Euclidean wormholes and the origin of alpha vacua

Wei-Chen Lin (Pusan National University)
Center of the Cosmological Constant Problem

In collaboration with: Pisin Chen (NTU), Kuan-Nan Lin (NTU), Dong-han Yeom (PNU) (in preparation)

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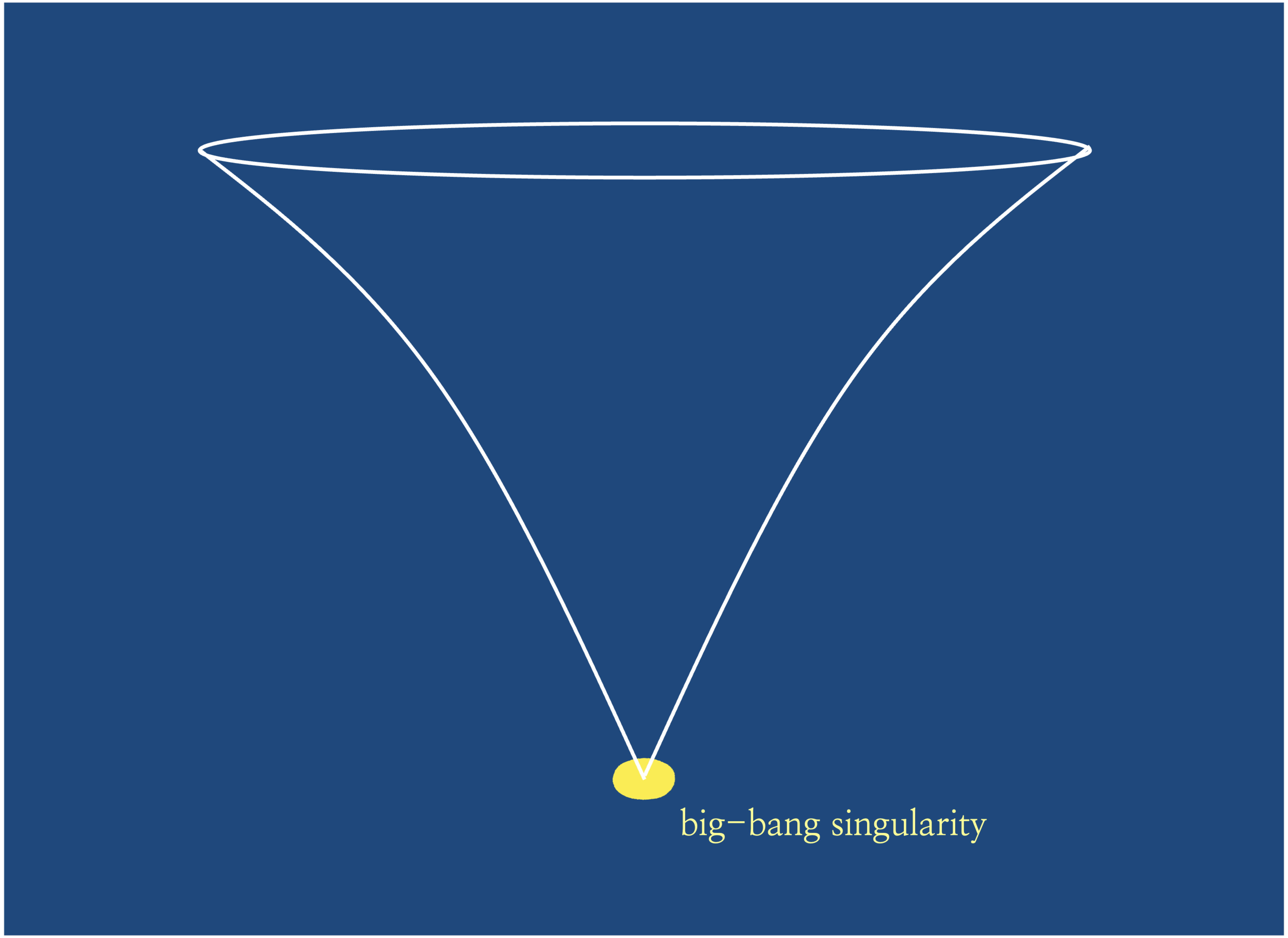
Outline

