On 2d CFTs and Cvitanovich-Deligne Series of Exceptional Lie Groups

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"A man who is tired of group theory is a man who is tired of life." – Sidney Coleman

Conformal Field Theory

Quantum Field Theory

Theory of Identical Particles

Special Relativity

Quantum Mechanics

The theoretical foundation of

the Standard Model of elementary particles and forces

perturbative approach: free particles + small interactions

non-perturbative effect: chiral condensation and confinement in QCD

Maxwell Theory

Lagrangian: $\mathscr{L} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2e^2}(E_i^2 - B_i^2)$

Symmetries:

Poincare Symmetry: Lorentz Symmetry+ ST translation

Discrete Symmetries: parity, time-reversal, charge conjugation

Gauge symmetry

Electro-Magnetic Duality: $(E, B) \rightarrow (B, -E)$

Maxwell Theory

Additional Symmetries:

electric and magnetic 1-form symmetries

$$\partial_\mu F^{\mu\nu} = 0,\ \partial_\mu {}^*F^{\mu\nu} = 0, \quad {}^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$$

conformal symmetry:

$$x^\mu \to \frac{x^\mu - a^\mu x^2}{1 - 2a\cdot x + a^2 x^2} : \quad x^\mu \to \frac{x^\mu}{x^2} - a^\mu$$

Noether

Symmetry leads to the Conserved Charge

Conserved Charge leads to the Symmetry Generator:

Poincare + Conformal Symmetry

$$P_\mu,\ M_{\mu\nu},\ D,\ K_\mu$$

$$[P_\mu, P_\nu] = 0,\ [K_\mu, K_\nu] = 0$$

$$[D, P_\mu] = P_\mu,\ [D, K_\mu] = -K_\mu$$

$$[K_\mu, P_\nu] = \eta_{\mu\nu} D - iM_{\mu\nu}$$

2d Conformal Field Theory

2d Conformal Symmetry

Euclidean Space-time

infinite dimensional: $\ell_n = z^{n+1}\partial_z,\ \bar{\ell}_n = \bar{z}^{n+1}\partial_{\bar{z}}$

Quantization

Virasoro Algebra of $L_m, \bar{L}_m,\ m \in \mathbb{Z}$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

c: central charge

Representation Theory

Highest Weight state $|h\rangle$: $L_{n>0}|h\rangle = 0,\ L_0|h\rangle = h|h\rangle$

Virasoro Descendants: $L_{-n_1}L_{-n_2}\cdots L_{-n_k}|h\rangle$

$$0 < n_1 \leq n_2 \leq \cdots \leq n_k,\ k \geq 0$$

Rational: finite number of primary states

For $0 < c < 1$, the minimal models are only possible.

$$c = 1 - \frac{6(p-q)^2}{pq} \text{ with } p > q \geq 2 \text{ coprime}$$

Representation Theory

Rational Unitary Representation: $p = m + 1,\ q = m,\ m = 2,3,...$

$$c = 1 - \frac{6}{m(m+1)} = \left\{ 0, \frac{1}{2}, \frac{7}{10}, \frac{4}{5}, \cdots \right\}$$

$$h = h_{r,s}(c) + \frac{((m+1)r - ms)^2 - 1}{4m(m+1}, \ r = 1,2...,m-1,\ s = 1,2,..,r$$

Ising model: $c = \frac{1}{2},\ \ h = 0, \frac{1}{2}, \frac{1}{16}$

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Operator-State correspondence

