Stringy Scaling of String Amplitudes in High Energy Limits

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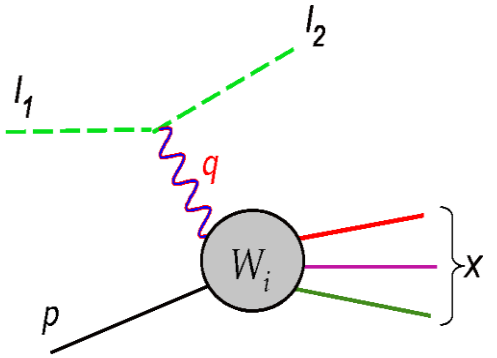
Outline

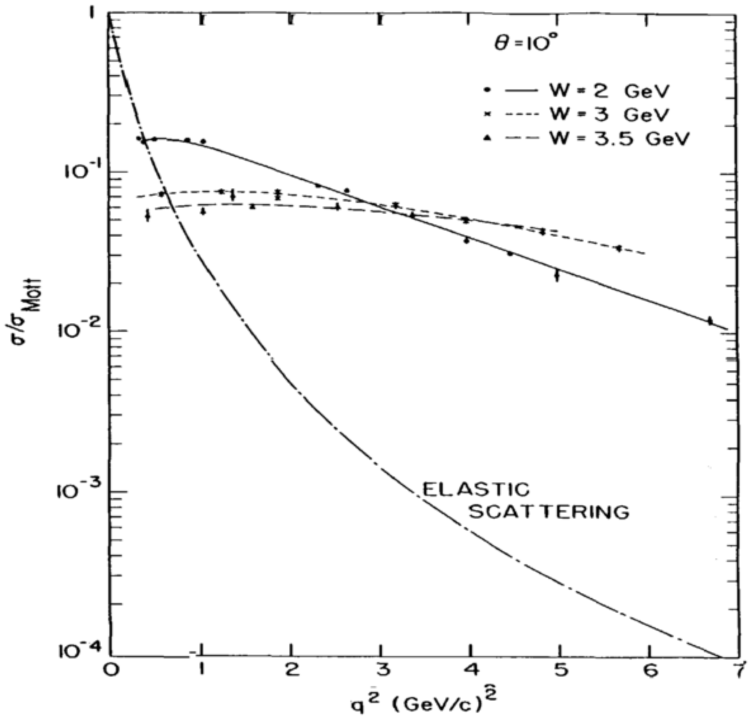
- Scaling behavior in field theory
- 4-point string amplitude: linear relation
- 5-point string amplitude: stringy scaling
- n-point string amplitude
- Summary

S.H Lai, J.C Lee and YY., JHEP 09 (2023)
S.H Lai, J.C Lee and YY., arXiv:2207.09236

Bjorken Scaling

- (1968 Bjorken) scaling behavior of DIS structure functions: $W(Q \to \infty, \nu) \sim F\left(\frac{Q^2}{2M\nu}\right)$
- A property of hadrons is determined not by the absolute energy but by dimensionless kinematic quantities, e.g. scattering angle
- Hadrons behave as collections of point-like constituents
- Bjorken Scaling and parton model leads to asymptotic freedom in QCD
- Stringy scaling?





4-Point String Amplitude

- Gross' conjecture: linear relation in hard limit
- 3 tachyons + a massive string state

$A =$

Kinematics in CM Frame: 4-Point

Scattering Plane: (e^L, e^P, e^T)

$$k_1 = \left(+\sqrt{p^2 + M_1^2}, -p, 0\right)$$

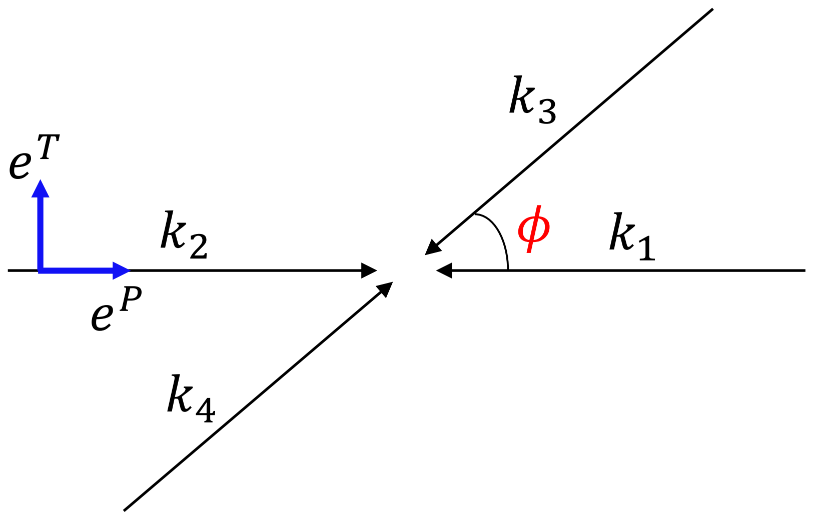
$$k_2 = \left(+\sqrt{p^2 + M_2^2}, +p, 0\right) \sim \frac{e^P}{M_2}$$

$$k_3 = \left(-\sqrt{q_3^2 + M_3^2}, -q_3 \cos\phi_3\,, -q_3 \sin\phi_3\right)$$

$$k_4 = \left(-\sqrt{q_4^2 + M_4^2}, +q_4 \cos\phi_4\,, +q_4 \sin\phi_4\right)$$

DOF:

