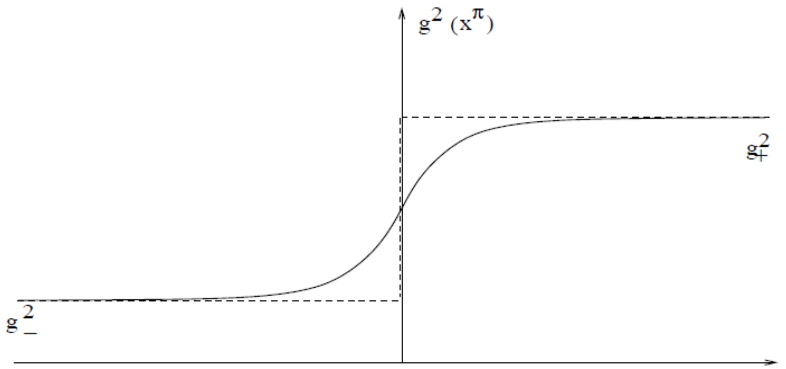
- In the context of string theory, mass and coupling parameters in low energy effective field theories can be understood as **non-dynamical traces (or background)** of high energy fields.

- Usually, these mass and coupling parameters are taken as constants in order **to maintain Poincare symmetry** of theories.

- Inhomogeneous non-dynamical traces break supersymmetry partially.

Janus field theories Vs Inhomogeneous field theories

- spatially varying coupling parameters

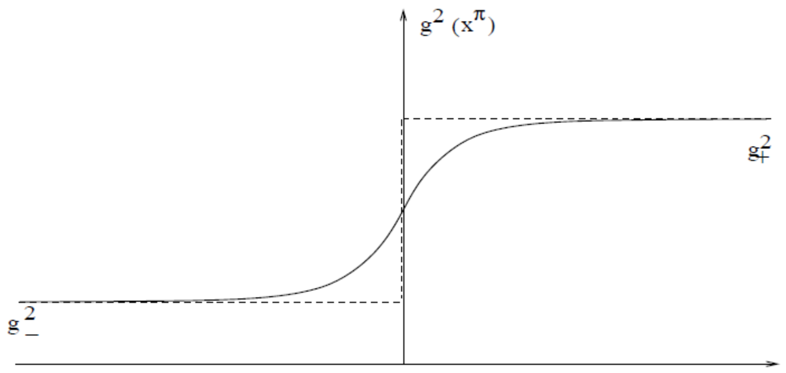


- space-dependent gauge coupling parameter g=g(x)
- Inhomogeneous mass m=m(x)

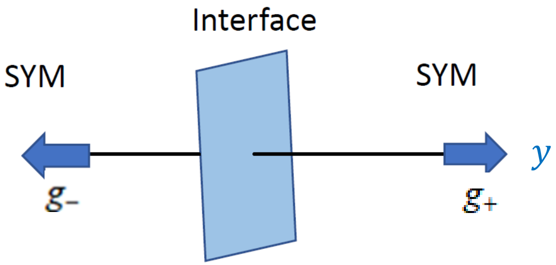
*Due to renormalizability of theory, possible number of parameter is limited.

Janus field theories Vs Inhomogeneous field theories

- **spatially varying coupling parameters**



Janus super Yang-Mills theory



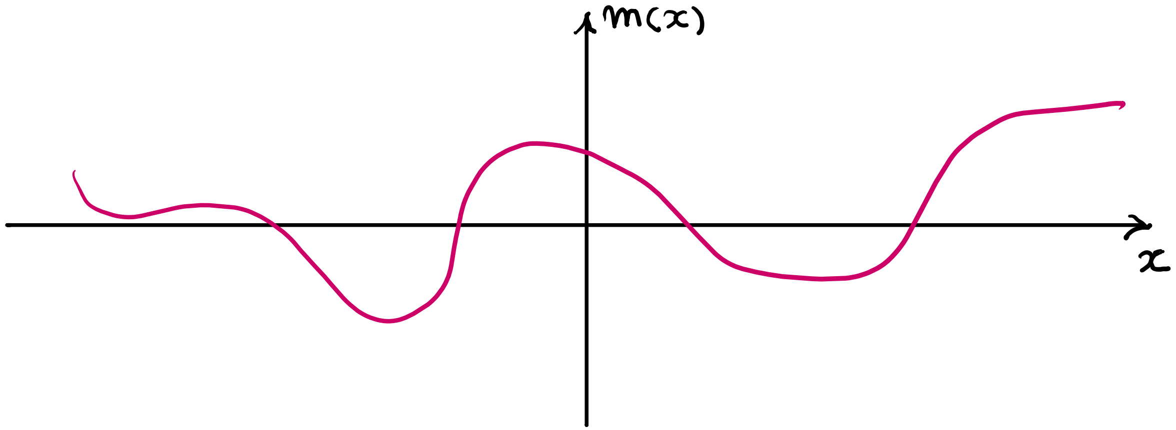
- usual coupling parameter g=g(x)

