Understanding cluster synchronizations in a unified framework using symmetry and its applications

Goverining equation

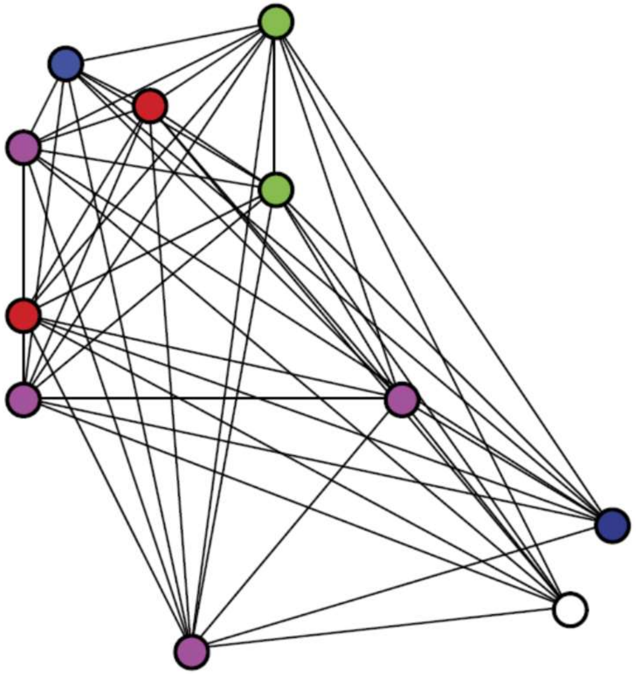
$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) + \sigma \sum_{j=1}^{N} A_{ij}\, \mathbf{H}(\mathbf{x}_j)$$

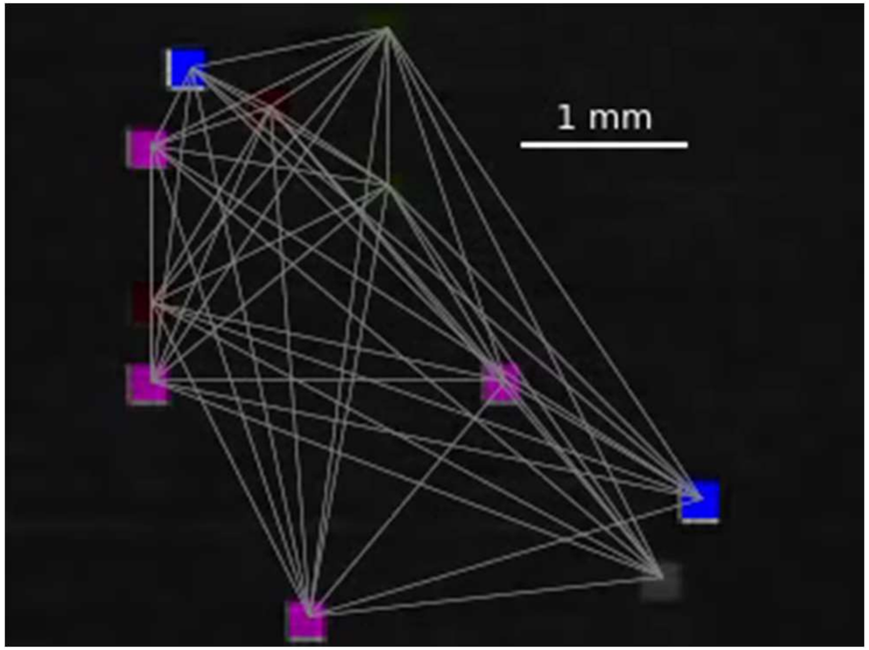
$\mathbf{x}_i$: **state** of each oscillator

A_{ij}: Adjacency matrix

Cluster synchronization (CS)

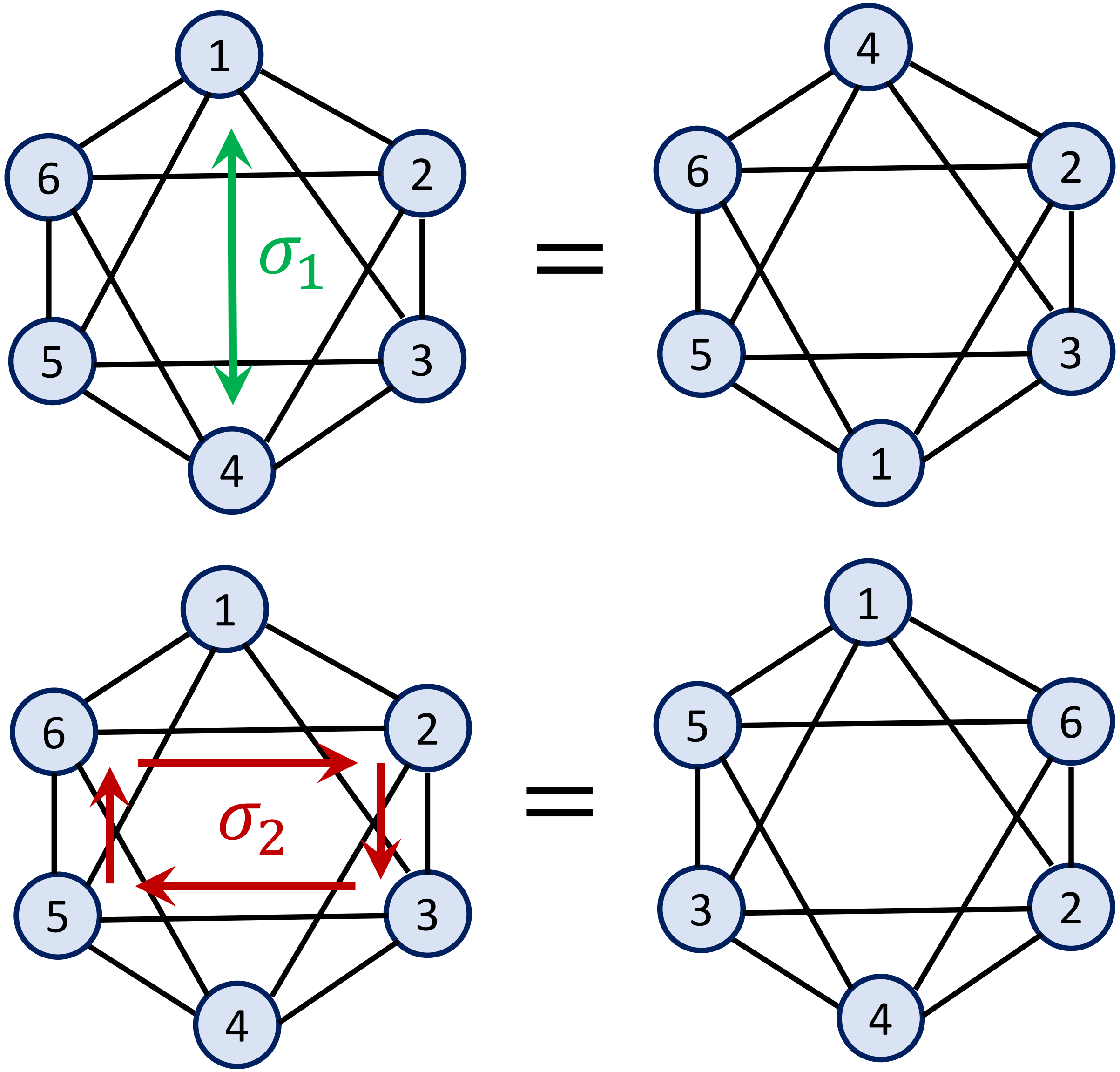
L. M. Pecora *et al.* Nat. Commun. **5**, 4079 (2014).





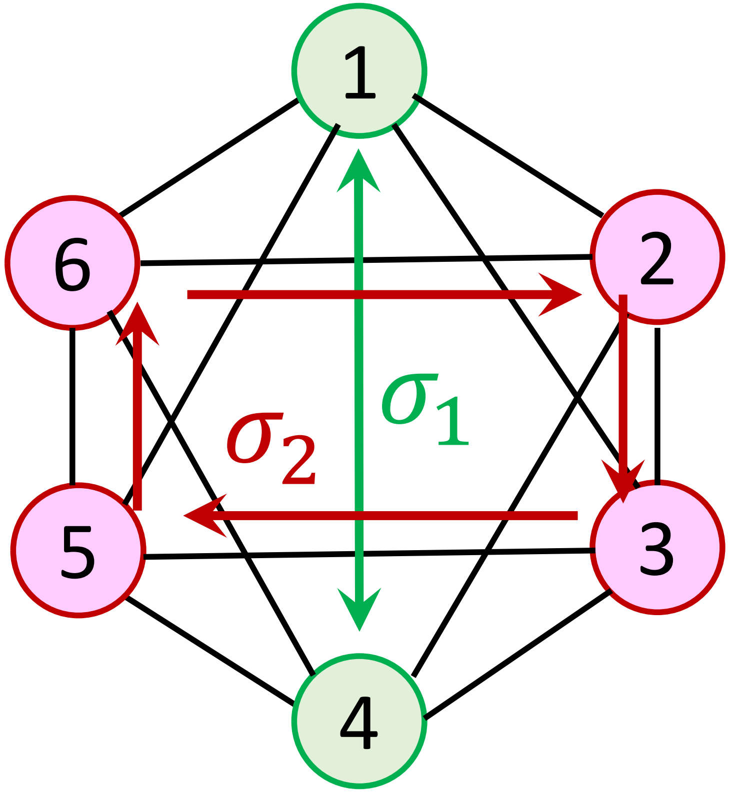
$\mathbf{x}_i = \mathbf{s}_m$ for $i \in C_m$ → C_m: cluster

Symmetry: A permutation not changing the given adjacency matrix



Diverse CSs are understood by symmetry of the given adjacency matrix.

L. M. Pecora *et al.* Nat. Commun. **5**, 4079 (2014).
F. Sorrentino *et al.* Sci. Adv. e1501737 (2016).

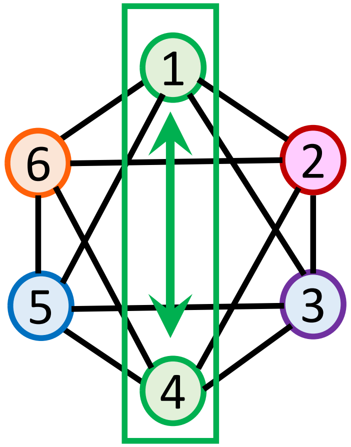


$\{C_1, C_2\}$

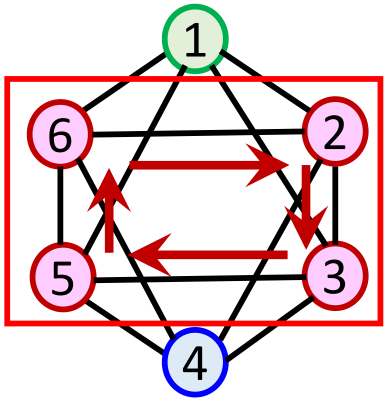
$C_1 = \{1,4\}$

$C_2 = \{2,3,5,6\}$

Independently synchronizable cluster (ISC)