$(p, q_3, q_4, \phi_3, \phi_4) \to (E, \phi)$



$$\sum k_i = 0, \quad \vec{k}_3 + \vec{k}_4 = 0$$

Hard Limit

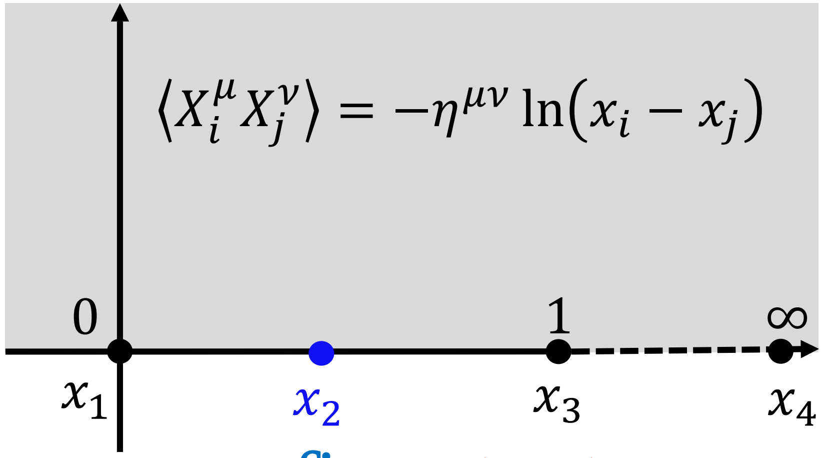
- High energy, fixed angle limit: $E \to \infty, \phi = \text{constant}, e^L \simeq e^P$
- Relevant states: $(\alpha_{-1}^T)^{N+p}(\alpha_{-1}^L)^{2m}(\alpha_{-2}^L)^q|0,k\rangle$

$$p + 2m + 2q = 0$$

$$A^{(p,2m,q)} = \int \prod_{i=1}^{4} dx_i \langle e^{ik_1X_1}(\partial X_2^T)^{N+p}(\partial X_2^L)^{2m}(\partial^2 X_2^L)^q e^{ik_2X_2}e^{ik_3X_3}e^{ik_4X_4}\rangle$$

$$= \int_0^1 dx_2\, u(x_2) e^{-\Lambda f(x_2)}$$

$$\Lambda = -k_1 \cdot k_2 \sim E^2 \to \infty$$



fix $SL(2,R)$

4-Point String Amplitude

$$A^{(p,2m,q)} = \int_0^1 dx_2\, u(x_2) e^{-\Lambda f(x_2)}, \qquad \Lambda = -k_1 \cdot k_2 \sim E^2 \to \infty$$

$$f(x_2) = \ln x_2 - \tau \ln(1 - x_2), \qquad \tau = -\frac{k_3 \cdot k_2}{k_1 \cdot k_2} \sim -\sin^2 \frac{\phi}{2}$$

$$u(x_2) = (K^T)^{N+p} (K^L)^{2m} \left(K'^L\right)^q, \qquad K(x_2) \equiv \frac{k_1}{x_2} - \frac{k_3}{1 - x_2}$$

$$K^L = K \cdot e^L = \frac{1}{M_2}\left(\frac{k_1 \cdot k_2}{x_2} - \frac{k_3 \cdot k_2}{1 - x_2}\right) = -\frac{\Lambda}{M_2} \partial_2 f(x_2)$$

Saddle Point Approximation

$$A^{(p,2m,q)} = \int_0^1 dx_2\, u(x_2) e^{-\Lambda f(x_2)}, \qquad \Lambda = -k_1 \cdot k_2 \sim E^2 \to \infty$$

$$f(x_2) = \ln x_2 - \tau \ln(1 - x_2), \qquad \partial_2 f(x_2) = 0 \Rightarrow \tilde{x}_2 = \frac{1}{1-\tau}$$

$$\tilde{u} = u(\tilde{x}_2) = \left(\widetilde{K}^T\right)^{N+p} \left(\widetilde{K}^L\right)^{2m} \left(\widetilde{K}'^L\right)^q, \qquad \widetilde{K}^L = -\frac{\Lambda}{M_2} \partial_2 f(\tilde{x}_2) = 0$$

$$\tilde{u} = \partial_2 \tilde{u} = \cdots = \partial_2^{2m-1} \tilde{u} = 0, \qquad \partial_2^{2m} \tilde{u} = (2m)! \left(\widetilde{K}^T\right)^{N+p} \left(\widetilde{K}'^L\right)^{2m+q}$$

4-Point String Amplitude

$$\tilde{u} = \partial_2 \tilde{u} = \cdots = \partial_2^{2m-1} \tilde{u} = 0, \qquad \partial_2^{2m} \tilde{u} = (2m)! \left(\widetilde{K}^T\right)^{N+p} \left(\widetilde{K}'^L\right)^{2m+q}$$

$$\begin{aligned} A^{(p,2m,q)} &= \int_0^1 dx_2\, u(x_2) e^{-\Lambda f(x_2)} \\ &= \int_0^1 dx_2 \frac{\partial_2^{2m} \tilde{u}}{(2m)!} (x_2 - \tilde{x}_2)^{2m} e^{-\Lambda\left[\tilde{f} + \frac{1}{2}\tilde{f}''(x_2 - \tilde{x}_2)^2 + \cdots\right]} \\ &\simeq e^{-\Lambda \tilde{f}} \sqrt{\frac{-2\pi}{M_2 \widetilde{K}'^L}} \frac{\left(\widetilde{K}^T\right)^{N+p} \left(\widetilde{K}'^L\right)^{m+q}}{2^m m!\, (-M_2)^m} \end{aligned}$$