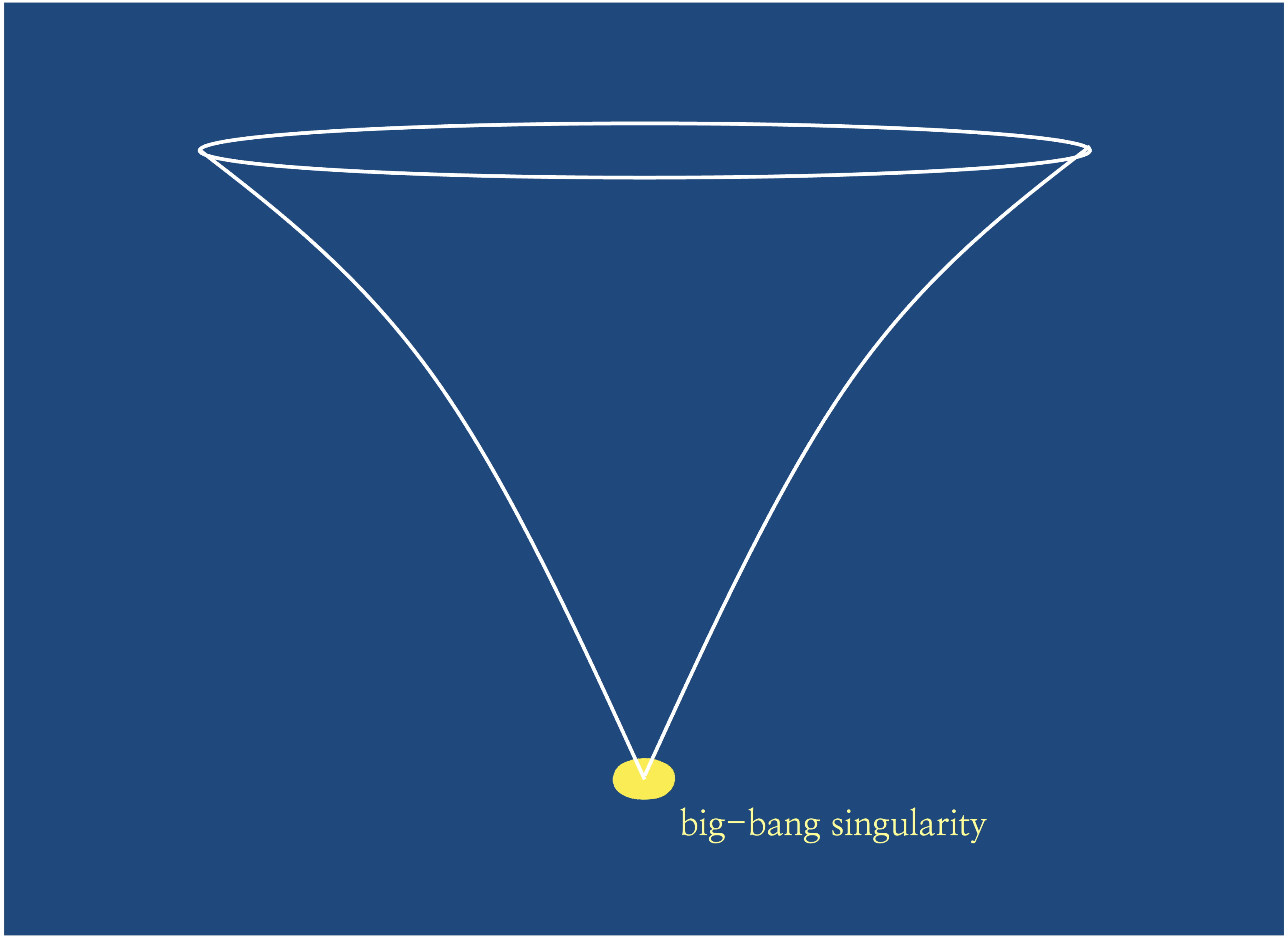
- Quantum cosmology (Euclidean path integral approach & No Boundary Proposal)
- Euclidean wormholes
- Bunch Davies limit in closed Universe
- Summary

(Many pretty pictures and some slides in the talk are credited to Dong-han)

Why quantum cosmology?