$C_1 = \{1,4\}$

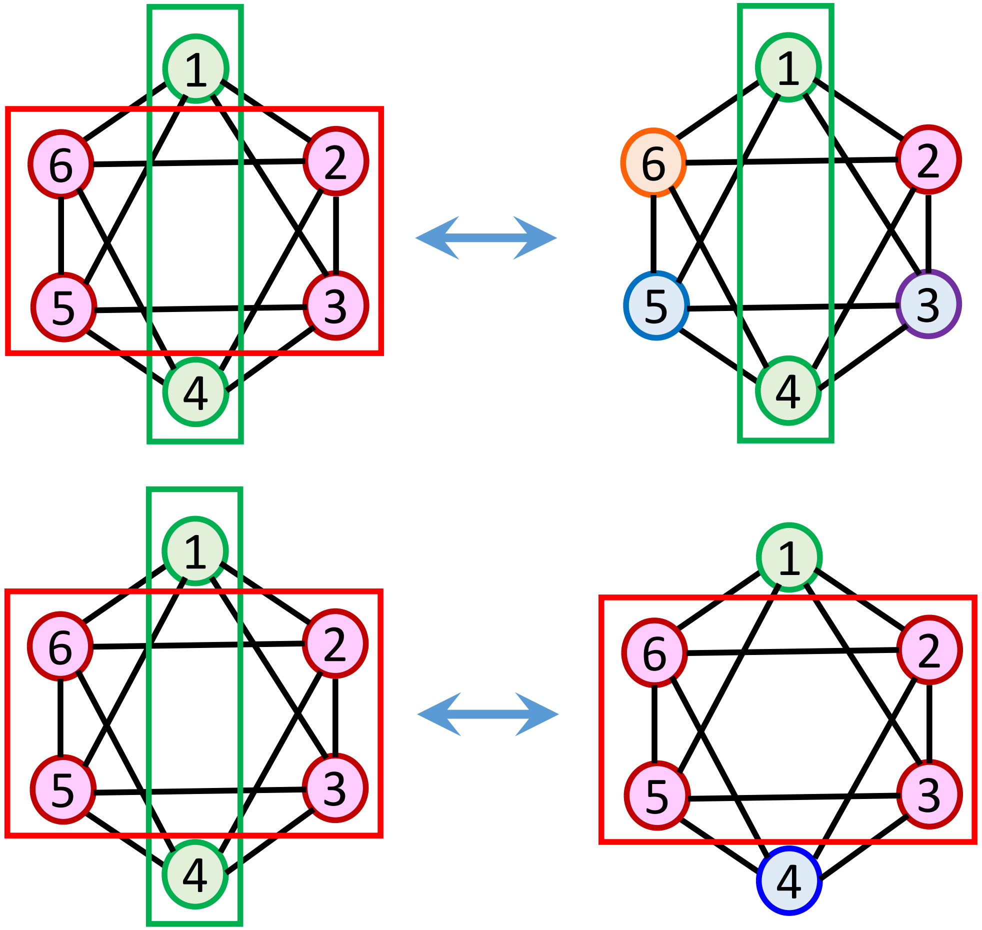


$C_2 = \{2,3,5,6\}$

$$\{|C_1|, |C_2|\}$$

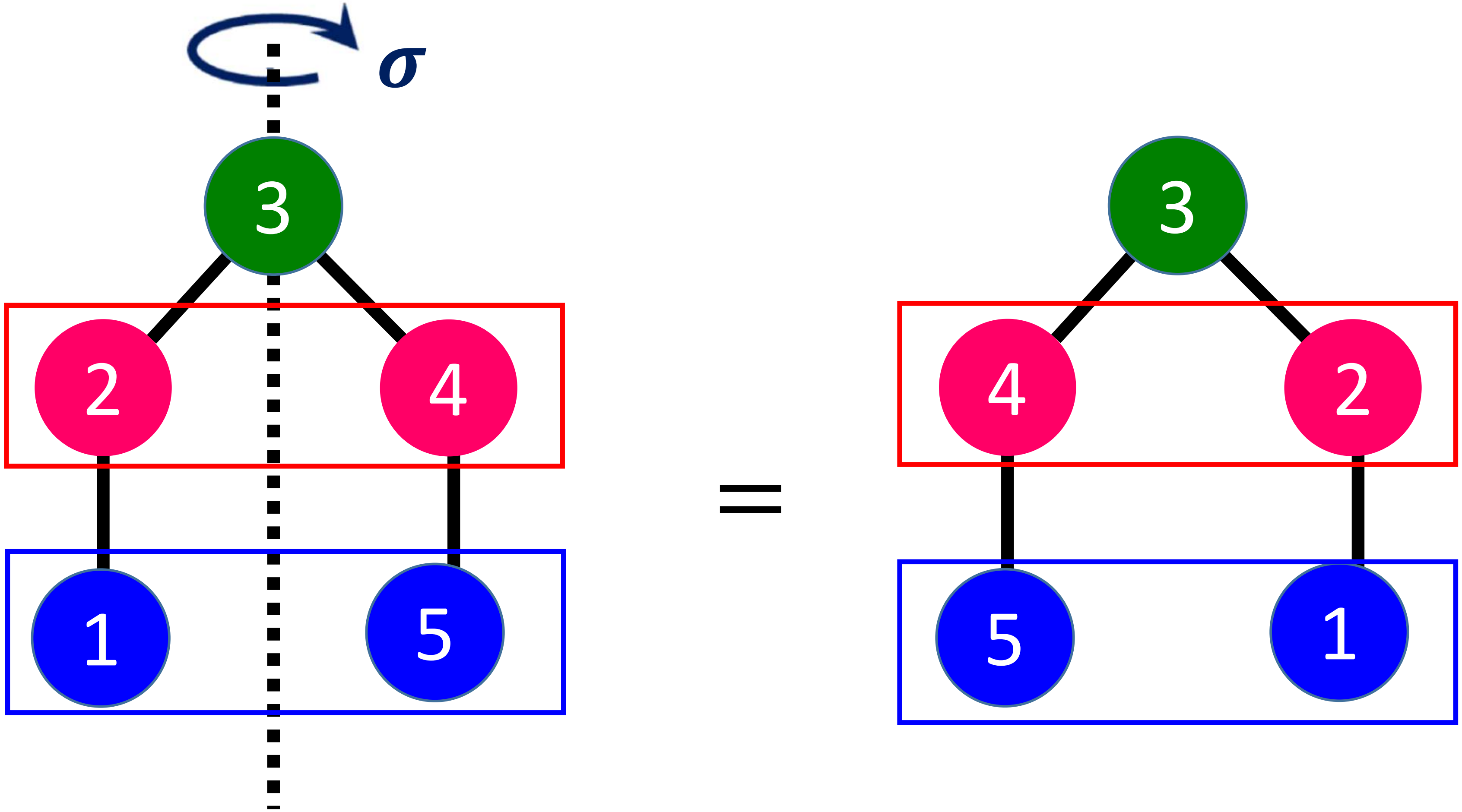
Two **ISCs**

Independently synchronizable cluster (ISC)



Each ISC can be **broken** (desynchronized) or **formed** (synchronized)
without disturbing the other clusters.

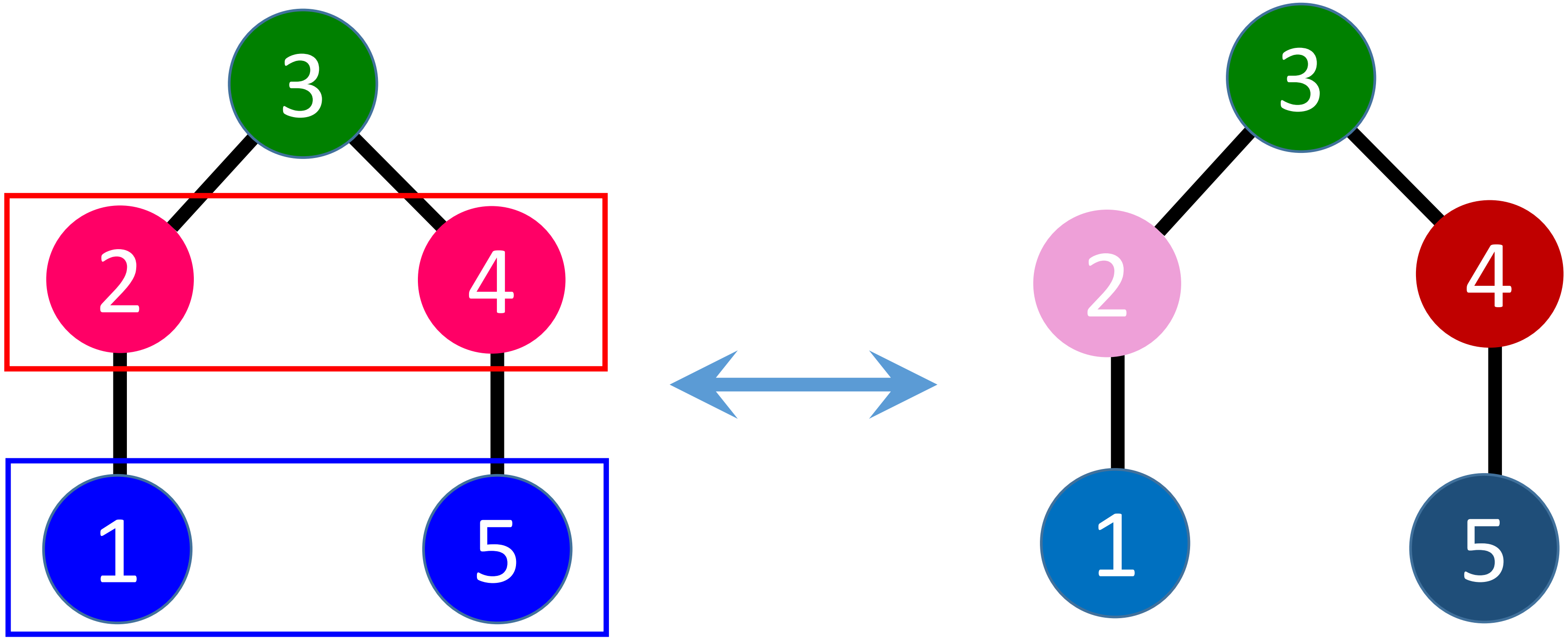
Intertwined cluster set



$$\{|C_1, C_2|, |C_3|\}$$

intertwined cluster set

Intertwined cluster set



Every cluster in **each intertwined cluster set**
should be **formed** (synchronized) or **broken** (desynchronized) **at the same time**.