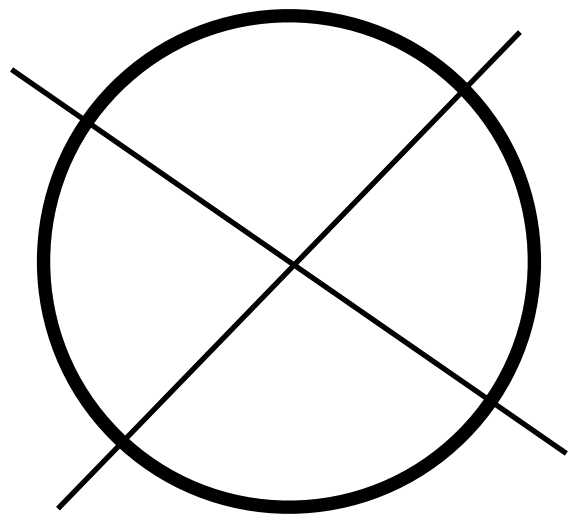
Conformal Field Theory:

Radial quantization

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\varphi^2$$

$$r = e^\tau, ds^2 = e^{2\tau}(d\tau^2 + d\varphi^2)$$

Operator at the origin of R^2 = the in-state at $\tau = -\infty$



Primary operator $\phi(z)$ and primary state $|\phi\rangle = \phi(0)|0\rangle$

Ising Model: $1,\ \epsilon_{\frac{1}{2}\frac{1}{2}}(z,\bar{z}),\ \sigma_{\frac{1}{16}\frac{1}{16}}(z,\bar{z})$

Hilbert space of Ising Model

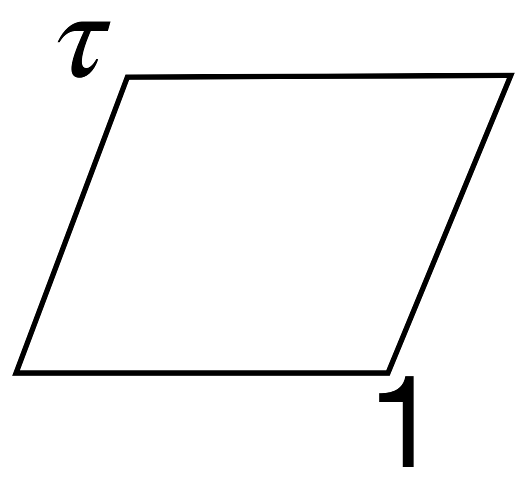
$$\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_\epsilon \oplus \mathscr{H}_\sigma$$

Hilbert space is a product of Hilbert space of two Virasoro algebras L_n, $\bar{L}_n$

Consider only the chiral part: L_n with c and h

The character of a primary operator or state:

$$\chi_h = \mathrm{Tr}_{\mathscr{H}_h}\, q^{L_0 - c/24},\ q = e^{2\pi i \tau}$$



Ising model:

$$\chi_0 = \frac{\sqrt{\theta_3} + \sqrt{\theta_4}}{2\sqrt{\eta}}, \quad \chi_{\frac{1}{2}} = \frac{\sqrt{\theta_3} - \sqrt{\theta_4}}{2\sqrt{\eta}}, \chi_{\frac{1}{16}} = \frac{\sqrt{\theta_2}}{2\sqrt{\eta}}$$

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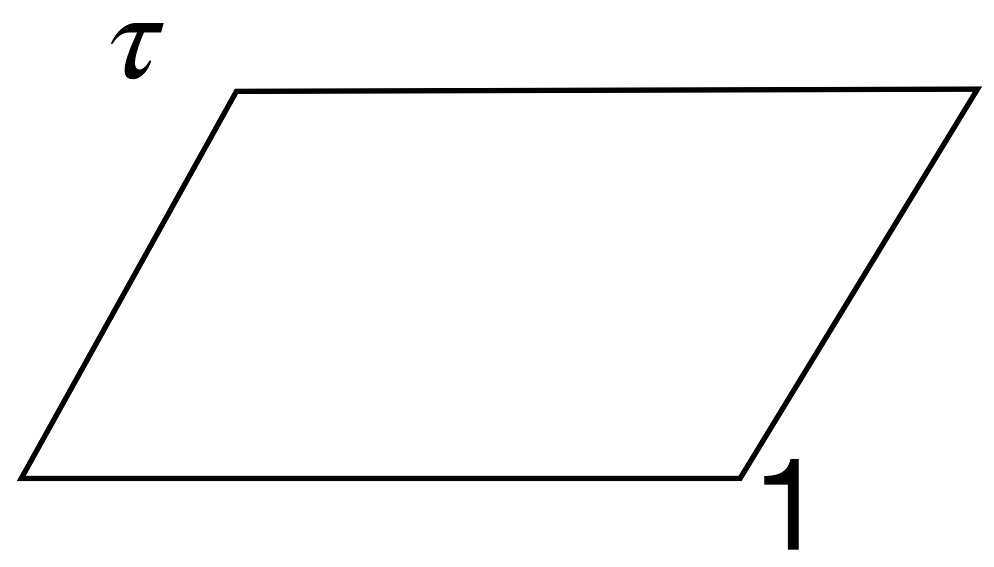