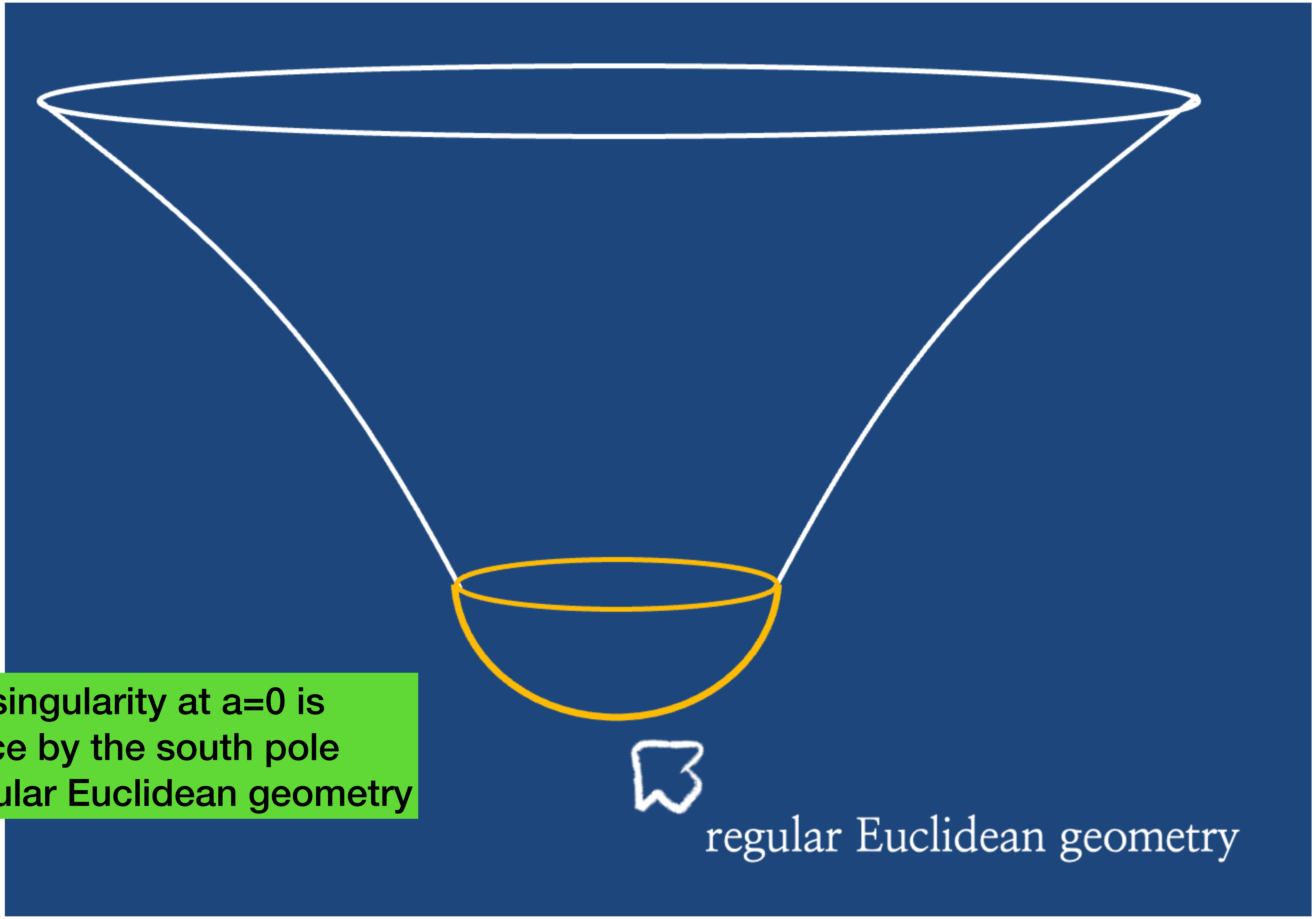


Inflationary spacetimes are not past-complete (BGV-theorem) (2003)



Quantum Cosmology (No Boundary proposal)

Geometries are smoothly rounded off and have no boundary in the past.



The singularity at a=0 is
replace by the south pole
of the regular Euclidean geometry

Wheeler–DeWitt equation

$$\hat{H}\Psi = 0$$

The Wheeler–DeWitt equation is the Schrodinger equation for gravity and all fields.