Understanding cluster synchronizations in a unified framework using symmetry

Y. S. Cho *et al.* Phys. Rev. Lett. **119**, 084101 (2017).

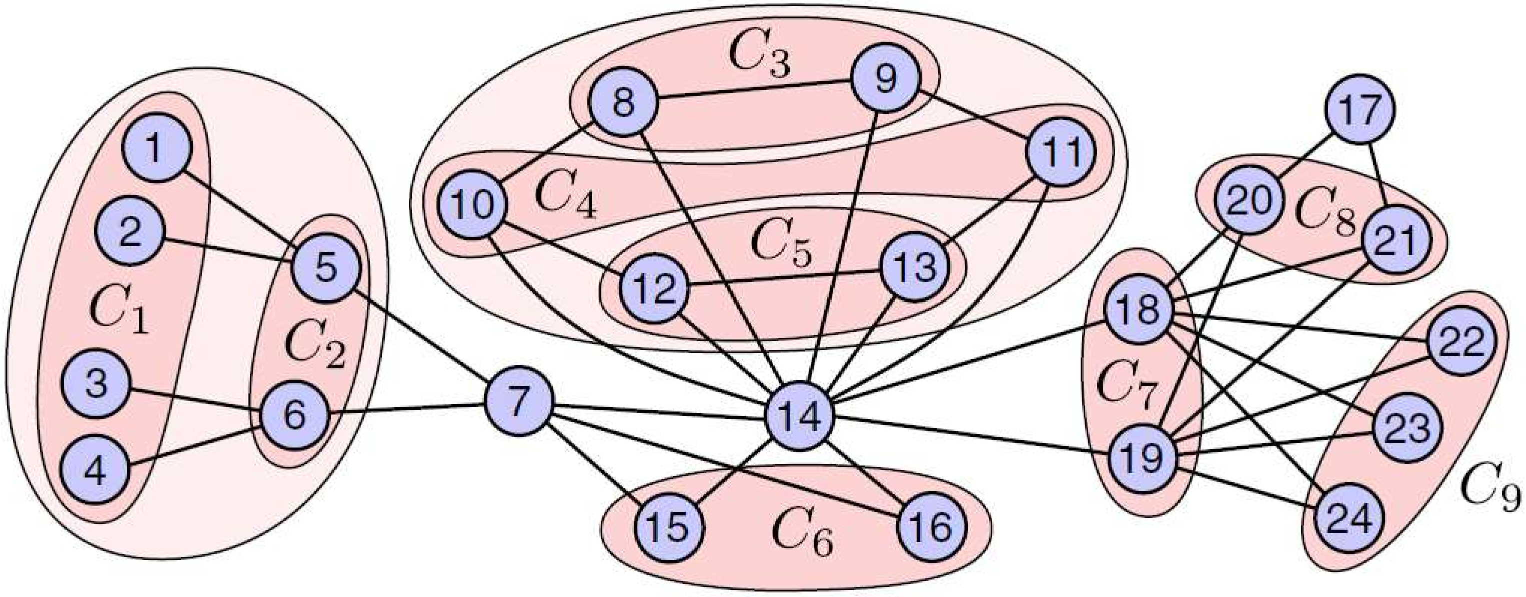
Unique partition of an arbitrary **symmetry induced** cluster synchronization (CS)

$$\{|C_1, \ldots, C_{M'}|, \ldots, |C_m|, \ldots\}$$

intertwined cluster set

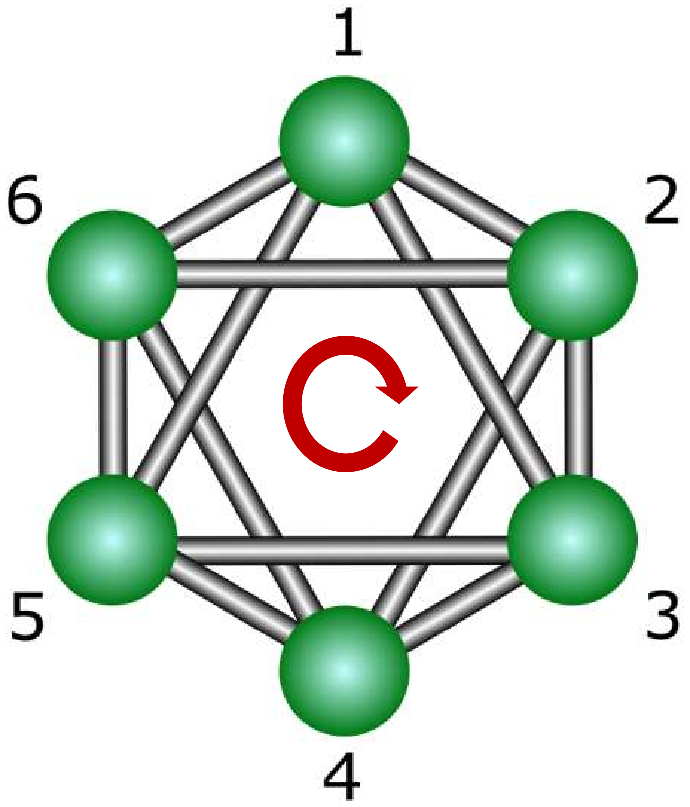
Independently synchronizable cluster (ISC)

Example



$$\{|C_1, C_2|, |C_3, C_4, C_5|, |C_6|, |C_7|, |C_8|, |C_9|\}$$

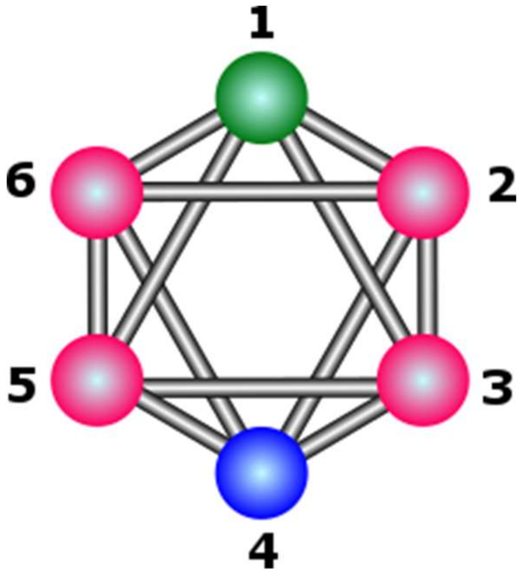
Stable chimera of finite size

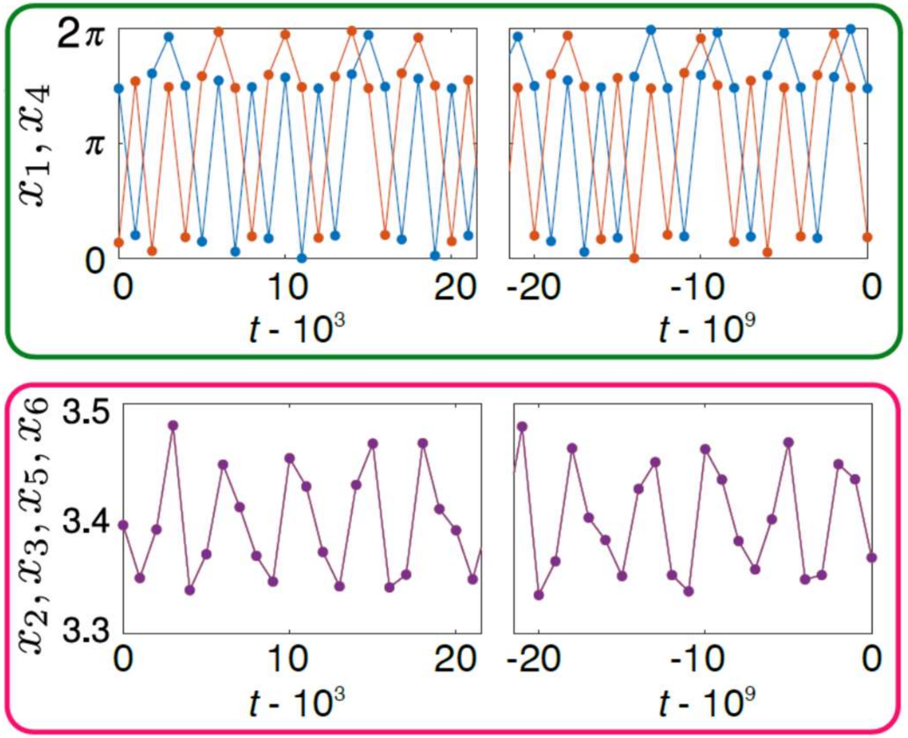


$\langle R_{60^\circ} \rangle$

Fully symmetric network

Stable chimera of finite size





Coexistence of sync. and desync. states

Chimera
(Hybrid creature)

Stable chimera of finite size

Motivation

(1) Stable chimera in infinite system size

D. M. Abrams *et al.* Phys. Rev. Lett. **93**, 174102 (2004).

D. M. Abrams *et al.* Phys. Rev. Lett. **101**, 084103 (2008).

(2) Chimeras in finite-size networks (ever found) can be long-lived but transient states

M. Wolfrum *et al.* Phys. Rev. E **84**, 015201 (2011).

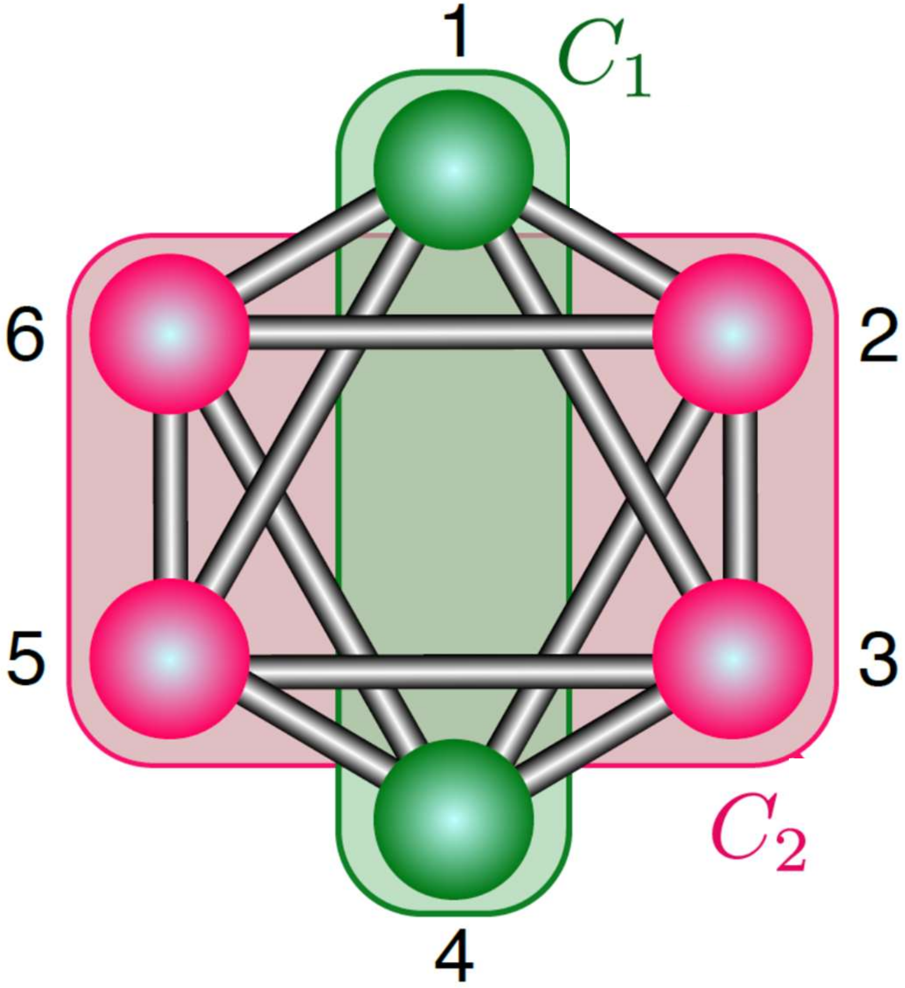
Is it possible to make stable chimera states in finite-size networks?