- Inhomogeneous mass m=m(x)

$$S_{\text{Janus}} = \int_{-\infty}^{0} \mathrm{dy} \int d^3x \mathcal{L}_{\text{SYM}}(g_-) + \int_{y=0} d^3x \mathcal{L}_{\text{interface}} + \int_{0}^{+\infty} \mathrm{dy} \int d^3x \mathcal{L}_{\text{SYM}}(g_+)$$

Janus field theories Vs Inhomogeneous field theories

- **spatially varying coupling parameters**



[Kyung Kiu Kim-**OK**]

Inhomogenous mass-deformed ABJM theory

$$\mathcal{L}_{\text{ImABJM}} = \mathcal{L}_{\text{ABJM}} - \hat{V}_{\text{ferm}} - \hat{V}_{\text{flux}} - \hat{V}_{\text{mass}} - \hat{V}_{J}$$

$$\hat{V}_J = m'\text{tr}\left(Y_a^\dagger Y^a - Y_i^\dagger Y^i\right)$$

- usual coupling constant g=g(x)
- **Inhomogeneous mass m=m(x)**

m=m(x)
arbitrary function

Janus field theories Vs Inhomogeneous field theories

- **Dual gravity origin (AdS/CFT correspondence)**

[Bak- Gutperle-Hirano, 2003]

usual coupling constant g=g(x)

turning on spatially varying background **dilaton field**

$$\frac{g(x)^2}{4\pi} = e^{\phi(x)}$$

[Kim-OK, 2018] [Arav et al, 2020]

Inhomogeneous mass m=m(x) [Kim-Kim-Kim-OK, 2019][Kim-OK-Tolla, 2020]

turning on spatially varying 4-form field strength in 11-dim. SUGRA (M-theory)

$$F_{AB\bar{C}\bar{D}} = T_{AB\bar{C}\bar{D}}(w_1)$$

RR 7-form field strength (IIB SUGRA)

$$T_{12\bar{1}\bar{2}} = -m, \quad T_{34\bar{3}\bar{4}} = m$$

Inhomogeneously mass-deformed ABJM(ImABJM)

- **Reduction of supersymmetry** $\mathcal{N} = 6 \longrightarrow \mathcal{N} = 3$

$a = 1, 2$ and $i = 3, 4,$

$$\gamma^1 \omega_{ab} = -\omega_{ab} \iff \omega^{ab}\gamma^1 = \omega^{ab},$$
$$\gamma^1 \omega_{ai} = \omega_{ai} \iff \omega^{ai}\gamma^1 = -\omega^{ai},$$

- **Deformation of the Lagrangian:**

$$-\frac{4\pi m}{k} M_B{}^D \left(Y_C^\dagger Y^C Y_D^\dagger Y^B - Y^C Y_C^\dagger Y^B Y_D^\dagger \right)$$ **: flux term**

$$\left(m^2 \delta_A^B + m' M_A{}^B \right) Y^A Y_B^\dagger$$ **: mass term**

m = m(x)

Gravity dual of the lmABJM

- **N=3 Inhomogeneously mass-deformed ABJM (Janus ABJM) model**

$m = m(x)$: **arbitrary mass function** [**OK**-K.Kim]
[K.Kim-Y.Kim-**OK**-C.Kim]

- **SUSY Q-lattice geometry in 11-dimensional gravity**

For a special mass function: [Gauntlett-Rosen]
$\boldsymbol{m(x) = m_0 \sin(kx)}$ [Arav-Gauntlett-Roberts-Rosen]

- Black brane solution dual to the N=3 lmABJM at finite temperature with the mass function [Ahn-Hyun-**OK**-Park]

- Extension to well-known supersymmetric field theories, such as N=2 Chern-Simon Higgs model in 3d and Abelian Higgs models in 3d and 4d, etc.

N=6 Mass deformed ABJM

Abelian projection

N=2 Chern-Simon Higgs

N=2* Mass deformed super Yang-Mills

Abelian projection

N=1 Abelian Higgs in 4d

*origin of space-dependent parameters

$$\bar{\mathcal{L}} = -D_\mu\bar{\phi}D^\mu\phi + i\bar{\psi}\gamma^\mu D_\mu\psi + \frac{k}{4\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho$$
$$+ i\mu\bar{\psi}\psi - 3i\lambda|\phi|^2\bar{\psi}\psi - |\phi|^2\left(\lambda|\phi|^2 - \mu\right)^2,$$

$$\mu = \mu(x)$$

$$\mathcal{L} = \bar{\mathcal{L}} - (\partial_x\mu(x))|\phi|^2$$

$$\mu = \mu(x,y)$$

$$\mathcal{L} = \bar{\mathcal{L}} + \frac{k}{\pi g}\mu(x,y)B$$

BPS solutions for non-constant μ, and other applications for various physical situations.