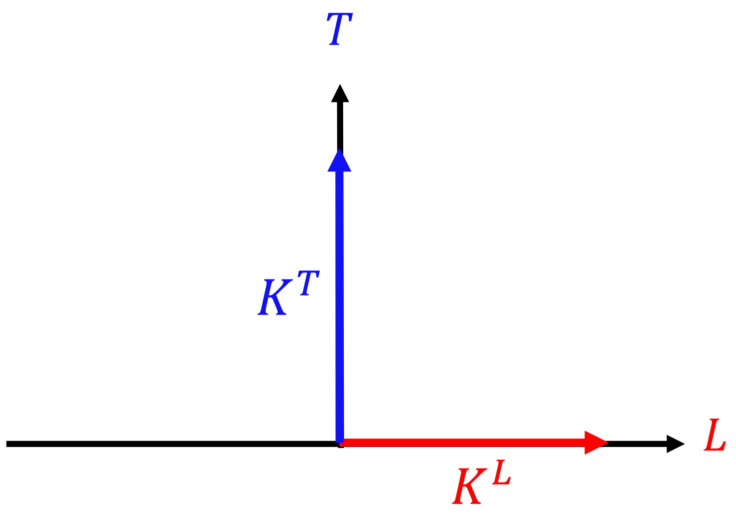
The Effective Momentum K

$$A^{(p,2m,q)} \simeq e^{-\Lambda\tilde{f}} \sqrt{\frac{-2\pi}{M_2 \widetilde{K}'^L}} \frac{\left(\widetilde{K}^T\right)^{N+p} \left(\widetilde{K}'^L\right)^{m+q}}{2^m m!\,(-M_2)^m}, \qquad \tilde{x}_2 = \frac{1}{1-\tau}$$

$$K(x_2) \equiv \frac{k_1}{x_2} - \frac{k_3}{1-x_2} = \left(K^L, K^T\right)$$

$$\widetilde{K}^L = -\frac{\Lambda}{M_2}\partial_2 f(\tilde{x}_2) = 0 \Rightarrow \widetilde{K}^T = \left|\widetilde{K}\right|$$

$$\widetilde{K}^2 + 2M_2\widetilde{K}'^L = 0 \Rightarrow \widetilde{K}'^L = -\frac{\widetilde{K}^2}{2M_2}$$



Linear Relation

$$A^{(p,2m,q)} = e^{-\Lambda\tilde{f}}\sqrt{\frac{-2\pi}{M_2\widetilde{K}'^L}}\frac{\left(\widetilde{K}^T\right)^{N+p}\left(\widetilde{K}'^L\right)^{m+q}}{2^m m!\,(-M_2)^m},\quad \widetilde{K}^T = \left|\widetilde{K}\right|, \widetilde{K}'^L = -\frac{\widetilde{K}^2}{2M_2}$$

$$= \sqrt{2\pi}e^{-\Lambda\tilde{f}}\left|\widetilde{K}\right|^{N-1}\frac{(2m)!}{m!}\left(\frac{-1}{2M_2}\right)^{2m+q}$$

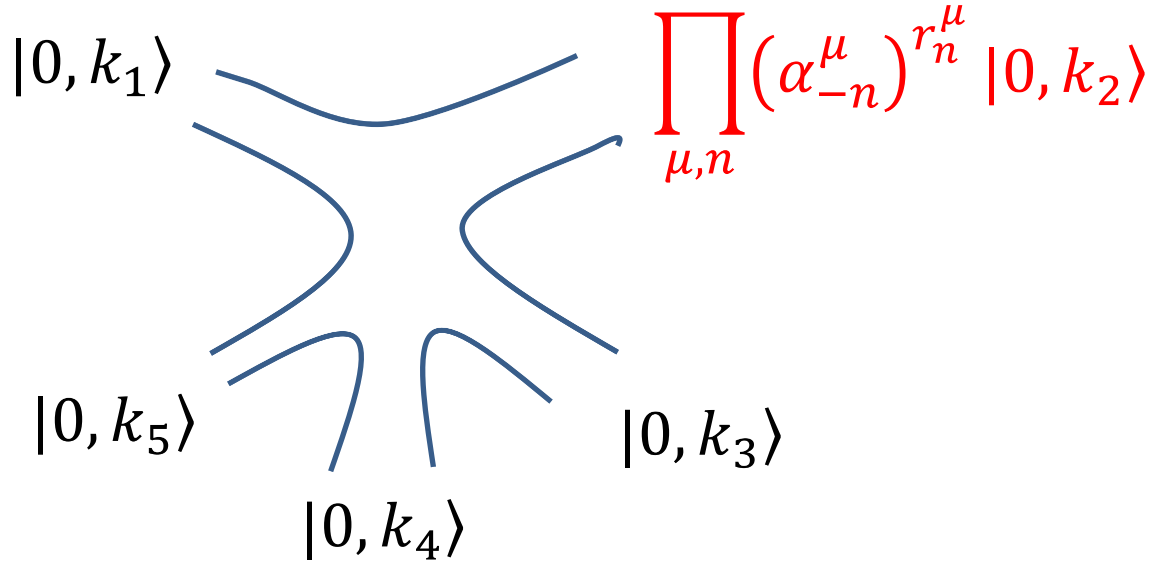
$$\frac{A^{(p,2m,q)}}{A^{(0,0,0)}} = \frac{(2m)!}{m!}\left(\frac{-1}{2M_2}\right)^{2m+q},$$

Independent of ϕ!