Stable chimera of finite size

Method

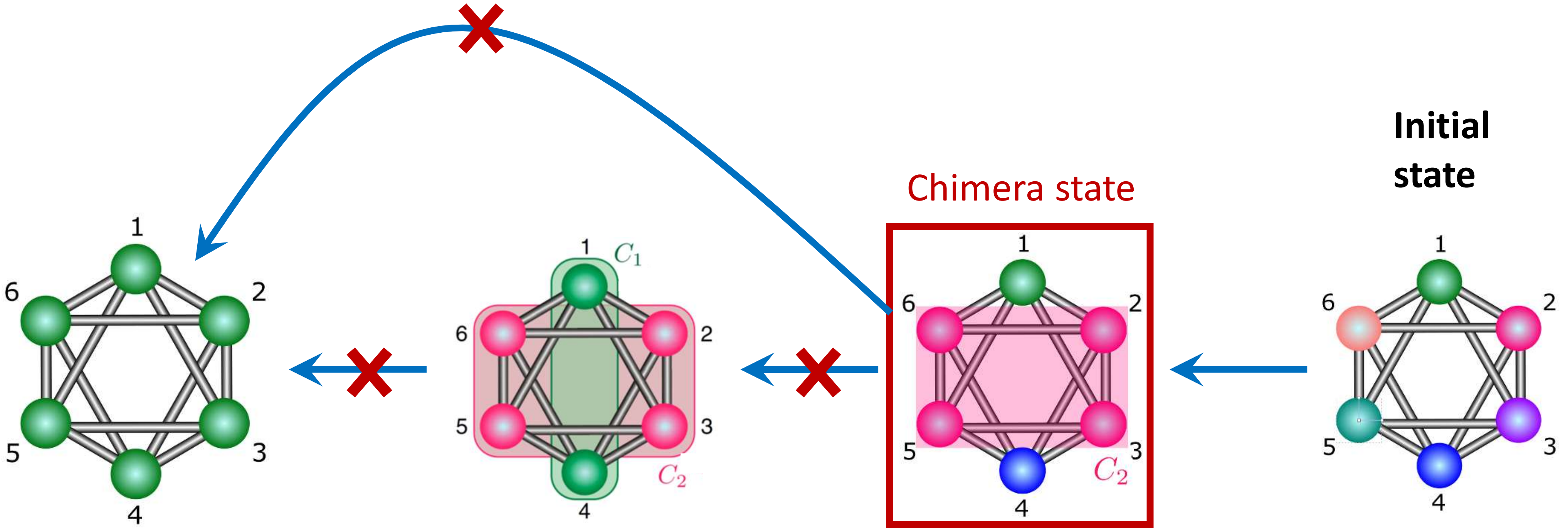
C_1, C_2

Independently synchronizable clusters



Stable chimera of finite size

Method



Initial state

Chimera state

C_2: stable

C_1: unstable

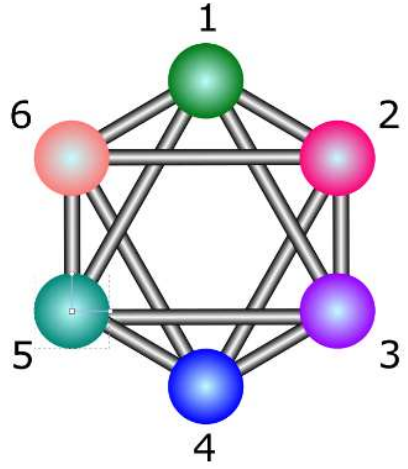
C_2: stable

Unstable

Stable chimera of finite size

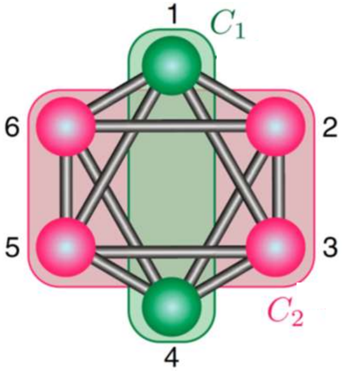
$$x_i(t+1) = \beta\left(\frac{1-\cos\left(x_i(t)\right)}{2}\right) + \delta + \sigma\sum_{j=1}^{N} A_{ij}\left(\frac{1-\cos(x_j(t))}{2}\right)$$

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$



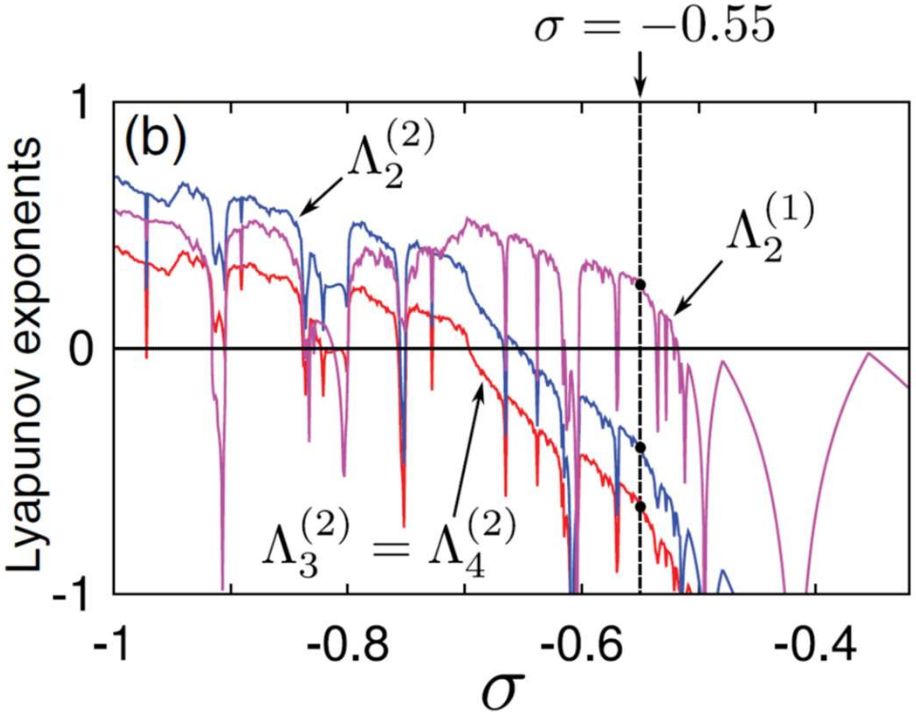
Stable chimera of finite size

$$\beta = \frac{2\pi}{3} - 4\sigma \qquad \delta = \frac{\pi}{6} \qquad \sigma = -0.55$$

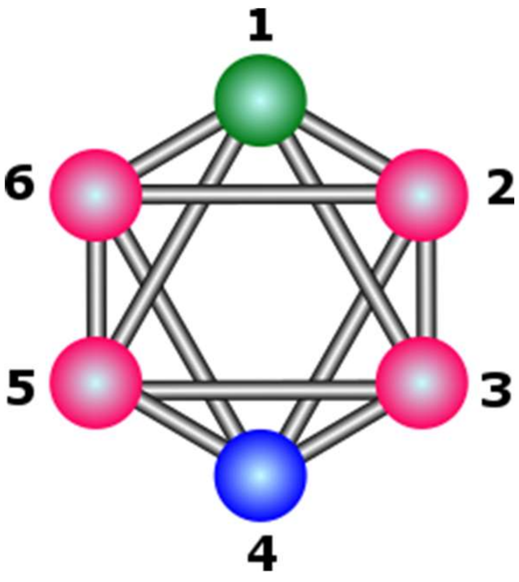


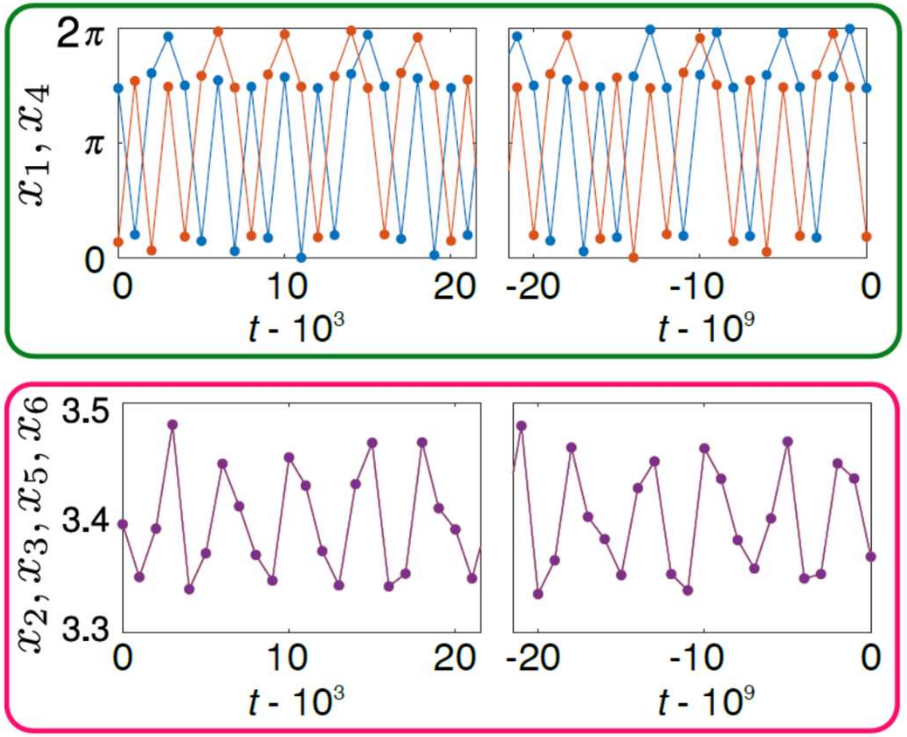
C_1: unstable

C_2: stable



Stable chimera of finite size





We check that the chimera state persists up to $t = 10^9$.

Cho *et al.* Phys. Rev. Lett. **119**, 084101 (2017)

Application 2

Nearly identical oscillators to reflect real systems

Nearly identical oscillators **with parameter mismatches**

parameter of each oscillator i

$|\boldsymbol{\mu}_i - \overline{\boldsymbol{\mu}}| \ll 1$ for $\forall_i$

$$\dot{\mathbf{x}}_i(t) = \mathbf{F}(\mathbf{x}_i(t), \boldsymbol{\mu}_i) + \sigma \sum_{j=1}^{N} A_{ij}\, \mathbf{H}\big(\mathbf{x}_j(t)\big)$$

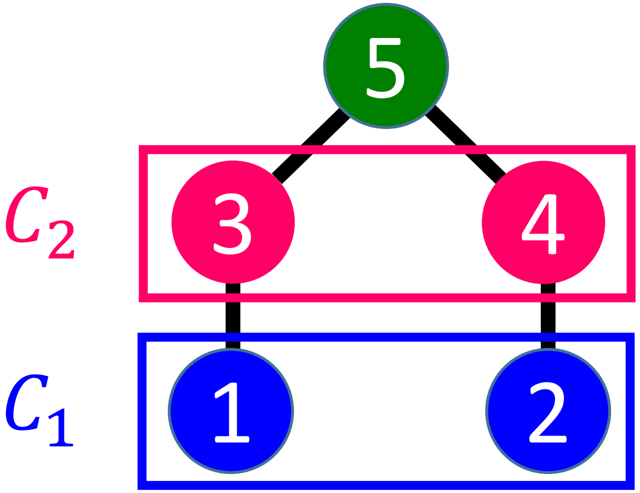
Concurrent formation of nearly synchronous clusters in each intertwined cluster set with parameter mismatches

General formalism is presented in

Y. S. Cho *et al.* PRE **99**, 052215 (2019)

Example

$$\dot{\mathbf{x}}_i(t) = \mathbf{F}(\mathbf{x}_i(t), a_i) + \sigma \sum_{j=1}^{N} A_{ij}\, \mathbf{H}\big(\mathbf{x}_j(t)\big)$$



$a_5 = a$

$a_3 = a \qquad a_4 = a$

$a_1 = a + \delta a \qquad a_2 = a - \delta a$

$|C_1, C_2|$: intertwined cluster set

Standard deviation of the states within each cluster C_m

$$\sigma_m(t) = \sqrt{\frac{1}{|C_m|} \sum_{i \in C_m} \|\delta \mathbf{x}_i(t)\|^2}$$