Inhomogeneous coupling constant deformations in 1+1 dimensions

- **2-dimensional *N*=1 supersymmetric real scalar field theory**

$$S = \int d^2x \Big[-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + i\bar{\psi}\gamma^\mu\partial_\mu\psi + iW''(\phi)\,\bar{\psi}\psi - \frac{1}{2}W'(\phi)^2 \Big] \qquad W' \equiv \frac{dW}{d\phi}$$

$$\delta\phi = i\psi\epsilon, \qquad \gamma^\mu = (i\sigma^2, \sigma^1) \text{ with } \mu = 0, 1$$

$$\delta\psi = -\frac{1}{2}\gamma^\mu\partial_\mu\phi\,\epsilon + \frac{1}{2}W'\epsilon$$

$$Q_\epsilon = \int dx\, J^0_\epsilon = i\epsilon_+ Q_+ + i\epsilon_- Q_- \quad \text{with} \quad Q_\pm = \int dx \Big((\partial_0\phi \pm \partial_1\phi)\psi_\pm \mp W'\psi_\mp \Big)$$

$$\bar{\epsilon}^\alpha = (\epsilon_+, \epsilon_-) \text{ with } \bar{\epsilon} \equiv \epsilon^\dagger = \epsilon^T$$

$$\{Q_\pm, Q_\pm^\dagger\} = 2(P^0 \mp P^1), \qquad \{Q_\pm, Q_\mp^\dagger\} = 2T$$

$$T = \int dx (\partial_1\phi) W'(\phi) = \int dx \frac{dW(\phi(x))}{dx} = W(\phi(\infty)) - W(\phi(-\infty))$$

Inhomogeneous coupling constant deformations in 1+1 dimensions

- **2-dimensional *N*=1 supersymmetric real scalar field theory**

$$E = P^0 = \frac{1}{4}\{Q_+ \pm Q_-, Q_+^\dagger \pm Q_-^\dagger\} \mp T$$

[Witten-Olive, 1978]

$$E \geq |T|$$

Inhomogeneous coupling constant deformations in 1+1 dimensions

- **2-dimensional *N*=1 supersymmetric real scalar field theory**

$$E = P^0 = \frac{1}{4}\{Q_+ \pm Q_-, Q_+^\dagger \pm Q_-^\dagger\} \mp T$$

[Witten-Olive, 1978]

$$E \geq |T|$$

- **Homogeneous QFT ➔ Inhomogeneous QFT (ImQFT)**

$$W(\phi) = \sum_i m_i \tilde{W}(\phi) \implies W(\phi, x) = \sum_i m_i(x) \tilde{W}(\phi)$$

Inhomogeneous coupling constant deformations in 1+1 dimensions

$$\delta\psi = -\frac{1}{2}\gamma^\mu\partial_\mu\phi\,\epsilon + \frac{1}{2}W'\epsilon \quad \longrightarrow \quad \delta\psi = -\frac{1}{2}\gamma^\mu\partial_\mu\phi\,\epsilon + \frac{1}{2}\frac{\partial W}{\partial\phi}\epsilon$$

$W' \equiv \frac{dW}{d\phi}$

Projection: $\gamma^1\epsilon = \pm\epsilon$ $\quad \epsilon_- = \pm\epsilon_+$

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + i\bar{\psi}\gamma^\mu\partial_\mu\psi + i\frac{\partial^2 W}{\partial\phi^2}\bar{\psi}\psi - \frac{1}{2}\left(\frac{\partial W}{\partial\phi}\right)^2 \boxed{\mp\frac{\partial W}{\partial x}}$$

$\bar{Q}_\epsilon = i\epsilon_+\bar{Q}$ $\quad \bar{Q} = \int dx\Big[(\partial_0\phi + \partial_1\phi - \partial_\phi W)\psi_+ - (\partial_0\phi - \partial_1\phi + \partial_\phi W)\psi_-\Big]$

new term

[Kim-Kim-OK]

Position dependent potential $\quad V(\phi, x) \equiv \frac{1}{2}\left(\frac{\partial W}{\partial\phi}\right)^2 + \frac{\partial W}{\partial x}$ $\quad E = \frac{1}{4}\{\bar{Q}, \bar{Q}^\dagger\} + T$

need not be nonnegative definite.