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WZW model

Lie algebra G

level k

central charge $c = \dfrac{kd_G}{k + h_G^\vee}$: $h_G^\vee$ dual Coxeter number

conformal weights: $h = \dfrac{j(j+2)}{4(k+2)}, \, j = 0,1,...k$ for $SU(2)$

Modular Property

Rational Conformal Field Theory:

central charge c, weight $h_i, i = 0,\ldots,r$

characters: $\chi_i(\tau) = \mathrm{Tr}_{\mathscr{H}_i} q^{L_0 - c/24} = q^{-c/24+h_i}(a_0 + a_1 q + a_2 q^2 + \cdots)$

$a_i \in \mathbb{Z}_+$

Under the $SL(2,Z)$ transform $\tau \to \tau' = \dfrac{a\tau + b}{c\tau + d}$,

$\chi_i(\tau) \to \chi_i(\tau') = S_{ij}\chi_j(\tau)$:

weight zero vector valued modular forms

Modular Linear Differential Equation

For rank two RCFT, the character χ_0, χ_1 satisfies a quadratic MLDE.

$$(D^2 + \mu E_4)\chi = 0, \;\; D = q\partial_q - \frac{n}{12}E_2$$

$E_{2k}(\tau)$: Eisenstein series of weight 2k

Solutions with non-negative coefficients: Mathur-Mukhi-Sen 1977

LY	SU(2)	SU(3)	G2	SO(8)	F4	E6	E7	?	E8
2/5	1	2	14/5	4	26/5	6	7	38/5	8

Table 1

l	μ	m_1	c	h	Identification
96	$\frac{11}{900}$	1	$\frac{2}{5}$	$\frac{1}{5}$	$c=-\frac{22}{5}$ minimal model ($c \leftrightarrow c-24$
90	$\frac{5}{144}$	3	1	$\frac{1}{4}$	$k=1$ SU(2) WZW model
80	$\frac{1}{12}$	8	2	$\frac{1}{3}$	$k=1$ SU(3) WZW model
72	$\frac{119}{900}$	14	$\frac{14}{5}$	$\frac{2}{5}$	$k=1$ G_2 WZW model
60	$\frac{2}{9}$	28	4	$\frac{1}{2}$	$k=1$ SO(8) WZW model
48	$\frac{299}{900}$	52	$\frac{26}{5}$	$\frac{3}{5}$	$k=1$ F_4 WZW model
40	$\frac{5}{12}$	78	6	$\frac{2}{3}$	$k=1$ E_6 WZW model
30	$\frac{77}{144}$	133	7	$\frac{3}{4}$	$k=1$ E_7 WZW model
24	$\frac{551}{900}$	190	$\frac{38}{5}$	$\frac{4}{5}$	?
20	$\frac{2}{3}$	248	8	$\frac{5}{6}$	$\supset k=1$ E_8 WZW model