Standard error of each cluster C_m

$$\Omega_m = \frac{1}{T} \int_0^T \sigma_m(t) dt$$

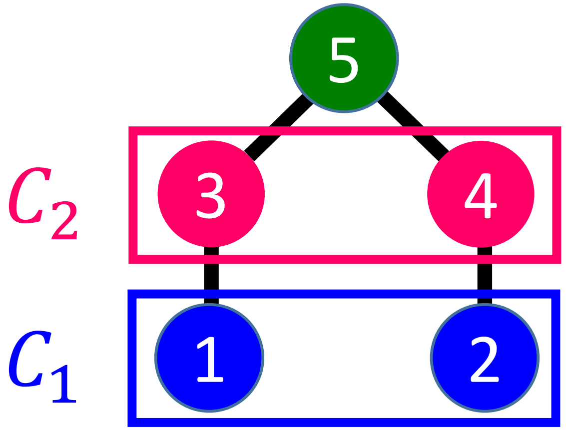
If $\Omega_m \ll 1$ → Nearly synchronous cluster

Otherwise → Asynchronous cluster

1. Concurrent formation of nearly synchronous clusters

General formalism is presented in

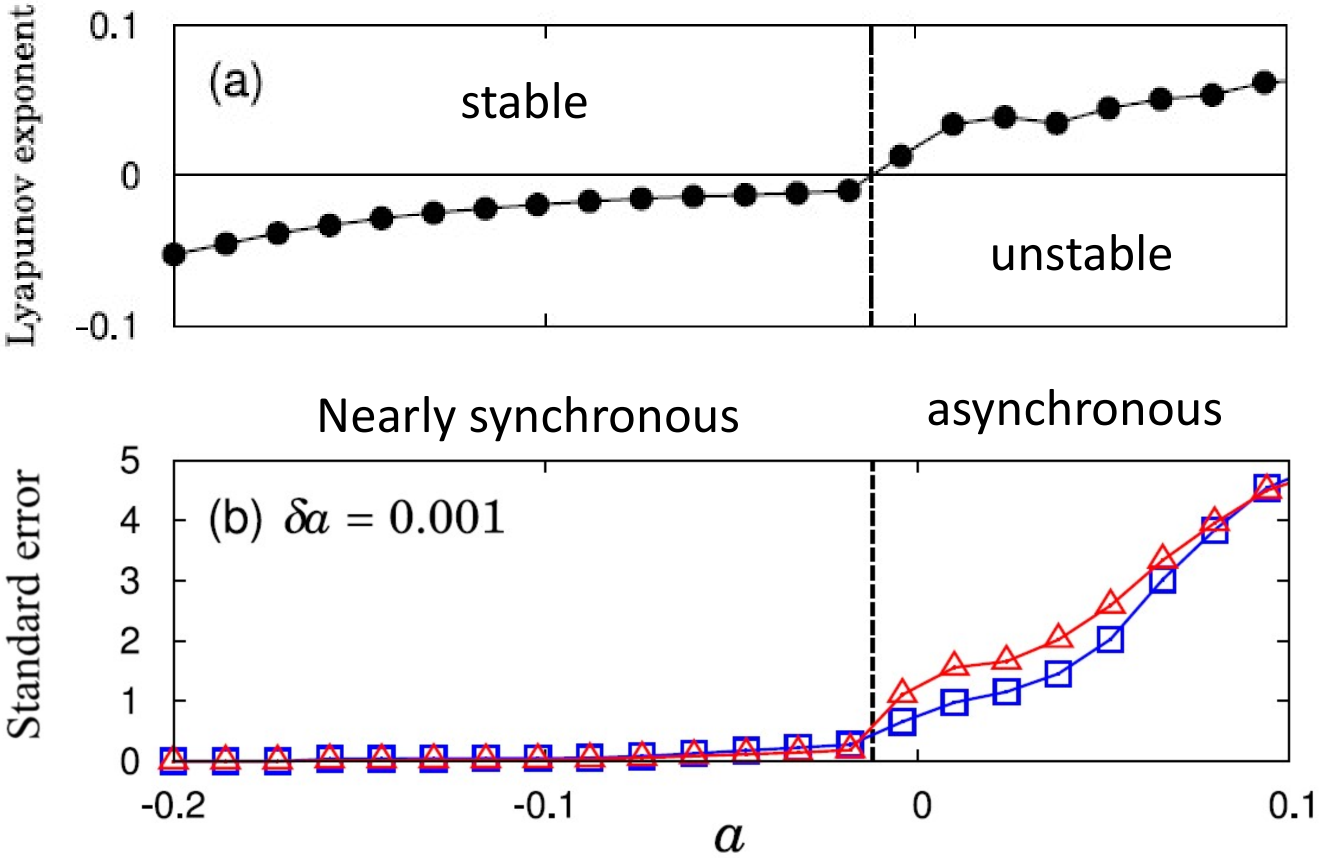
Y. S. Cho *et al.* PRE **99**, 052215 (2019)



$$a_5 = a$$

$$a_3 = a \qquad a_4 = a$$

$$a_1 = a + \delta a \qquad a_2 = a - \delta a$$

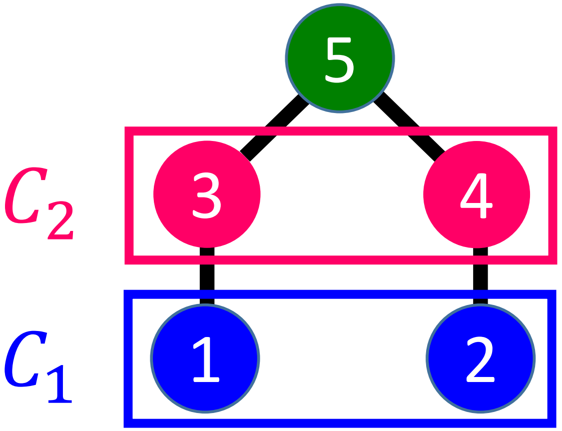


Ω_1: □

Ω_2: △

General formalism is presented in
Y. S. Cho *et al.* PRE **99**, 052215 (2019)

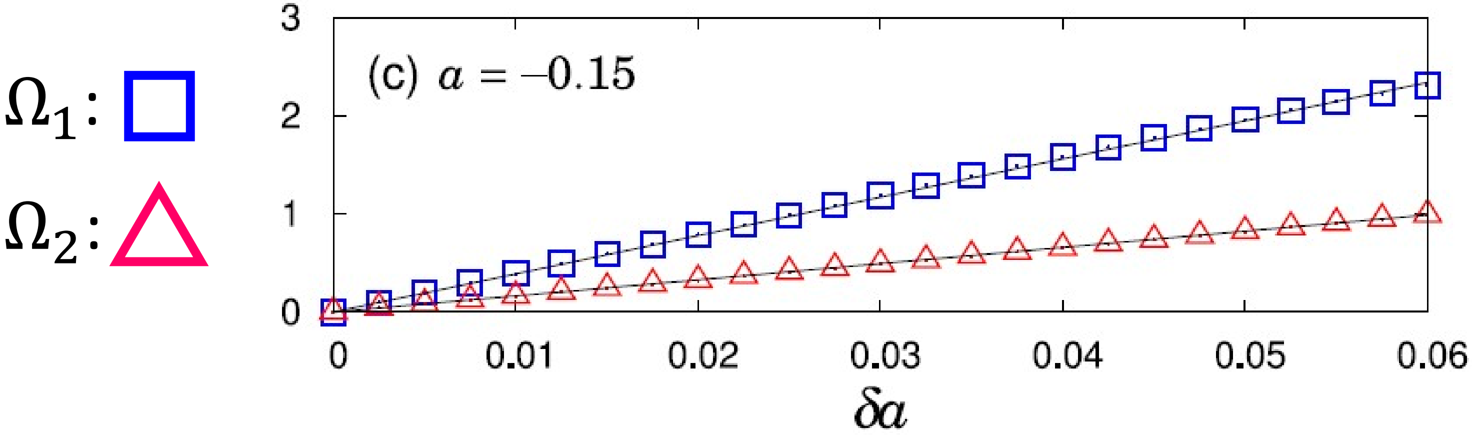
2. Standard error Ω_m of every cluster in the same set increases **linearly** with the magnitude of parameter mismatch of the single cluster.



$a_5 = a$

$a_3 = a \quad a_4 = a$

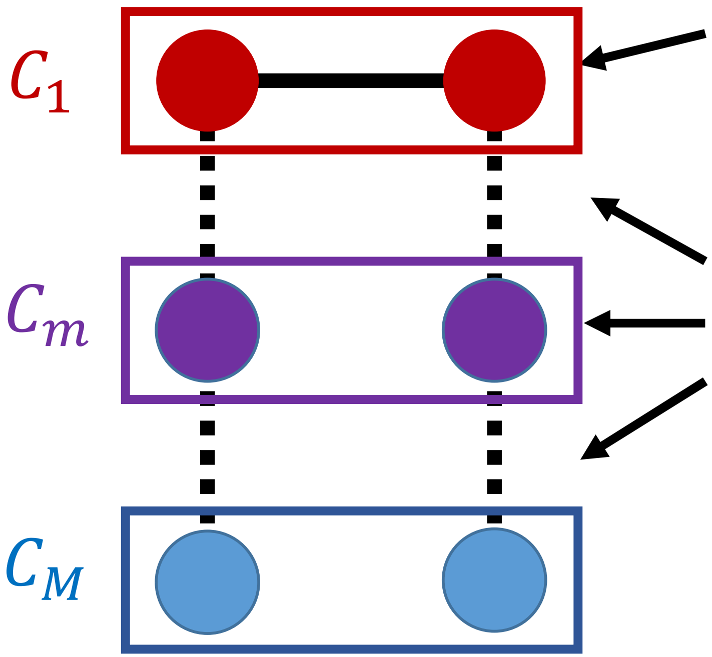
$a_1 = a + \delta a \quad a_2 = a - \delta a$



Extension

Intertwined cluster set of a large number of clusters

$$|C_1, \ldots, C_m, \ldots, C_M|$$

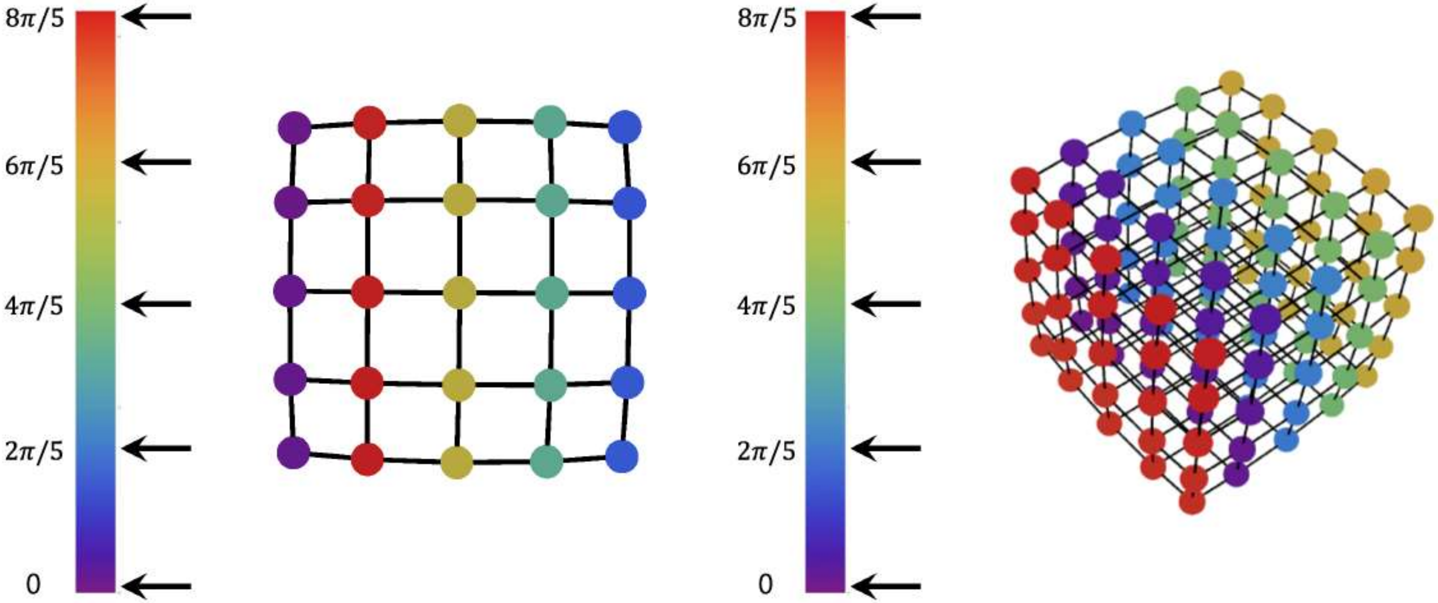


Only one cluster with parameter mismatch

can break **all the other clusters at the same time** by increasing parameter mismatch.