Equivalence between IFT and FTCS

- Instead, we explore another way for a quantization of the above IFT.
- For this purpose, we consider a (1 + 1)-dimensional scalar FTCS:

$$S_{\text{FTCS}} = \int d^2x\sqrt{-g}\mathcal{L}_{\text{FTCS}} = \int d^2x\sqrt{-g}\Big[-\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{2}m_0^2\phi^2 - \sum_{\ell=1} f_\ell(\mathcal{R})\phi^\ell\Big]$$

m_0 is a constant, $\mathcal{R}$ denotes the scalar curvature of the background metric

S_{FTCS} satisfies the general covariance : Under the coordinate transformation $x^\mu \to X^\mu(x)$

$$\bar{g}_{\mu\nu}(X) \equiv \frac{\partial x^\rho}{\partial X^\mu}\frac{\partial x^\sigma}{\partial X^\nu}g_{\rho\sigma}(x)$$

- Conformal form of the metric in (1+1) dimensions:

$$ds^2 = e^{2\omega(x)}(-dt^2 + dx^2)$$

➔ Spatial inhomogeneity in IFT

Equivalence between IFT and FTCS

- Conversion from IFT to FTCS

$$\sqrt{-g} = e^{2w}, \qquad g^{tt} = -g^{xx} = -e^{2w}, \qquad \mathcal{R} = -2w''e^{-2w}$$

$$\sqrt{-g}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi = e^{2\omega}(e^{-2\omega})\left(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right) = \eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

Regardless w(x), the kinetic terms in both sides are always identical in (1+1) dimension

- IFT is converted to FTCS with the parameter matching:

$$m^2(x) = \sqrt{-g}\left(m_0^2 + 2f_2(\mathcal{R})\right)$$

$$g_n(x) = \sqrt{-g}f_n(\mathcal{R})\,,$$

$$J(x) = -\sqrt{-g}f_1(\mathcal{R})\,.$$

Supersymmetric Field Theory on Curved spacetime (SFTCS)

- IFT and FTCS

$$S_{\rm IFT} = \int d^2x \mathcal{L}_{\rm IFT} = \int d^2x \Big[-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2(x)\phi^2 - \sum_{n=3} g_n(x)\phi^n + J(x)\phi \Big],$$

$$S_{\rm FTCS} = \int d^2x\sqrt{-g}\,\mathcal{L}_{\rm FTCS} = \int d^2x\sqrt{-g}\,\Big[-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}m_0^2\phi^2 - \sum_{\ell=1} h_\ell(\mathcal{R})\phi^\ell \Big],$$

- Wess-Zumino model (real scalar + quadratic Majorana fermion) $\mathcal{N}$ =(1,1) with two real supercharges

$$S_{\rm SFT} = \int d^2x \mathcal{L}_{\rm SFT} = \int d^2x \Big[-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{i}{2}\bar{\psi}\gamma^\mu_{\rm F}\partial_\mu\psi + \frac{i}{2}\left(\frac{\partial^2 W}{\partial\phi^2}\right)\bar{\psi}\psi - \frac{1}{2}\left(\frac{\partial W}{\partial\phi}\right)^2 \Big] \qquad W(\phi) = \sum_{n\geq 2}\lambda_n\phi^n$$

- SIFT version of Wess-Zumion model (position-dependence only)

$$\mathcal{L}_{\rm SIFT} = \mathcal{L}_{\rm SIFT}|_{\lambda_n\to\lambda_n(x)} - \frac{\partial W(\phi,x)}{\partial x} \qquad \frac{\partial W(\phi,x)}{\partial x} \equiv \sum_n \frac{\partial \lambda_n(x)}{\partial x}\phi^n$$ [Kim-Kim-Kwon]

Supersymmetric Field Theory on Curved spacetime (SFTCS)

- Supersymmetric field theory on curved spacetime (SFTCS) : One may ask whether there is a relation between SFTCS and SIFT just like the bosonic case. However, it is well-known that a rigid background allowing supersymmetric field theory is not abundant.
- A tentative Wess-Zumino model on curved background (which is obtained simply by replacing the flat metric $\eta_{\mu\nu} \longrightarrow g_{\mu\nu}$

$$S^g_{\rm SFT} = \int d^2x \sqrt{-g}\, \mathcal{L}_0$$

$$\mathcal{L}_0 = -\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + \frac{i}{2} \bar{\Psi} \gamma^\mu \nabla_\mu \Psi + \frac{i}{2} \left(\frac{\partial^2 W}{\partial \phi^2} \right) \bar{\Psi} \Psi - \frac{1}{2} \left(\frac{\partial W}{\partial \phi} \right)^2$$