5-Point String Amplitude

- 4 tachyons + a massive string state

$A =$



Kinematics in CM Frame: 5-Point

- Polarization: $(e^L, e^P, e^{T_1}, e^{T_2})$

DOF:
$(p, q_3, q_4, q_5, \phi_3^1, \phi_4^1, \phi_4^2, \phi_5^1, \phi_5^2) \rightarrow 5$

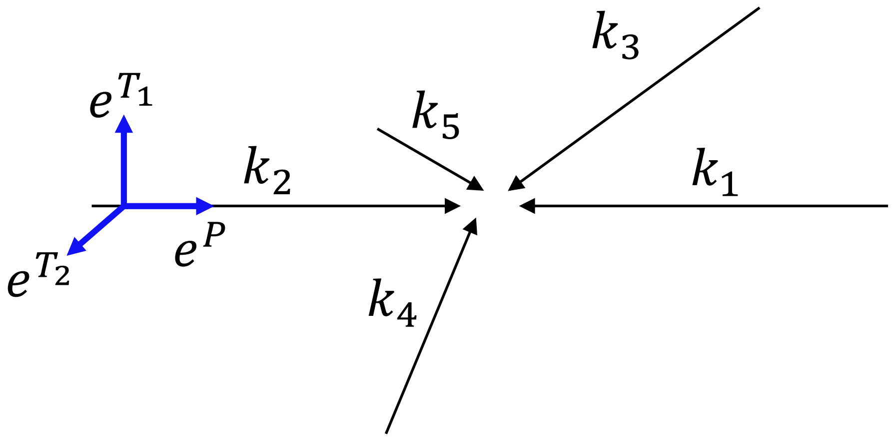
$$k_1 = \left(\sqrt{p^2 + M_1^2}, -p, 0,0\right)$$

$$k_2 = \left(\sqrt{p^2 + M_2^2}, +p, 0,0\right) \sim \frac{e^P}{M_2}$$

$$k_3 = \left(-\sqrt{q_3^2 + M_3^2}, -q_3 \cos\phi_3^1, -q_3 \sin\phi_3^1, 0\right)$$

$$k_4 = \left(-\sqrt{q_4^2 + M_4^2}, -q_4 \cos\phi_4^1, -q_4 \cos\phi_4^2 \sin\phi_4^1, -q_4 \sin\phi_4^2 \sin\phi_4^1\right)$$

$$k_5 = \left(-\sqrt{q_5^2 + M_4^2}, -q_5 \cos\phi_5^1, -q_5 \cos\phi_5^2 \sin\phi_5^1, -q_5 \sin\phi_5^2 \sin\phi_5^1\right)$$



$$\sum k_i = 0$$

$$\vec{k}_3 + \vec{k}_4 + \vec{k}_5 = 0$$

Hard Limit

- High energy, fixed angle limit: $p, q_i \to \infty, \quad \phi_i^j$ fixed, $e^L \simeq e^P$
- Relevant states: $|p_1, p_2, 2m, q\rangle \sim \left(\alpha_{-1}^{T_1}\right)^{N+p_1}\left(\alpha_{-1}^{T_2}\right)^{p_2}(\alpha_{-1}^{L})^{2m}(\alpha_{-2}^{L})^{q}|0, k\rangle$

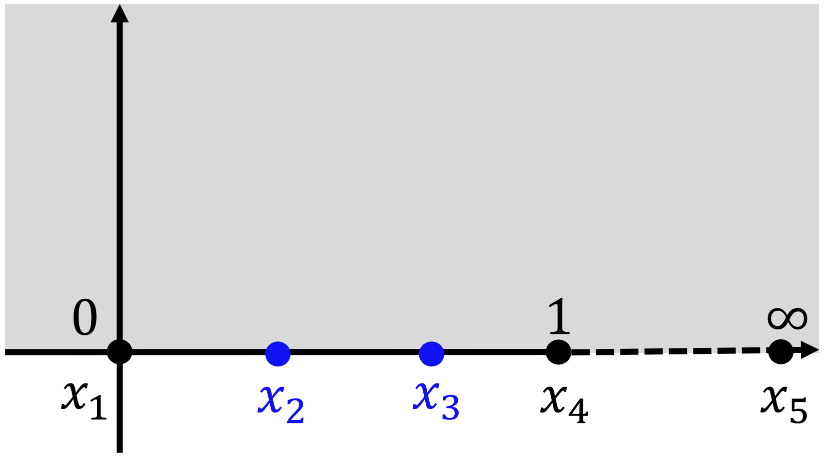
$p_1 + p_2 + 2m + 2q = 0$

$$A^{(p_1,p_2,2m,q)}$$

$$= \int \prod_{i=1}^{5} dx_i \left\langle e^{ik_1X_1}\left(\partial X_2^{T_1}\right)^{N+p_1}\left(\partial X_2^{T_2}\right)^{p_2}(\partial X_2^{L})^{2m}(\partial^2 X_2^{L})^{q} e^{ik_2X_2} e^{ik_3X_3} e^{ik_4X_4} e^{ik_5X_5}\right\rangle$$

$$= \int_0^1 dx_3 \int_0^{x_3} dx_2\, u(x_2, x_3) e^{-\Lambda f(x_2,x_3)}$$

$$\Lambda = -k_1 \cdot k_2 \sim E^2 \to \infty$$



5-Point String Amplitude

$$A^{(p_1,p_2,2m,q)} = \int_0^1 dx_3 \int_0^{x_3} dx_2\, u(x_2,x_3) e^{-\Lambda f(x_2,x_3)}, \qquad \Lambda = -k_1 \cdot k_2 \sim E^2 \to \infty$$

$$f = \ln x_2 - \frac{k_1 \cdot k_3}{\Lambda} \ln x_3 - \frac{k_2 \cdot k_3}{\Lambda} \ln(x_3 - x_2) - \frac{k_2 \cdot k_4}{\Lambda} \ln(1 - x_2) - \frac{k_3 \cdot k_4}{\Lambda} \ln(1 - x_3)$$

$$u = (K^{T_1})^{N+p_1} (K^{T_2})^{p_2} (K^L)^{2m} \left(K'^L\right)^q$$

$$K(x_2, x_3) \equiv \frac{k_1}{x_2} - \frac{k_3}{x_3 - x_2} - \frac{k_4}{1 - x_2}$$

$$\Rightarrow K^L = K \cdot e^L = \frac{1}{M_2}\left(\frac{k_1 \cdot k_2}{x_2} - \frac{k_3 \cdot k_2}{x_3 - x_2} - \frac{k_4 \cdot k_2}{1 - x_2}\right) = -\frac{\Lambda}{M_2} \partial_2 f(x_2, x_3)$$