Application 3

Twisted states in low-dimensional hypercubic lattices



S. Lee *et al.* Phys. Rev. E **98**, 062221 (2018).

Kuramoto model

$$\dot{\phi}_i = \omega_i + K \sum_{j=1}^{N} A_{ij} \sin\left(\phi_j - \phi_i\right)$$

$$\phi_i \in [\mathbf{0}, \mathbf{2}\boldsymbol{\pi}) \quad i = 1,2, \dots, N$$

(1) Identical oscillators

$$\omega_i = \omega \text{ for } \forall_i$$

$$\dot{\phi}_i = \omega + K \sum_{j=1}^{N} A_{ij} \sin\left(\phi_j - \phi_i\right)$$

For a rotating frame $\phi_i \rightarrow \phi_i + \omega t$

$$\boxed{\dot{\phi}_i = K \sum_{j=1}^{N} A_{ij} \sin\left(\phi_j - \phi_i\right)}$$

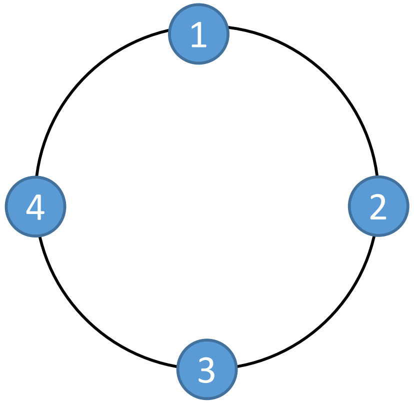
(1) Identical oscillators

in a **ring**

$$\dot{\phi}_i = K[\sin(\phi_{i+1} - \phi_i) + \sin(\phi_{i-1} - \phi_i)]$$

Periodic boundary condition

$\phi_{N+1} \equiv \phi_1 \quad \phi_0 \equiv \phi_N$



ring of $N = 4$

(1) Identical oscillators

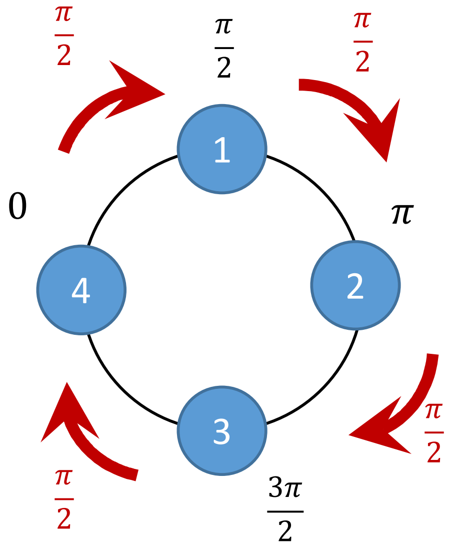
D. A. Wiley *et al.* Chaos **16**, 015103 (2006).

q-twisted state in a ring

$$\phi_i = \left[\frac{2\pi q}{N} i\right] \bmod 2\pi$$

(1) is fully phase-locked ($\dot{\phi}_i = 0$ for $\forall i$).

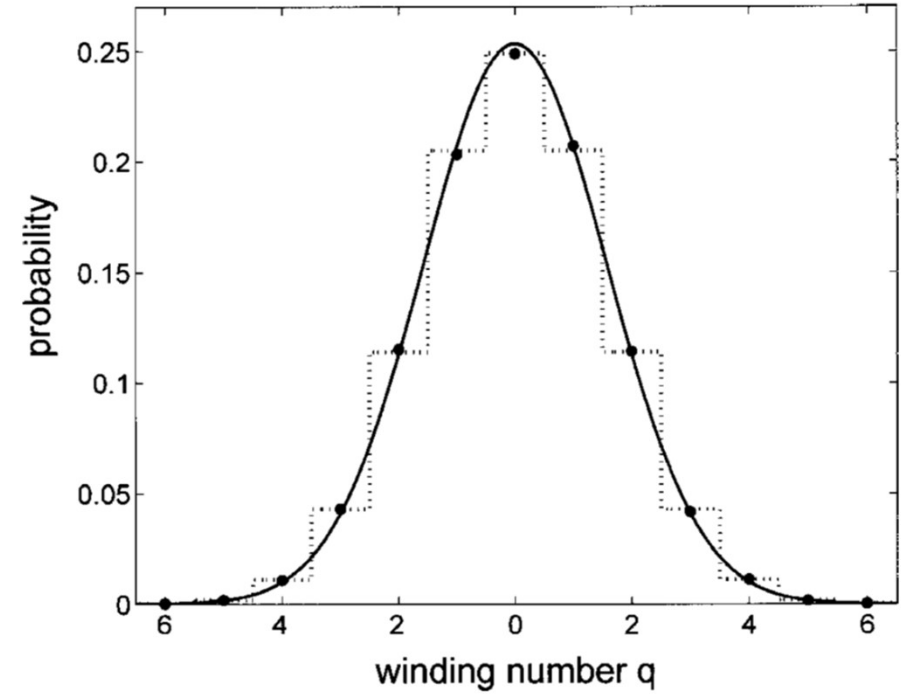
$q = 1$



(1) Identical oscillators

D. A. Wiley *et al.* Chaos **16**, 015103 (2006).

Basin stability



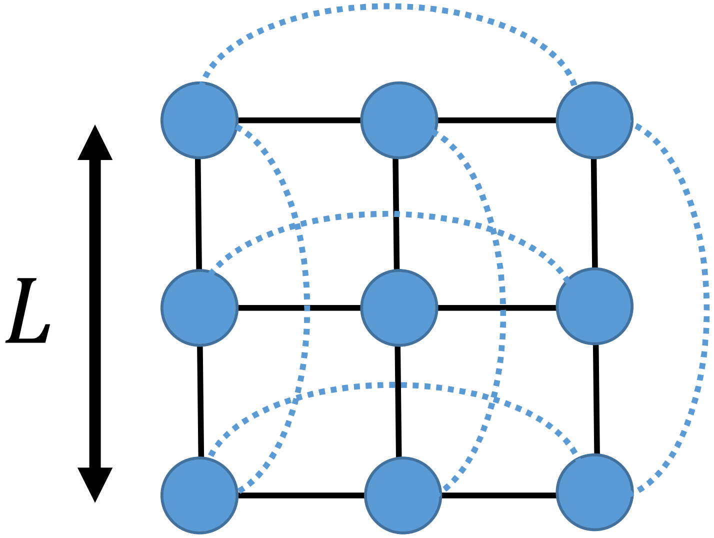
$N = 80$

(2) When $N > 4|q|$ → Stable

(1) Identical oscillators

S. Lee *et al.* Phys. Rev. E **98**, 062221 (2018).

We consider d-dimensional hypercubic lattice for
$d = 2$ (square lattice) and $d = 3$ (cubic lattice).



$N = L^d$

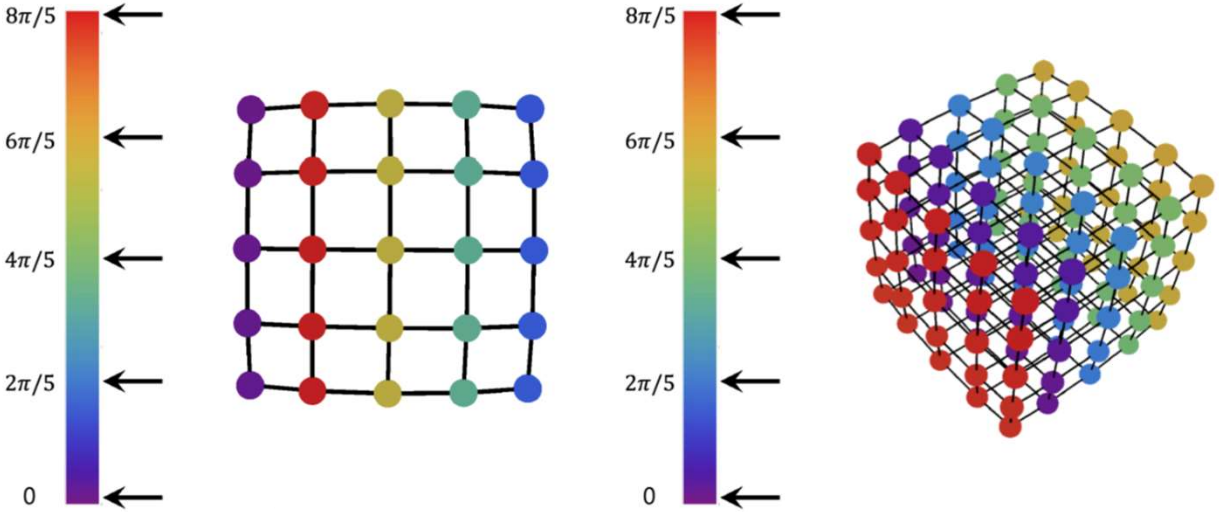
Periodic boundary condition

(1) Identical oscillators

S. Lee *et al.* Phys. Rev. E **98**, 062221 (2018).

Observation of q**-twisted state (in** d**-dimensional hypercubic lattices)**

$$\phi_i = \left(\frac{2\pi q}{L} \left\lfloor \frac{i-1}{L^{d-1}} \right\rfloor \right) \bmod 2\pi$$