Deligne Series
of
Exceptional Lie Groups

Deligne Exceptional Series

classical lie algebra: $SU(N)$, $SO(N)$, $Sp(N) = USp(2N)$

Deligne Series: $SU(2)$, $SU(3)$, G_2, $SO(8)$, F_4, E_6, E_7, E_8

Dual Coxeter number: $h_G^\vee$

Single instanton zero modes: $4h_G^\vee$

SU(N)	SO(N)	Sp(N)	E6	E7	E8	F4	G2
N	N-2	N+1	12	18	30	9	4

Cvitanovic- Exceptional Series

Dual Coxeter number: $h_G^\vee$

parameter: $\lambda = -6/h_G^\vee$

dimension of group: $d_\theta = \dfrac{2(h^\vee + 1)(5h^\vee - 6)}{h^\vee + 6}$

$$d_{2\theta} = \frac{5(h^\vee)^2(2h^\vee + 3)(5h^\vee - 6)}{(h^\vee + 6)(h^\vee + 12)}$$

For $h^\vee = 24$, the above dimension becomes $d_\theta = 190$, $d_{2\theta} = 15504$

$$E_7 \subset E_{7+\frac{1}{2}} \subset E_8$$

248= **190**+**57**+**1** of $E_{7+1/2}$

190=**133**+56+1, 57=56+1 of E_7

$$[D^2 - \frac{55}{3600}E_4]\chi = 0$$

$$\chi_0 = q^{-\frac{19}{60}}(1 + 190q + 2831q^2 + 22306q^3 + 129276q^4 + \dots),$$
$$\chi_{\frac{4}{5}} = q^{\frac{29}{60}}(57 + 1102q + 9367q^2 + 57362q^3 + 280459q^4 + \dots).$$

RCFT with c=38/5, h=0,4/5

$$(E_{7+1/2})_1 = \frac{(E_7)_1}{M(5,3)} = (E_7)_1 \otimes M_{\rm eff}(5,3).$$

$$E_7 \subset E_{7+\frac{1}{2}} \subset E_8$$

Hecke Image of M(5,2) (Harvey-Wu 18)

$$(E_{7+1/2})_1 = \mathrm{T}_{19}\, M_{\mathrm{eff}}(5,2)$$

The characters can be written in terms of those of LY

$$\chi_0 = \phi_1^{19} + 171\phi_1^{14}\phi_2^5 + 247\phi_1^9\phi_2^{10} - 57\phi_1^4\phi_2^{15},$$
$$\chi_{\frac{4}{5}} = \phi_2^{19} - 171\phi_2^{14}\phi_1^5 + 247\phi_2^9\phi_1^{10} + 57\phi_2^4\phi_1^{15}, \text{ with}$$

$$\phi_1 = q^{-\frac{1}{60}} \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}, \qquad \phi_2 = q^{\frac{11}{60}} \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}.$$

$E_{7+\frac{1}{2}}$

Rank 2: Duan, Lee, Sun 2022

$$(E_{7+1/2})_2 = \mathrm{T}_{19}\ M_{\mathrm{eff}}(13,2)$$

central charge c= 190/23

weights: h= 0, 10/13, 12/13,18/13, 19/13, 21/13

$$\chi_0 = q^{-\frac{95}{156}}(1 + 190q + 18335q^2 + 448210q^3 + 6264585q^4 + \dots),$$
$$\chi_{\frac{10}{13}} = q^{\frac{25}{156}}(57 + 10830q + 321575q^2 + 4979330q^3 + 53025295q^4 + \dots),$$
$$\chi_{\frac{12}{13}} = q^{\frac{49}{156}}(190 + 20596q + 537890q^2 + 7761500q^3 + 79066030q^4 + \dots),$$
$$\chi_{\frac{18}{13}} = q^{\frac{121}{156}}(1045 + 48070q + 910955q^2 + 10983690q^3 + 99272435q^4 + \dots),$$
$$\chi_{\frac{19}{13}} = q^{\frac{133}{156}}(2640 + 109155q + 1979610q^2 + 23245740q^3 + 206319480q^4 + \dots),$$
$$\chi_{\frac{21}{13}} = q^{\frac{157}{156}}(1520 + 51395q + 860890q^2 + 9606457q^3 + 82347710q^4 + \dots).$$