Saddle Point

$$f = \ln x_2 - \frac{k_1 \cdot k_3}{\Lambda} \ln x_3 - \frac{k_2 \cdot k_3}{\Lambda} \ln(x_3 - x_2) - \frac{k_2 \cdot k_4}{\Lambda} \ln(1 - x_2) - \frac{k_3 \cdot k_4}{\Lambda} \ln(1 - x_3)$$

$$\partial_2 f(x_2, x_3) = \frac{1}{x_2} + \frac{k_2 \cdot k_3}{\Lambda(x_3 - x_2)} + \frac{k_2 \cdot k_4}{\Lambda(1 - x_2)} = 0$$

$$\partial_3 f(x_2, x_3) = -\frac{k_1 \cdot k_3}{\Lambda x_3} - \frac{k_2 \cdot k_3}{\Lambda(x_3 - x_2)} + \frac{k_3 \cdot k_4}{\Lambda(1 - x_3)} = 0$$

No analytic solutions for $\tilde{x}_2, \tilde{x}_3$!

Saddle Point Approximation

$$A^{(p_1,p_2,2m,q)} = \int_0^1 dx_3 \int_0^{x_3} dx_2\, u(x_2,x_3) e^{-\Lambda f(x_2,x_3)}, \qquad \Lambda = -k_1 \cdot k_2 \sim E^2 \to \infty$$

$$\tilde{u} = \left(\widetilde{K}^{T_1}\right)^{N+p_1} \left(\widetilde{K}^{T_2}\right)^{p_2} \left(\widetilde{K}^{L}\right)^{2m} \left(\widetilde{K}'^{L}\right)^{q}, \quad \widetilde{K}^{L} = -\frac{\Lambda}{M_2} \partial_2 f(\tilde{x}_2, \tilde{x}_3) = 0$$

$$\tilde{u} = \partial_2 \tilde{u} = \cdots = \partial_2^{2m-1} \tilde{u} = 0$$

$$\partial_2^{2m} \tilde{u} = (2m)! \left(\widetilde{K}^{T_1}\right)^{N+p_1} \left(\widetilde{K}^{T_2}\right)^{p_2} \left(\widetilde{K}'^{L}\right)^{2m+q}$$

5-Point String Amplitude

$$\tilde{u} = \partial_2 \tilde{u} = \cdots = \partial_2^{2m-1} \tilde{u} = 0, \qquad \partial_2^{2m} \tilde{u} = (2m)! \left(\widetilde{K}^{T_1}\right)^{N+p_1} \left(\widetilde{K}^{T_2}\right)^{p_2} \left(\widetilde{K}'^{L}\right)^{2m+q}$$

$$A^{(p_1, p_2, 2m, q)} = \int_0^1 dx_3 \int_0^{x_3} dx_2 \, u(x_2, x_3) e^{-\Lambda f(x_2, x_3)}$$

$$= \int_0^1 dx_3 \int_0^{x_3} dx_2 \left[\frac{\partial_2^{2m} \tilde{u}}{(2m)!} (x_2 - \tilde{x}_2)^{2m} \right] e^{-\Lambda\left[\tilde{f} + \frac{1}{2}\tilde{f}''(x_2 - \tilde{x}_2)^2 + \cdots\right]}$$

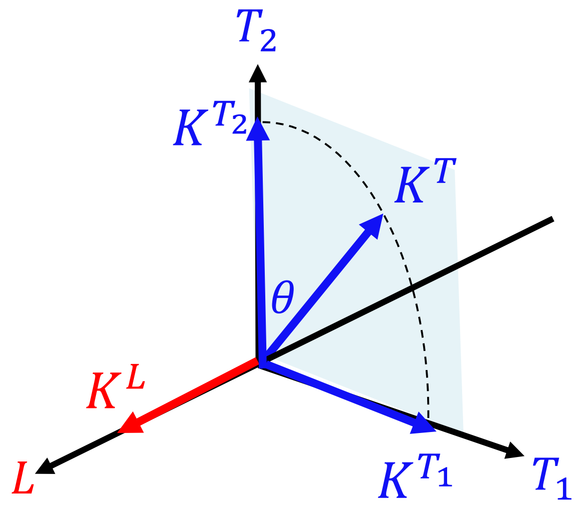
$$\simeq e^{-\Lambda \tilde{f}} \sqrt{\frac{-2\pi}{M_2 \widetilde{K}'^{L}}} \frac{\left(\widetilde{K}^{T_1}\right)^{N+p_1} \left(\widetilde{K}^{T_2}\right)^{p_2} \left(\widetilde{K}'^{L}\right)^{m+q}}{2^m m! \, (-M_2)^m}$$

The Effective Momentum K

$$A^{(p_1,p_2,2m,q)} \simeq e^{-\Lambda\tilde{f}}\sqrt{\frac{-2\pi}{M_2\widetilde{K}'^L}}\,\frac{\left(\widetilde{K}^{T_1}\right)^{N+p_1}\left(\widetilde{K}^{T_2}\right)^{p_2}\left(\widetilde{K}'^L\right)^{m+q}}{2^m m!\,(-M_2)^m}$$

$$K(x_2,x_3) \equiv \frac{k_1}{x_2} - \frac{k_3}{x_3-x_2} - \frac{k_4}{1-x_2} = (K^L, K^T)$$

$$\widetilde{K}^L = -\frac{\Lambda}{M_2}\partial_2 f(\tilde{x}_2,\tilde{x}_3) = 0 \Rightarrow \begin{cases}\widetilde{K}^{T_1} = \left|\widetilde{K}\right|\sin\theta \\ \widetilde{K}^{T_2} = \left|\widetilde{K}\right|\cos\theta\end{cases}$$



$$\widetilde{K}^2 + 2M_2\widetilde{K}'^L = 0 \Rightarrow \widetilde{K}'^L = -\frac{\widetilde{K}^2}{2M_2},$$ (numerical proof)