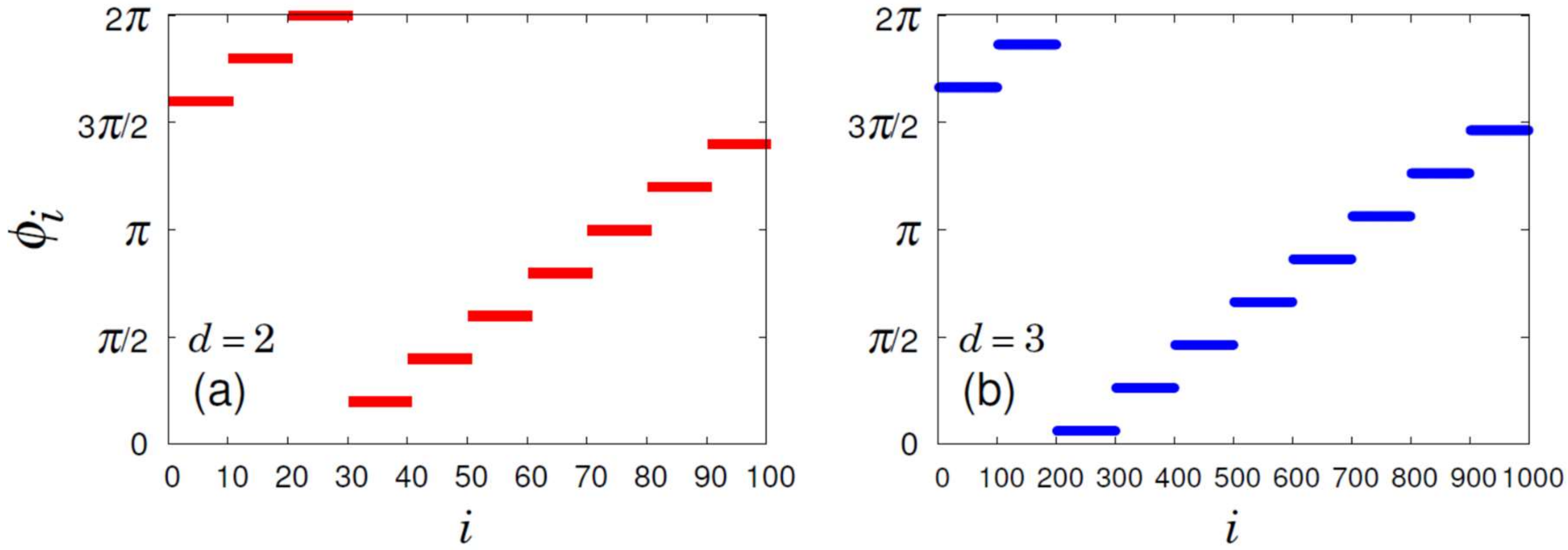
$(q = 1)$-twisted state

(1) Identical oscillators

S. Lee *et al.* Phys. Rev. E **98**, 062221 (2018).

Observation of **q-twisted state (in d-dimensional hypercubic lattices)**

$$\phi_i = \left(\frac{2\pi q}{L}\left\lfloor\frac{i-1}{L^{d-1}}\right\rfloor\right) \bmod 2\pi$$

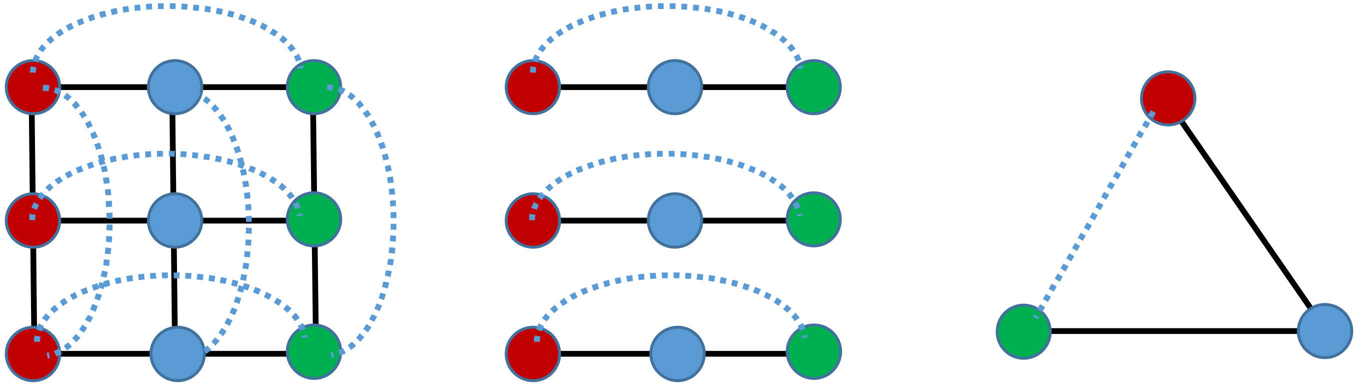


$(q = 1)$-twisted state

(1) Identical oscillators

S. Lee *et al.* Phys. Rev. E **98**, 062221 (2018).

q-twisted state (in d-dimensional hypercubic lattices)
is reduced into q-twisted state in a ring of length L.



$$L = 3$$

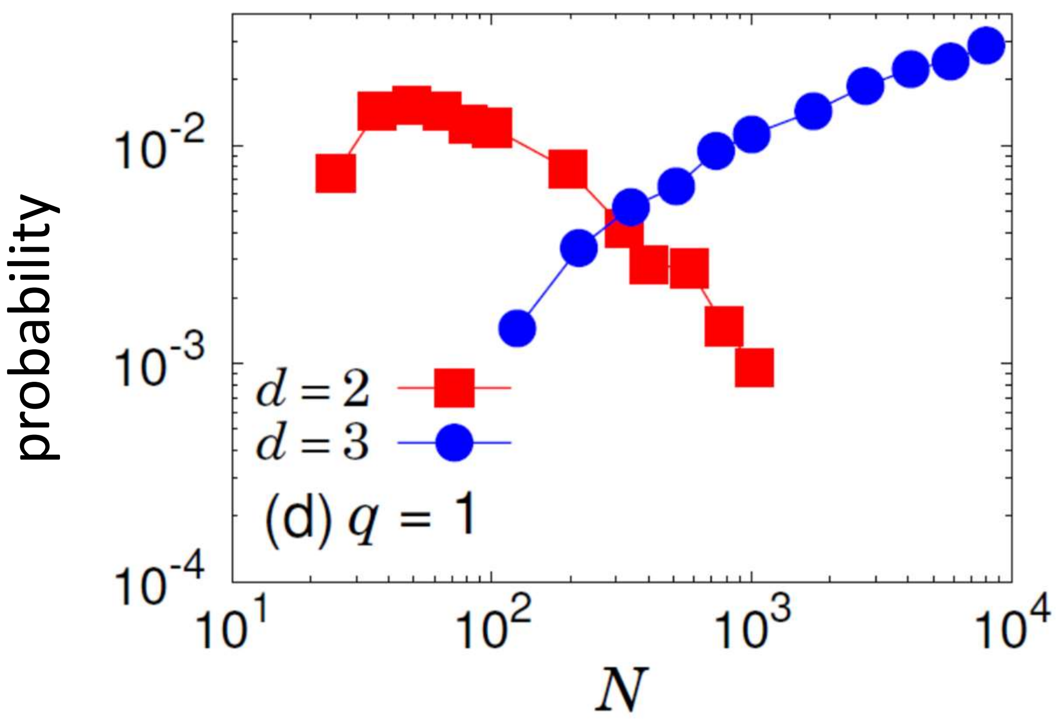
(1) Fully phase-locked ($\dot{\phi}_i = 0$ for $\forall i$).

(2) When $L > 4|q|$ → Stable

(1) Identical oscillators

S. Lee *et al.* Phys. Rev. E **98**, 062221 (2018).

Basin stability



We observe ($q = 1$)-twisted state for random configurations when $L > 4 = 4q$.

(2) Non-identical oscillators

S. Lee *et al.* Phys. Rev. E **98**, 062221 (2018).

$$\dot{\phi}_i = \omega_i + K \sum_{j=1}^{N} A_{ij} \sin(\phi_j - \phi_i)$$

Heterogeneous natural frequencies of Gaussian distribution

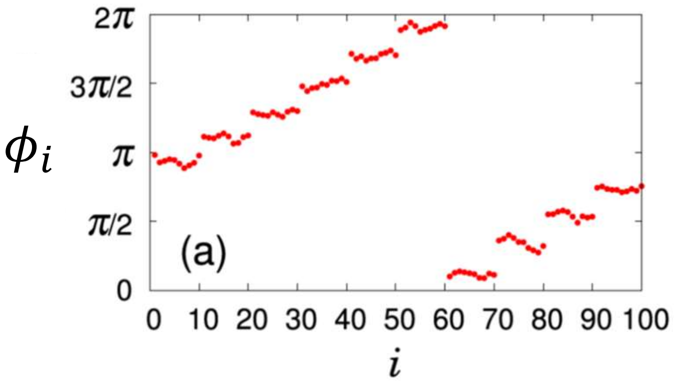
$$g(\omega) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\omega^2/2\sigma^2}$$

(2) Non-identical oscillators

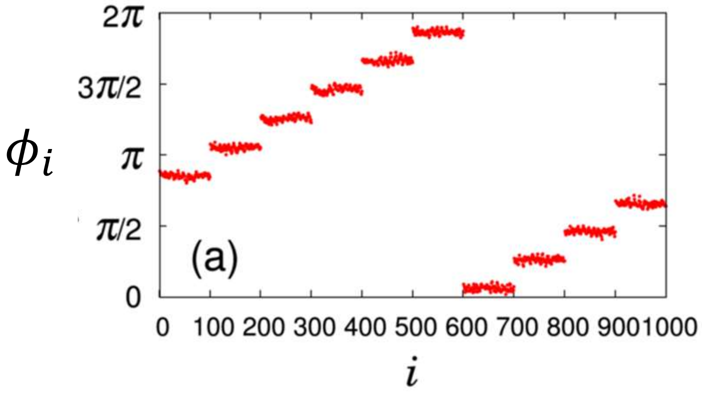
S. Lee *et al.* Phys. Rev. E **98**, 062221 (2018).

$$\sigma = 1$$

$d = 2$



$d = 3$

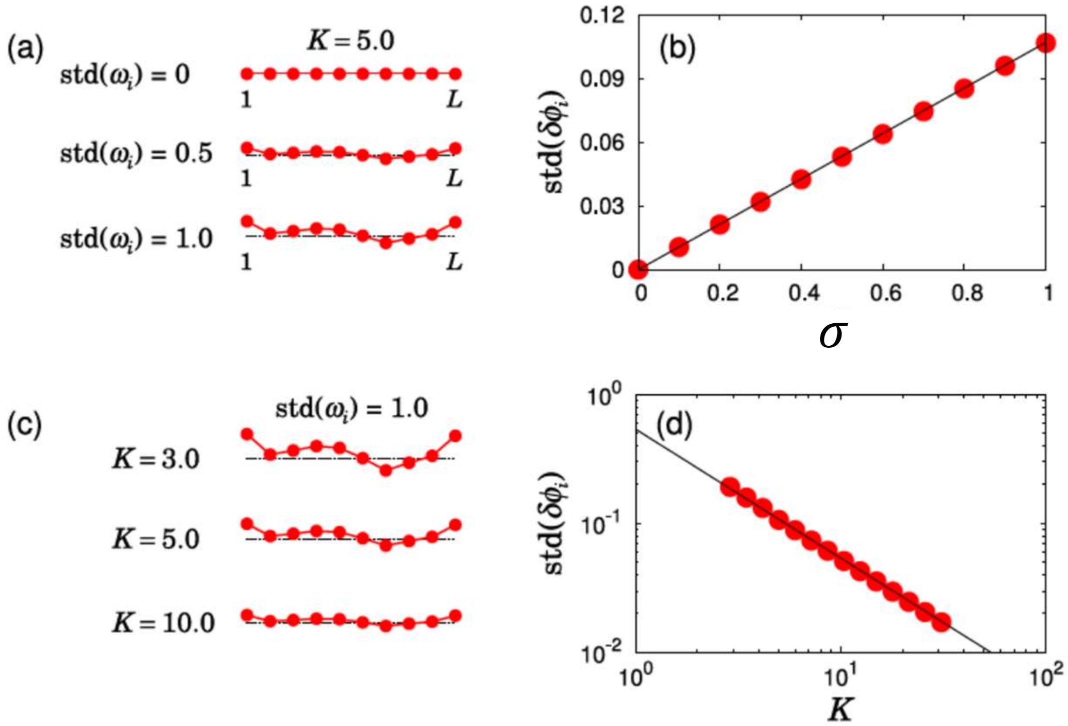


$$L = 10$$

(2) Non-identical oscillators

S. Lee *et al.* Phys. Rev. E **98**, 062221 (2018).

Roughness: $\sqrt{\left\langle \delta\phi_i{}^2 \right\rangle} \propto \frac{\sigma}{K}$