- This is not supersymmetric in general. As a method of supersymmetrization of supersymmetrization of the action preserving covariance, we extend the superpotential as $W(\phi) \longrightarrow \mathcal{W}(\phi, \mathcal{R}) = \sum_{n \geq 1} \mathcal{F}_n(\mathcal{R}) \phi^n$

Supersymmetric Field Theory on Curved spacetime (SFTCS)

- Under the supersymmetric variation:

$$\delta\phi = i\bar{\Psi}\epsilon\,,$$

$$\delta\Psi = -\gamma^{\mu}\nabla_{\mu}\phi\;\epsilon + \left(\frac{\partial\mathcal{W}}{\partial\phi}\right)\epsilon$$

the variation of the Lagrangian results in

$$\delta(\sqrt{-g}\mathcal{L}_0) = -i\sqrt{-g}\;\nabla_{\mu}\phi\,\bar{\Psi}(g^{\mu\nu} - \gamma^{\mu\nu})\nabla_{\nu}\epsilon + i\sqrt{-g}\sum_{n} n\phi^{n-1}\bar{\Psi}\gamma^{\mu}\nabla_{\mu}\big(\mathcal{F}_n\epsilon\big)$$

Conditions to be supersymmetric: $\nabla_{\mu}\epsilon = \frac{1}{2}f\gamma_{\mu}\epsilon$ $f = f(\mathcal{R})$

$$\nabla_{\mu}\mathcal{F}_n(\mathcal{R})\gamma^{\mu}\epsilon = \mathcal{G}_n(\mathcal{R})\epsilon$$

Generalized Killing spinor equation

$$\sqrt{-g}\,\mathcal{L}_{\mathrm{SFTCS}} = \sqrt{-g}\big(\mathcal{L}_0 - f(\mathcal{R})\mathcal{W}(\phi,\mathcal{R}) - \mathcal{U}(\phi,\mathcal{R})\big)$$

$$\mathcal{U}(\phi,\mathcal{R}) \equiv \sum_{n\geq 1}\mathcal{G}_n(\mathcal{R})\phi^n$$

- Our results by solving the **generalized Killing spinor** equation:

1. Flat background (Minkowski, Rindler, etc….) ➔ two susy
2. AdS2 background ➔ two susy dS2 ➔ no susy
3. $m = m(t)$, $g = g(t)$ ➔ no susy
4. $m = m(x)$, $g = g(x)$ ➔ one susy

A supersymmetric background

- To be specific, let us take:

$$ds^2 = e^{2\omega(t,x)}\left(-dt^2 + dx^2\right)$$

$$f(\mathcal{R}) = \frac{\xi}{m_0}\mathcal{R}$$

$$f(\mathcal{R}) = \pm\omega' e^{-\omega}$$:supersymmetric condition

$$\omega'' + \frac{m_0}{2\xi}e^{\omega}\omega' = 0$$

$$e^{\omega(x)} = \frac{1}{a + e^{-bx}}$$ $ab = \frac{m_0}{2\xi}$ $b > 0$

$$\mathcal{R} = 2ab^2 e^{-bx}$$

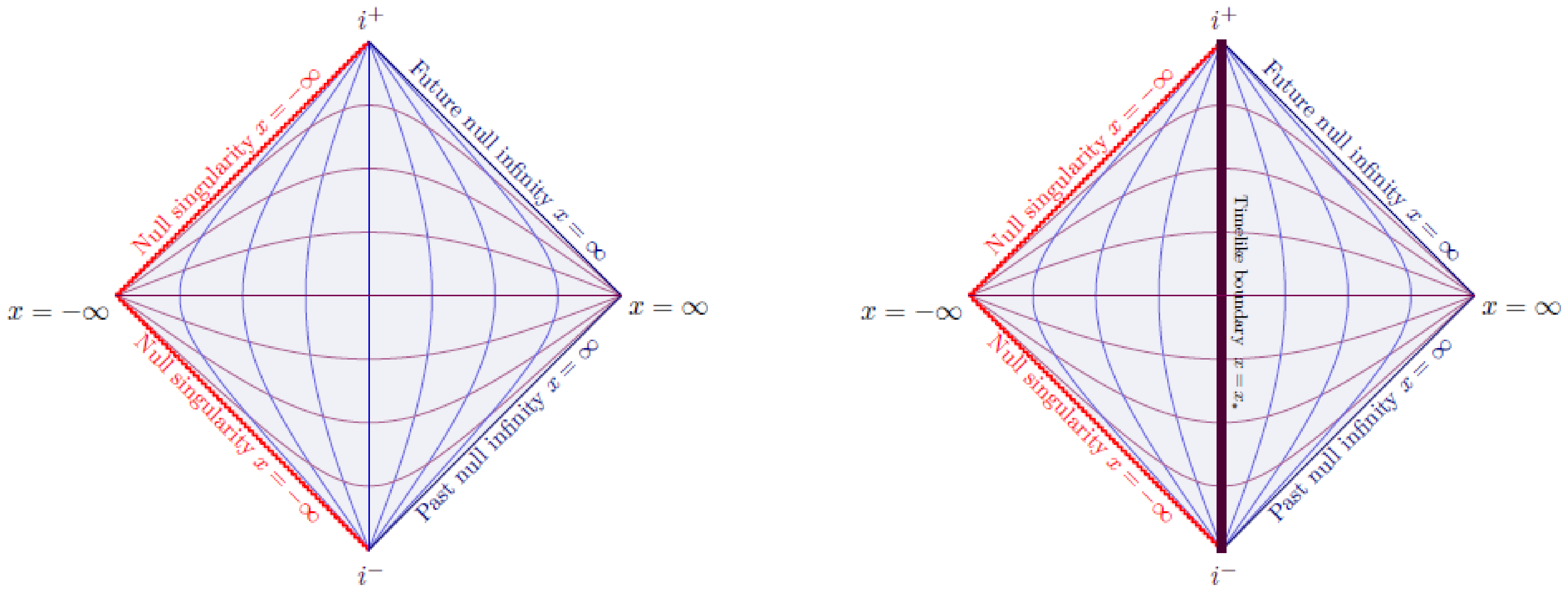
- Interestingly, the rigid background described by the above metric allows various field theories, such as Sine-Gordon, Liouville, ϕ^6 theory, etc.