Stringy Scaling

$$A^{(p_1,p_2,2m,q)} \simeq e^{-\Lambda\tilde{f}} \sqrt{\frac{-2\pi}{M_2\tilde{K}'^L}} \frac{\left(\tilde{K}^{T_1}\right)^{N+p_1}\left(\tilde{K}^{T_2}\right)^{p_2}\left(\tilde{K}'^L\right)^{m+q}}{2^m m!\,(-M_2)^m}, \begin{cases} \tilde{K}'^L = -\tilde{K}^2/2M_2 \\ \tilde{K}^{T_1} = \left|\tilde{K}\right| \sin\theta \\ \tilde{K}^{T_2} = \left|\tilde{K}\right| \cos\theta \end{cases}$$

$$= 2\sqrt{\pi} e^{-\Lambda\tilde{f}} \left|\tilde{K}\right|^{N-1} \frac{(2m)!}{m!}\left(\frac{-1}{2M_2}\right)^{2m+q} (\sin\theta)^{N+p_1}(\cos\theta)^{p_2}$$

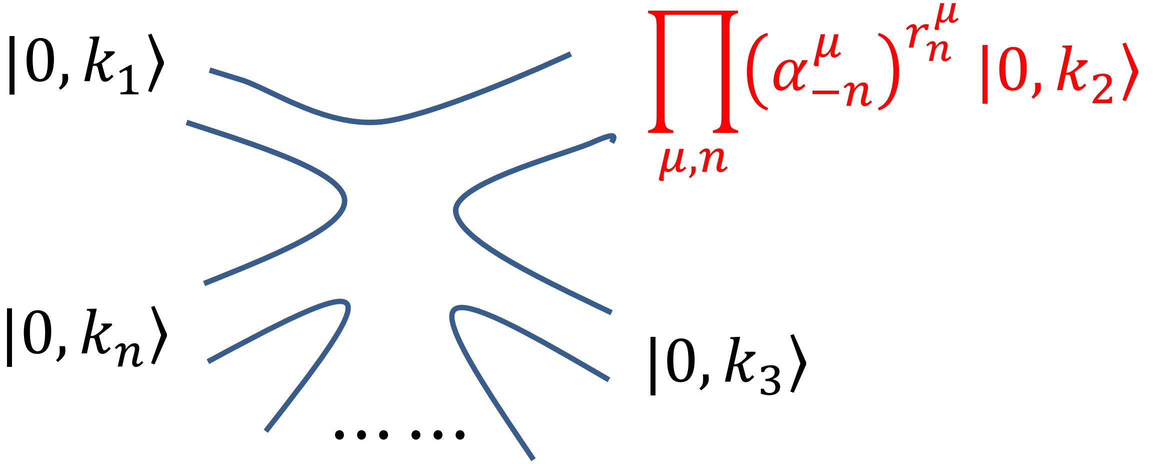
$$\frac{A^{(p_1,p_2,2m,q)}}{A^{(N,0,0)}} = \frac{(2m)!}{m!}\left(\frac{-1}{2M_2}\right)^{2m+q} (\sin\theta)^{N+p_1}(\cos\theta)^{p_2}$$

DOF: $\left(p, q_3, q_4, q_5, \phi_3^1, \phi_4^1, \phi_4^2, \phi_5^1, \phi_5^2\right) \rightarrow 5 \rightarrow 1$

n-Point String Amplitude

- $(n-1)$ tachyons + a massive string state

$A =$



Kinematics in CM Frame: n-Point

- Polarization: $(e^L, e^P, e^{T_{i=1,\cdots,r}})$

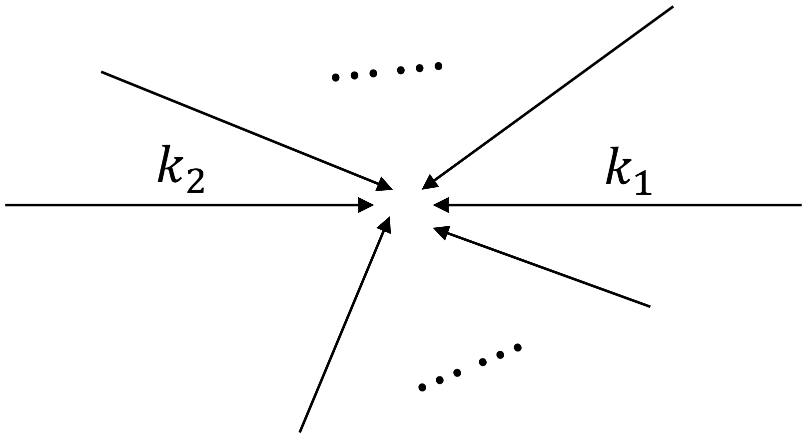
DOF:

$$\left(p, q_{j=3,\cdots,n}, \Omega^{i=1,\cdots,r+1}_{j=3,\cdots,n}\right) \rightarrow \frac{(r+1)(2n-r-4)}{2} - 1$$

$$k_1 = \left(\sqrt{p^2 + M_1^2}, -p, 0, \cdots, 0\right)$$

$$k_2 = \left(\sqrt{p^2 + M_2^2}, +p, 0, \cdots, 0\right) \sim \frac{e^P}{M_2}$$

$$k_j = \left(-\sqrt{q_j^2 + M_j^2}, -q_j\Omega_j^1, \cdots, -q_j\Omega_j^{r+1}\right), j = 3, \cdots, n$$