(2) Non-identical oscillators

S. Lee *et al.* Phys. Rev. E **98**, 062221 (2018).

Measurement of basin stability

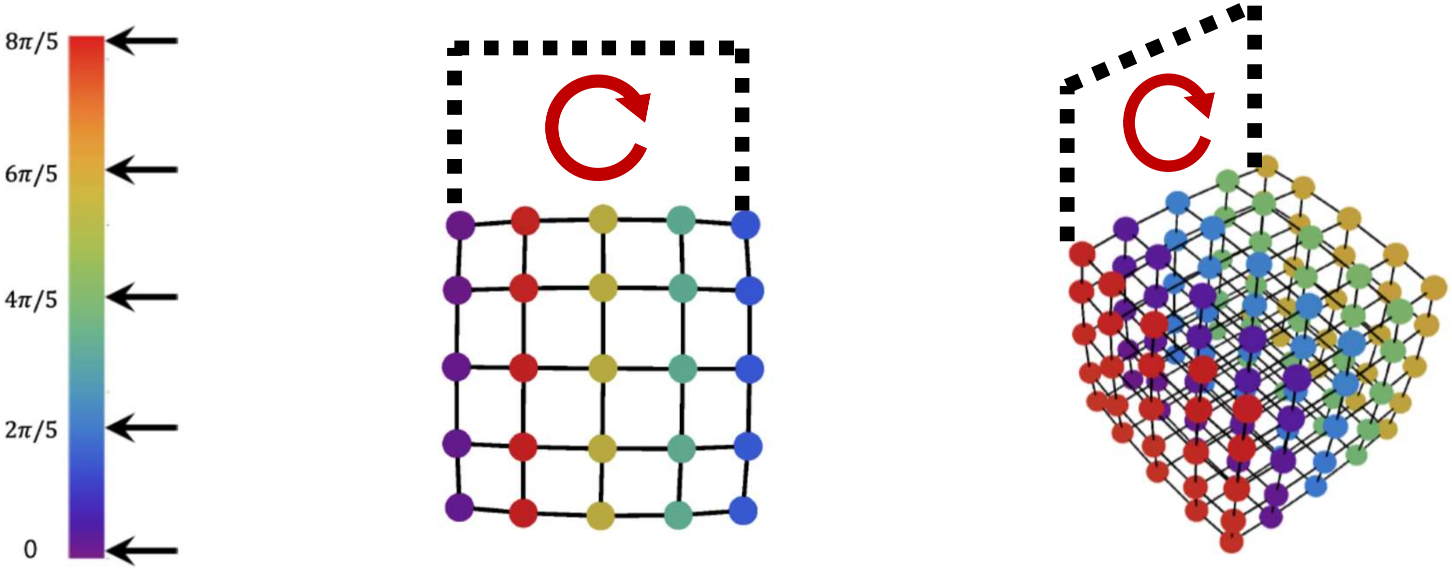
:How much fraction of random initial conditions induce such states?

To do this, we should define the states clearly.

(2) Non-identical oscillators

S. Lee *et al.* Phys. Rev. E **98**, 062221 (2018).

Winding number: 1

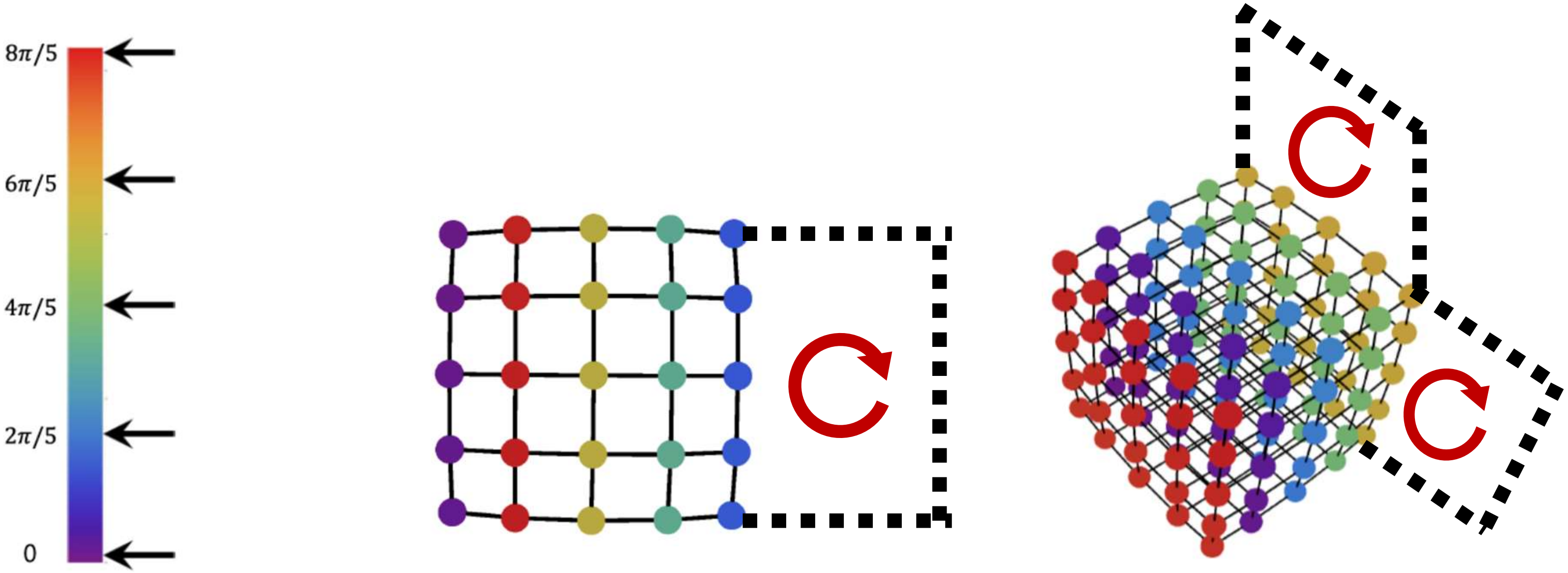


Periodic boundary condition

(2) Non-identical oscillators

S. Lee *et al.* Phys. Rev. E **98**, 062221 (2018).

Winding number : 0

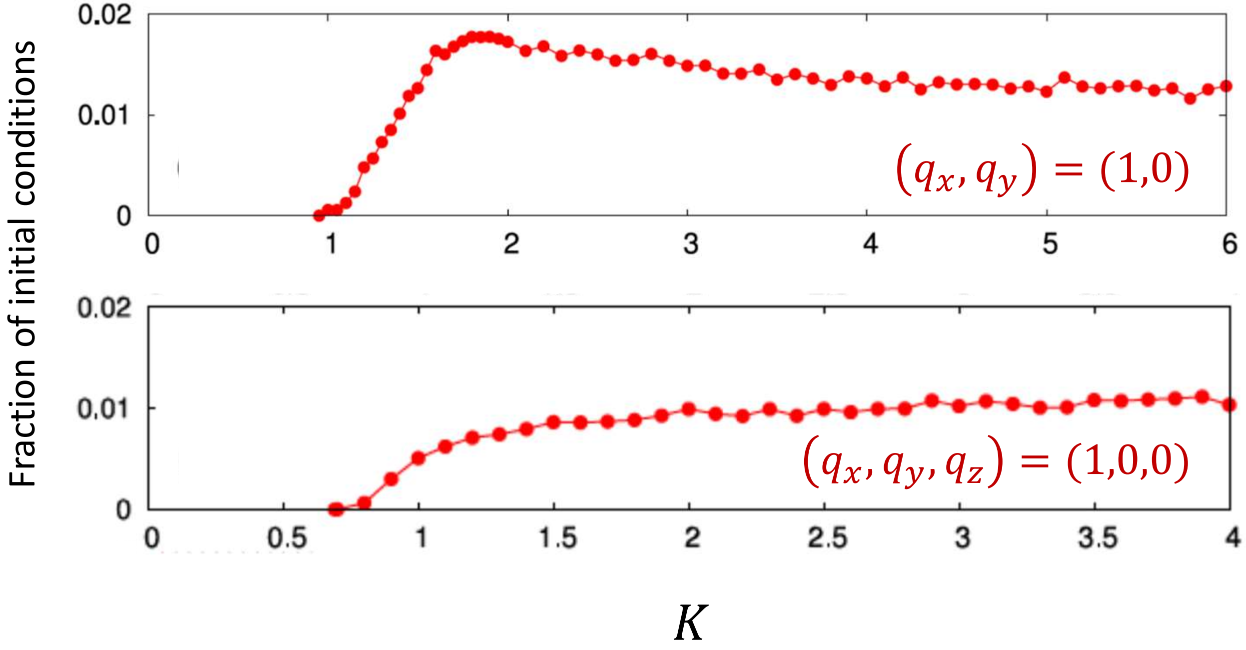


Periodic boundary condition

(2) Non-identical oscillators

S. Lee *et al.* Phys. Rev. E **98**, 062221 (2018).

Basin stability

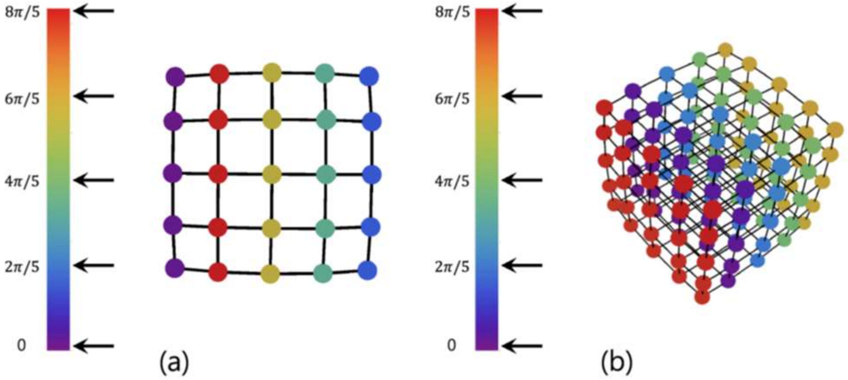


These states appear in random configurations for sufficiently large K.

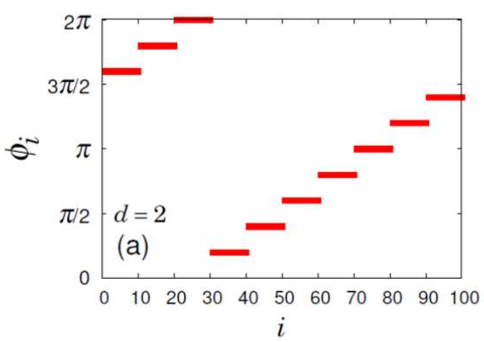
Summary

S. Lee *et al.* Phys. Rev. E **98**, 062221 (2018).

Observation of twisted states in (d=2) square and (d=3) cubic lattices **for random configurations**



Identical oscillators



Non-identical oscillators