A supersymmetric background

- The supersymmetric background metric $e^{\omega(x)} = \dfrac{1}{a + e^{-bx}}$

  has curvature singularity at $x \to -\infty$

$$\mathcal{R} = 2ab^2 e^{-bx}$$



(a) (b)

Free scalar FTCS and free scalar IFT

$$S_{\mathrm{FTCS}} = \int d^2x\sqrt{-g}\Big[-\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{2}m_0^2\phi^2 - \frac{\xi}{2}\mathcal{R}\phi^2\Big]$$

$$ds^2 = e^{2\omega(t,x)}\left(-dt^2 + dx^2\right)$$

Klein-Gordon eq: $(\Box + m_0^2 + \xi\mathcal{R})\phi = 0$

$$e^{\omega(x)} = \frac{1}{a + e^{-bx}}$$

$$\partial_t^2\phi = -A\phi\,, \qquad A = -\partial_x^2 + e^{2\omega}(m_0^2 + \xi\mathcal{R})$$

$$\mathcal{L}_{\mathrm{SIFT}} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{i}{2}\bar{\psi}\gamma_{\mathrm{F}}^\mu\partial_\mu\psi + \frac{i}{2}m(x)\bar{\psi}\psi - \frac{1}{2}\big(m^2(x) + m'(x)\big)\phi^2\,.$$

$$\Big[-\frac{d^2}{dx^2} + V_{\mathrm{eff}}(x)\Big]\phi_\omega(x) = \omega^2\phi_\omega(x)\,, \qquad V_{\mathrm{eff}}(x) \equiv m_{\mathrm{eff}}^2(x) = \frac{(m_0^2e^{bx} + 2\xi ab^2)e^{bx}}{(ae^{bx}+1)^2}$$

$$m_{\mathrm{eff}}^2 = e^{2\omega}(m_0^2 + \xi\mathcal{R}) = m^2 + m'$$

Free scalar FTCS and free scalar IFT

The operator *A* can be identified with Haimiltonian in Supersymmetric quantum mechanics

$$\left[-\frac{d^2}{dx^2}+V_{\text{eff}}(x)\right]\phi_\omega(x)=\omega^2\phi_\omega(x)\,,\qquad V_{\text{eff}}(x)\equiv m_{\text{eff}}^2(x)=\frac{(m_0^2e^{bx}+2\xi ab^2)e^{bx}}{(ae^{bx}+1)^2}$$

$$m_{\text{eff}}^2=e^{2\omega}(m_0^2+\xi\mathcal{R})=m^2+m'$$

$$V_{\text{eff}}=m^2+m'=V_{\text{QM}}^2-\frac{dW_{\text{QM}}}{dx}$$

$$W_{\text{QM}}=-m(x)$$

a>0 : Rosen-Morse potential
a<0 : Eckart potential ➔ Exactly solvable!

$$A=D_-D_+,\qquad D_\pm\equiv\pm\frac{d}{dx}-m(x)\qquad(\phi,A\phi)=\int_{-\infty}^{\infty}|D_+\phi|^2dx\,,$$