Ω_j^i's are solid angles in the $(r+1)$ dimensional space.

$$\sum k_i = 0, \sum_{j=3}^{n} \vec{k}_j = 0, \sum_{i=1}^{j-2} \left(\Omega_j^i\right)^2 = 1$$

Hard Limit

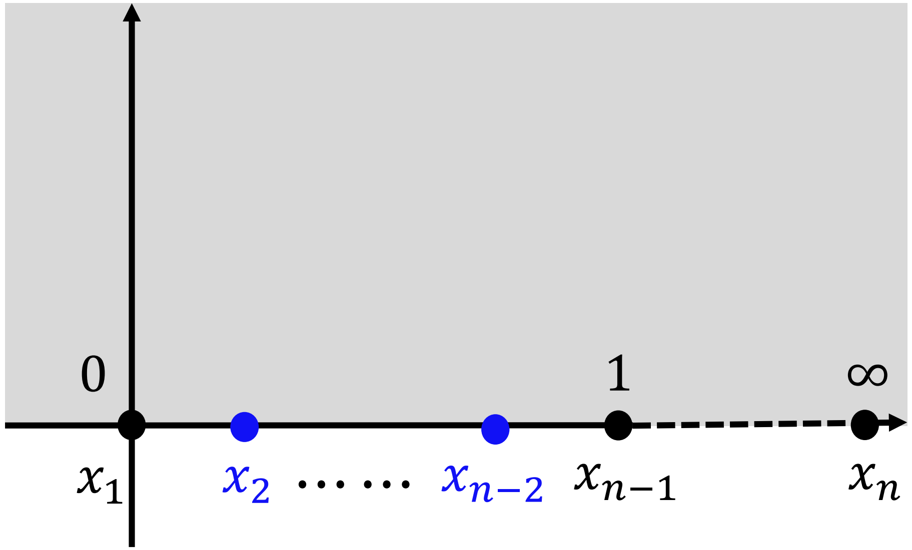
- High energy, fixed angle limit: $p, q_i \to \infty, \quad \phi_i^j$ fixed, $e^L \simeq e^P$
- Relevant states

$$|\{p_i\}, 2m, q\rangle \sim \left(\alpha_{-1}^{T_1}\right)^{N+p_1}\left(\alpha_{-1}^{T_2}\right)^{p_2}\cdots\left(\alpha_{-1}^{T_r}\right)^{p_r}(\alpha_{-1}^{L})^{2m}(\alpha_{-2}^{L})^{q}|0,k\rangle$$

$$A^{(\{p_i\},2m,q)}$$

$$= \int_0^1 dx_{n-2}\cdots\int_0^{x_4} dx_3 \int_0^{x_3} dx_2\, u(x_i) e^{-\Lambda f(x_i)}$$

$$\Lambda = -k_1 \cdot k_2 \sim E^2 \to \infty$$



n-Point String Amplitude

$$A^{(\{p_i\},2m,q)} = \int_0^1 dx_{n-2} \cdots \int_0^{x_4} dx_3 \int_0^{x_3} dx_2\, u(x_i) e^{-\Lambda f(x_i)}, \Lambda = -k_1 \cdot k_2 \sim E^2 \to \infty$$

$$f(x_i) = -\sum_{i<j} \frac{k_i \cdot k_j}{\Lambda} \ln\left(x_j - x_i\right)$$

$$u(x_i) = (K^{T_1})^{N+p_1} (K^{T_2})^{p_2} \dots (K^{T_r})^{p_r} (K^L)^{2m} \left(K'^L\right)^q$$

$$K(x_i) = \sum_{i \neq 2,n} \frac{-k_i}{x_i - x_2} \Rightarrow K^L = K \cdot e^L = \frac{1}{M} \sum_{i=j+2}^{n-1} \frac{-k_i \cdot k_2}{x_i - x_2} = -\frac{\Lambda}{M_2} \partial_2 f(x_i)$$

Saddle Point

$$A^{(\{p_i\},2m,q)} = \int_0^1 dx_{n-2} \cdots \int_0^{x_4} dx_3 \int_0^{x_3} dx_2\, u(x_i) e^{-\Lambda f(x_i)}, \Lambda = -k_1 \cdot k_2 \sim E^2 \to \infty$$

$$f(x_i) = -\sum_{i<j} \frac{k_i \cdot k_j}{\Lambda} \ln(x_j - x_i), \qquad \partial_k f(\tilde{x}_i) = 0$$

$$\tilde{u} = (\widetilde{K}^{T_1})^{N+p_1} (\widetilde{K}^{T_2})^{p_2} \cdots (\widetilde{K}^{T_r})^{p_r} (\widetilde{K}^{L})^{2m} (\widetilde{K}'^{L})^{q}, \qquad \widetilde{K}^L = -\frac{\Lambda}{M_2} \partial_2 f(\tilde{x}_i) = 0$$

$$\tilde{u} = \partial_2 \tilde{u} = \cdots = \partial_2^{2m-1} \tilde{u} = 0$$

$$\partial_2^{2m} \tilde{u} = (2m)!\, (\widetilde{K}^{T_1})^{N+p_1} (\widetilde{K}^{T_2})^{p_2} \cdots (\widetilde{K}^{T_r})^{p_r} (\widetilde{K}'^{L})^{2m+q}$$

n-Point String Amplitude

$$\tilde{u} = \partial_2 \tilde{u} = \cdots = \partial_2^{2m-1}\tilde{u} = 0, \partial_2^{2m}\tilde{u} = (2m)!\left(\widetilde{K}^{T_1}\right)^{N+p_1}\left(\widetilde{K}^{T_2}\right)^{p_2}\cdots\left(\widetilde{K}^{T_r}\right)^{p_r}\left(\widetilde{K}'^{L}\right)^{2m+q}$$