$E_{7+\frac{1}{2}}$

Possible only up to rank $1 \leq k \leq 5$

Weyl dimension formula is known partially

Most of higher weight representations are not known yet

$k = -5$ theory is related to 4d N=2 theory

Additional surprises on the dimension of fermionic representations

Conclusion

Freudenthal Magic Square

Freudenthal, Tits, Vinberg, (54~64) Cvitanovich

A \ B	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	A_1	A_2	C_3	F_4
$\mathbb{C}$	A_2	$A_2 \times A_2$	A_5	E_6
$\mathbb{H}$	C_3	A_5	D_6	E_7
$\mathbb{O}$	F_4	E_6	E_7	E_8

New Magic Square

Borsten and Marrani 2017, KL and Sun Work in Progress

	0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{T}$	$\mathbb{H}$	$\mathbb{S}$	$\mathbb{O}$	series
0								
R	0	$\mathfrak{a}_1$	$\mathfrak{a}_2$	$\mathfrak{a}_{2\frac{1}{4}}$	$\mathfrak{c}_3^{(21,4)}$	$\mathfrak{c}_{3\frac{1}{2}}^{(36,13/2)}$	$\mathfrak{f}_4^{(52,9)}$	
$\mathbb{C}$	0	$\mathfrak{a}_2$	$\mathfrak{a}_2 \oplus \mathfrak{a}_2$	$[\mathfrak{a}_2 \oplus \mathfrak{a}_2]_{\frac{1}{4}}$	$\mathfrak{a}_5^{(35,6)}$	$\mathfrak{a}_{5\frac{1}{2}}^{(56,9)}$	$\mathfrak{e}_6^{(78,12)}$	Severi
$\mathbb{T}$		$\mathfrak{a}_{2\frac{1}{4}}$	$[\mathfrak{a}_2 \oplus \mathfrak{a}_2]_{\frac{1}{4}}$	$[\mathfrak{a}_2 \oplus \mathfrak{a}_2]_{\frac{1}{4}+\frac{1}{4}}$	$\mathfrak{a}_{5\frac{1}{4}}^{(45,22/3)}$	$\mathfrak{a}_{5(\frac{1}{4}+\frac{1}{2})}^{(70,32/3)}$	$\mathfrak{e}_{6\frac{1}{2}}^{(96,14)}$	subsub
$\mathbb{H}$		$\mathfrak{c}_3^{(21,4)}$	$\mathfrak{a}_5^{(35,6)}$	$\mathfrak{a}_{5\frac{1}{4}}^{(45,22/3)}$	$\mathfrak{d}_6^{(66,10)}$	$\mathfrak{d}_{6\frac{1}{2}}^{(99,14)}$	$\mathfrak{e}_7^{(133,18)}$	subexceptional
$\mathbb{S}$	$AD_{3\frac{1}{2}}^{(18,4)}$	$\mathfrak{c}_{3\frac{1}{2}}^{(36,13/2)}$	$\mathfrak{a}_{5\frac{1}{2}}^{(56,9)}$	$\mathfrak{a}_{5(\frac{1}{4}+\frac{1}{2})}^{(70,32/3)}$	$\mathfrak{d}_{6\frac{1}{2}}^{(99,14)}$	$\mathfrak{d}_{6(\frac{1}{2}+\frac{1}{2})}^{(144,19)}$	$\mathfrak{e}_{7\frac{1}{2}}^{(190,24)}$	IES
$\mathbb{O}$	$\mathfrak{d}_4$	$\mathfrak{f}_4^{(52,9)}$	$\mathfrak{e}_6^{(78,12)}$	$\mathfrak{e}_{6\frac{1}{2}}^{(96,14)}$	$\mathfrak{e}_7^{(133,18)}$	$\mathfrak{e}_{7\frac{1}{2}}^{(190,24)}$	$\mathfrak{e}_8^{(248,30)}$	Deligne-C

Table 6. 1707.02072