Solution:

$$y = ae^{bx} = e^{b(x-x_0)} \qquad \phi_\omega(y) \equiv y^\alpha (1+y)^\gamma f_\omega(y)$$

$$\left[y(1+y)\frac{d^2}{dy^2} + \Big(2\alpha + 1 + (2\gamma + 2\alpha + 1)y\Big)\frac{d}{dy} + \gamma(2\alpha+1) - 2\xi\right] f_\omega(y) = 0\,.$$

$$\phi_\omega(y) = (1+y)^\gamma \left[a_1 y^\alpha F(A,B\,;\,C\,|\,-y) + a_2\, y^{\alpha+1-C} F(A-C+1, B-C+1\,;\,2-C\,|\,-y)\right]$$

$$F(A,B\,;\,C\,|\,z) = (1-z)^{C-A-B} F(C-A, C-B\,;\,C\,|\,z)$$

$$A = \frac{i}{b}(\omega - k) + \beta\,, \qquad B = \frac{i}{b}(\omega + k) + \beta\,, \qquad C = 1 + 2\frac{i}{b}\omega\,,$$

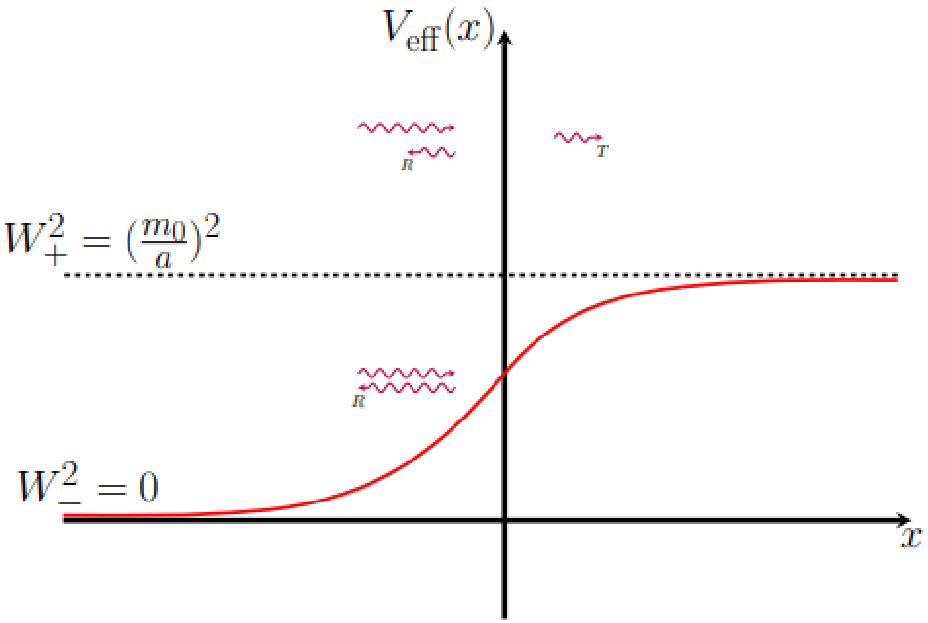
where k is defined by

$$k^2 \equiv \omega^2 - (2b\xi)^2\,.$$

Note that this choice implies $A - B = -2i\frac{k}{b}$.

Proposal on the quantization of IFT

- Explicit example: $e^{\omega(x)} = \frac{1}{a + e^{-bx}}$ $\quad a > 0, \quad \xi \geq \frac{1}{4}$



Proposal on the quantization of IFT

- L(eft)-quantization: $$\phi_{\rm L}(\boldsymbol{x}) = \int_0^\infty \frac{d\omega}{\sqrt{2\pi}}\frac{1}{\sqrt{2\omega}}\sum_{i=\pm}\left[a_\omega^{(i)}u_\omega^{(i)}(\boldsymbol{x}) + (a_\omega^{(i)})^\dagger (u_\omega^{(i)}(\boldsymbol{x}))^*\right]$$

$$\phi_\omega(x) \underset{x\to-\infty}{\longrightarrow} a_1\, e^{i\omega(x-x_0)} + a_2\, e^{-i\omega(x-x_0)}$$

$$u_\omega^{(-)}(\boldsymbol{x}) = (1+e^{bx})^{2\xi}F(A,B\,;\,C\,|-e^{bx})e^{-i\omega(t-x)}\,,$$
$$u_\omega^{(+)}(\boldsymbol{x}) = (1+e^{bx})^{2\xi}F(A-C+1,B-C+1\,;\,2-C\,|-e^{bx})e^{-i\omega(t+x)}$$

$$(u_\omega^{(-)}(\boldsymbol{x}))^* = (1+e^{bx})^{2\xi}F(A-C+1,B-C+1\,;\,2-C\,|-e^{bx})e^{i\omega(t-x)}$$
$$(u_\omega^{(+)}(\boldsymbol{x}))^* = (1+e^{bx})^{2\xi}F(A,B\,;\,C\,|-e^{bx})e^{i\omega(t+x)}\,.$$

$$[a_\omega^{(i)}, (a_{\omega'}^{(j)})^\dagger] = \delta^{ij}\delta(\omega-\omega') \qquad u_\omega^{(\mp)}(\boldsymbol{x}) \underset{x\to-\infty}{\longrightarrow} e^{-i\omega(t\mp x)} \qquad a_\omega^{(\mp)}|0\rangle_{\rm L} = 0$$