$$A^{(\{p_i\},2m,q)} = \int_0^1 dx_{n-2}\cdots\int_0^{x_4} dx_3 \int_0^{x_3} dx_2\, u(x_i) e^{-\Lambda f(x_i)}$$

$$= \int_0^1 dx_{n-2}\cdots\int_0^{x_4} dx_3 \int_0^{x_3} dx_2 \left[\frac{\partial_2^{2m}\tilde{u}}{(2m)!}(x_2-\tilde{x}_2)^{2m}\right] e^{-\Lambda\left[\tilde{f}+\frac{1}{2}\tilde{f}''(x_2-\tilde{x}_2)^2+\cdots\right]}$$

$$\simeq e^{-\Lambda\tilde{f}}\sqrt{\frac{-2\pi}{M_2\widetilde{K}'^{L}}}\frac{\left(\widetilde{K}^{T_1}\right)^{N+p_1}\left(\widetilde{K}^{T_2}\right)^{p_2}\cdots\left(\widetilde{K}^{T_r}\right)^{p_r}\left(\widetilde{K}'^{L}\right)^{m+q}}{2^m m!\,(-M_2)^m}$$

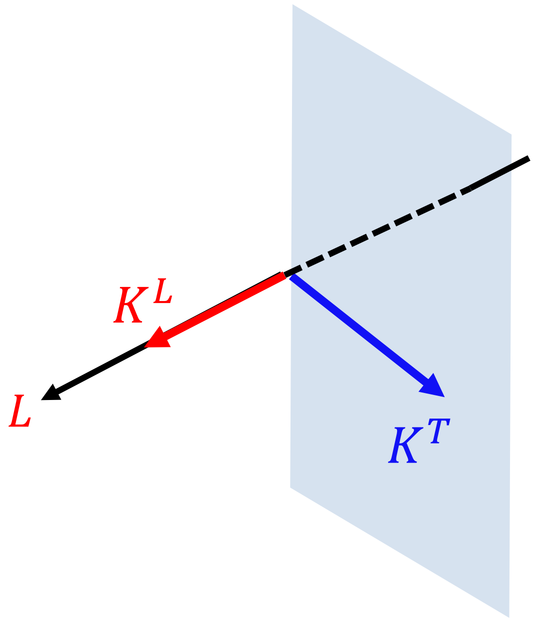
The Effective Momentum K

$$A^{(p_1,p_2,2m,q)} \simeq e^{-\Lambda\tilde{f}} \sqrt{\frac{-2\pi}{M_2\widetilde{K}'^L}} \frac{\left(\widetilde{K}^{T_1}\right)^{N+p_1}\left(\widetilde{K}^{T_2}\right)^{p_2}\cdots\left(\widetilde{K}^{T_r}\right)^{p_r}\left(\widetilde{K}'^L\right)^{m+q}}{2^m m!\,(-M_2)^m}$$

$$K(x_i) \equiv \sum_{i\neq 2,n} \frac{-k_i}{x_i - x_2} = (K^L, K^T)$$

$$\widetilde{K}^L = -\frac{\Lambda}{M_2}\partial_2 f(\tilde{x}_i) = 0 \Rightarrow \widetilde{K}^{T_i} = |\widetilde{K}|\omega_i, \sum_{i=1}^{r}\omega_i^2 = 1$$

$$\widetilde{K}^2 + 2M_2\widetilde{K}'^L = 0 \Rightarrow \widetilde{K}'^L = -\frac{\widetilde{K}^2}{2M_2},$$ (conjectured)



Stringy Scaling

$$A^{(p_1,p_2,2m,q)} \simeq e^{-\Lambda\tilde{f}}\sqrt{\frac{-2\pi}{M_2\widetilde{K}'^L}}\frac{\left(\widetilde{K}^{T_1}\right)^{N+p_1}\left(\widetilde{K}^{T_2}\right)^{p_2}\cdots\left(\widetilde{K}^{T_r}\right)^{p_r}\left(\widetilde{K}'^L\right)^{m+q}}{2^m m!\,(-M_2)^m}, \qquad \begin{cases}\widetilde{K}'^L = -\widetilde{K}^2/2M_2\\ \widetilde{K}^{T_i} = |\widetilde{K}|\omega_i\end{cases}$$

$$= 2\sqrt{\pi}e^{-\Lambda\tilde{f}}\left|\widetilde{K}\right|^{N-1}\frac{(2m)!}{m!}\left(\frac{-1}{2M}\right)^{2m+q} \quad {\omega_1}^{p_1}{\omega_2}^{p_2}\cdots{\omega_r}^{p_r}$$

$$\frac{A^{(\{p_i\},2m,q)}}{A^{(N,0,0)}} = \frac{(2m)!}{m!}\left(\frac{-1}{2M}\right)^{2m+q} \quad {\omega_1}^{p_1}{\omega_2}^{p_2}\cdots{\omega_r}^{p_r}, \qquad \sum_{i=1}^{r}\omega_i^2 = 1$$

DOF: $\left(p, q_{j=3,\cdots,n}, \Omega_{j=3,\cdots,n}^{i=1,\cdots,r+1}\right) \rightarrow \frac{(r+1)(2n-r-4)}{2} - 1 \rightarrow r-1$

Summary

- Linear relations of 4-point string amplitudes in hard limit
- Stringy scaling of n-point string amplitudes in hard limit
- Saddle point method
- The effective momentum: $K = \sum_{i \neq 2,n} \frac{-k_i}{x_i - x_2} \Rightarrow \tilde{K}^L = 0$
- The identity in hard limit: $\tilde{K}^2 + 2M\tilde{K}'^L = 0$