Stochastic Processes : an Introduction
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Solutions for selected problems

Lévy distribution

$$F(k) := \sqrt{2\pi}C(k) = \int_0^\infty x^{-3/2}\exp\left(-\frac{1}{2}\left(x^{-1}+2kx\right)\right) = 2\int_0^\infty \exp\left(-\frac{1}{2}\left(y^2+\frac{2k}{y^2}\right)\right)dy$$
$$= 2\int_0^\infty \frac{\sqrt{2k}}{t^2}\exp\left(-\frac{1}{2}\left(\frac{2k}{t^2}+t^2\right)\right)dt,$$

where the second equality is obtained by the change of variables $y^{-2}=x$ and third is by $2ky^{-2}=t^2$. Hence,

$$F(k) = \int_0^\infty \left(1+\frac{\sqrt{2k}}{t^2}\right)\exp\left(-\frac{1}{2}\left(\frac{2k}{t^2}+t^2\right)\right)dt = e^{-\sqrt{2k}}\int_0^\infty \left(1+\frac{\sqrt{2k}}{t^2}\right)\exp\left(-\frac{1}{2}\left(t-\frac{\sqrt{2k}}{t}\right)^2\right)dt$$
$$= e^{-\sqrt{2k}}\int_{-\infty}^\infty \exp\left(-\frac{1}{2}z^2\right)dz = \sqrt{2\pi}e^{-\sqrt{2k}},$$

where we have made changes of variables $z = t-\sqrt{2k}t^{-1}$. Hence, we have $C(k) = \exp\left(-\sqrt{2k}\right)$.

RW The master equation is

$$\frac{\partial P_n(t)}{\partial t} = rP_{n-1}(t) + lP_{n+1}(t) - (r+l)P_n(t)$$

Let $\mathcal{G}(z,t) = \sum_n z^n P_n(t)$ with $\mathcal{G}(z,0)=1$. Then,

$$\frac{\partial \mathcal{G}(z,t)}{\partial t} = \left(rz + \frac{l}{z} - (r+l)\right)\mathcal{G}(z,t) \Rightarrow \mathcal{G}(z,t) = \exp\left(rtz + \frac{lt}{z} - (r+l)t\right)$$

Discrete approach: Let $p := r/(r+l)$ and $q := 1-p$. For $n = 2m-1$ (m integer), the number of steps up to t should be odd ($2N+1$, say). Therefore,

$$P_n(t) \equiv P_{2m-1}(t) = \sum_{N=0}^\infty \frac{(r+l)^{2N+1}}{(2N+1)!}e^{-(r+l)t}\binom{2N+1}{N+m}p^{N+m}q^{N-m+1}.$$

For $n=2m$ (m integer), the number of steps up to t should be even ($2N$, say). Therefore,

$$P_n(t) \equiv P_{2m}(t) = \sum_{N=0}^\infty \frac{(r+l)^{2N}}{(2N)!}e^{-(r+l)t}\binom{2N}{N+m}p^{N+m}q^{N-m}.$$

Now we write

$$\mathcal{G}(z,t) = \sum_{m=-\infty}^\infty \left(P_{2m}(t)z^{2m} + P_{2m-1}(t)z^{2m-1}\right).$$

Since

$$\begin{aligned}\sum_{m=-\infty}^{\infty} P_{2m}(t)z^{2m} &= \sum_{N=-\infty}^{\infty} \frac{(rt+lt)^{2N}}{(2N)!} e^{-(r+l)t} \sum_{m=-\infty}^{\infty} \binom{2N}{N+m} p^{N+m} q^{N-m} z^{2m} \\ &= \sum_{N=-\infty}^{\infty} \frac{(rt+lt)^{2N}}{(2N)!} e^{-(r+l)t} \sum_{k=-\infty}^{\infty} \binom{2N}{k} p^k q^{2N-k} z^{2k-2N} \\ &= \sum_{N=-\infty}^{\infty} \frac{(rt+lt)^{2N}}{z^{2N}(2N)!} e^{-(r+l)t} \sum_{k=-\infty}^{\infty} \binom{2N}{k} (z^2 p)^k q^{2N-k} \\ &= e^{-(r+l)t} \sum_{N=0}^{\infty} \frac{1}{(2N)!} \left(rtz + \frac{lt}{z}\right)^{2N} = e^{-(r+l)t} \cosh\left(rtz + \frac{lt}{z}\right),\end{aligned}$$

and

$$\begin{aligned}\sum_{m=-\infty}^{\infty} P_{2m-1}(t)z^{2m-1} &= \sum_{N=-\infty}^{\infty} \frac{(rt+lt)^{2N+1}}{(2N+1)!} e^{-(r+l)t} \sum_{m=-\infty}^{\infty} \binom{2N+1}{N+m} p^{N+m} q^{N-m+1} z^{2m-1} \\ &= \sum_{N=-\infty}^{\infty} \frac{(rt+lt)^{2N+1}}{(2N+1)!} e^{-(r+l)t} \sum_{k=-\infty}^{\infty} \binom{2N+1}{k} p^k q^{2N+1-k} z^{2k-2N-1} \\ &= \sum_{N=-\infty}^{\infty} \frac{(rt+lt)^{2N+1}}{z^{2N+1}(2N+1)!} e^{-(r+l)t} \sum_{k=-\infty}^{\infty} \binom{2N}{k} (z^2 p)^k q^{2N+1-k} \\ &= e^{-(r+l)t} \sum_{N=0}^{\infty} \frac{1}{(2N+1)!} \left(rtz + \frac{lt}{z}\right)^{2N+1} = e^{-(r+l)t} \sinh\left(rtz + \frac{lt}{z}\right),\end{aligned}$$

we have

$$\mathcal{G}(z,t) = \exp\left(rtz + \frac{lt}{z} - (r+l)t\right).$$

Project Check the following paper

S.-C. Park, J. Stat. Mech. (2015) P10009 (https://doi.org/10.1088/1742-5468/2015/10/P10009).

same paper in arXiv:1507.05537 (https://doi.org/10.48550/arXiv.1507.05537)

In particular, see Eqs. (4,6,7,8,10,11) and Figures 1,2,3