Fock space $\mathscr{F}_{\rm L}$ is constructed by these operators.

$$\phi_{\rm L}(\boldsymbol{x}) \underset{x\to-\infty}{\simeq} \int_0^\infty \frac{d\omega}{\sqrt{2\pi}}\frac{1}{\sqrt{2\omega}}\left[a_\omega^{(+)}e^{-i\omega(t+x)} + a_\omega^{(-)}e^{-i\omega(t-x)} + (a_\omega^{(+)})^\dagger e^{i\omega(t+x)} + (a_\omega^{(-)})^\dagger e^{i\omega(t-x)}\right]$$

Proposal on the quantization of IFT

- R(right)-quantization:

$$\phi_{\mathrm{R}}(\boldsymbol{x}) = \int_0^\infty \frac{dk}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} \sum_{i=\pm} \left[b_k^{(i)} v_k^{(i)}(\boldsymbol{x}) + (b_k^{(i)})^\dagger (v_k^{(i)}(\boldsymbol{x}))^* \right]$$

$$\phi_\omega(x) \underset{x\to\infty}{\longrightarrow} b_1 e^{ik(x-x_0)} + b_2 e^{-ik(x-x_0)}$$

$$v_k^{(-)}(\boldsymbol{x}) = (1+e^{-bx})^{2\xi} F\left(A, A-C+1\,;\, A-B+1 \,\middle|\, -e^{-bx}\right) e^{-i(\omega t - kx)}$$

$$v_k^{(+)}(\boldsymbol{x}) = (1+e^{-bx})^{2\xi} F\left(B, B-C+1\,;\, B-A+1 \,\middle|\, -e^{-bx}\right) e^{-i(\omega t + kx)}$$

$$\omega = \sqrt{k^2 + b^2\beta^2}.$$

$$[b_k^{(i)}, (b_{k'}^{(j)})^\dagger] = \delta^{ij}\delta(k-k') \qquad b_k^{(\mp)}|0\rangle_{\mathrm{R}} = 0$$

Fock space $\mathscr{F}_{\mathrm{R}}$ is constructed by these operators.

$$\phi_{\mathrm{R}}(\boldsymbol{x}) \underset{x\to\infty}{\simeq} \int_0^\infty \frac{dk}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} \left[b_k^{(+)} e^{-i(\omega t + kx)} + b_k^{(-)} e^{-i(\omega t - kx)} + (b_k^{(+)})^\dagger e^{i(\omega t + kx)} + (b_k^{(-)})^\dagger e^{i(\omega t - kx)} \right]$$

Some comments:

- Two vacua $|0\rangle_{\mathrm{L}}$ and $|0\rangle_{\mathrm{R}}$ are inequivalent due to the lack of an invertible transformation connecting $a_{\omega}^{(i)}$ and $b_{k}^{(i)}$ ➔ no unitary transformation between two
- Both quantization schemes are distinct, and neither vacuum is preferred.
- It is natural to interpret the Fock spaces $\mathscr{F}_{\mathrm{L}}$ and $\mathscr{F}_{\mathrm{R}}$ as local Hilbert space, rather than global ones.
- Even though (local) $\mathscr{F}_{\mathrm{L}}$ and $\mathscr{F}_{\mathrm{R}}$ are not unitarily equivalent, they share the same algebraic relation among the field operators.
- In this algebraic viewpoint, one may consider some extended (algebraic) states from the local fock spaces.

Discussions

- Inhomogeneous couplings in IFT can be understood as relics of some fields of enlarged theory, dilaton ➔ g(x) in SYM, form field ➔ m(x) in mABJM
- IFT = FTCS in (1+1) dimensions
- A new supersymmetric background
- SQM interpretation of the results (Rosen-Morse /Eckart potential)
- Quantization of IFT: algebraic quantization approach is natural

Future directions

- Calculation of two-point function in the supersymmetric background?
- How to understand the finite temperature in IFT
- Integrability for the field theory on the supersymmetric vacuum
- Algebraic understanding for the quantization in our background

Thank you for attention!!