The wave function of the Universe gives a probability distribution for a certain stage of our universe, e.g., a probability before inflation

Euclidean path integral approach

(Hartle and Hawking, 1983)

$$|f\rangle = \sum_j a_j |f^j\rangle$$

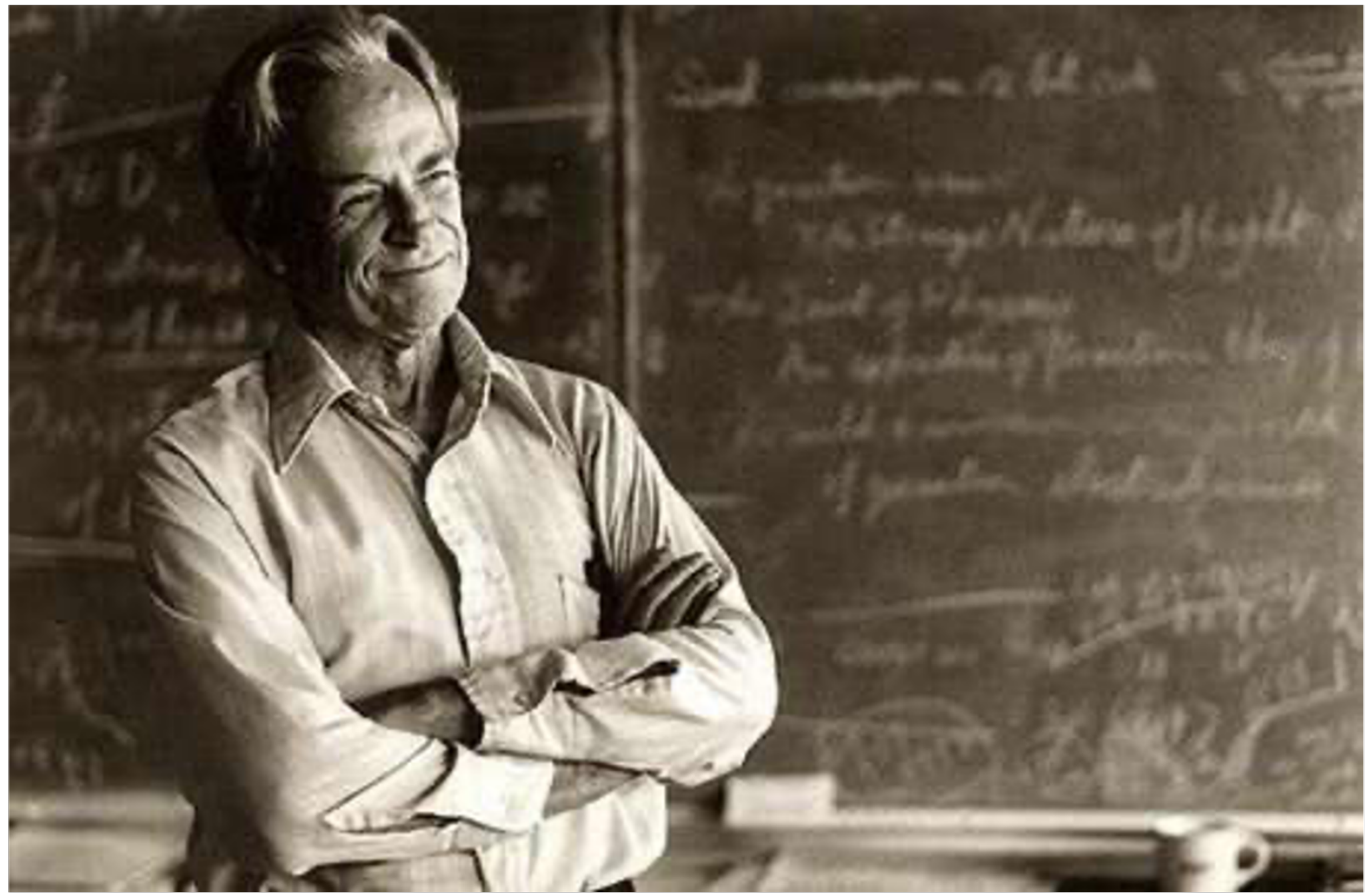
$$\langle f|i\rangle = \int_{i\to f^j} DgD\phi\, e^{iS}$$

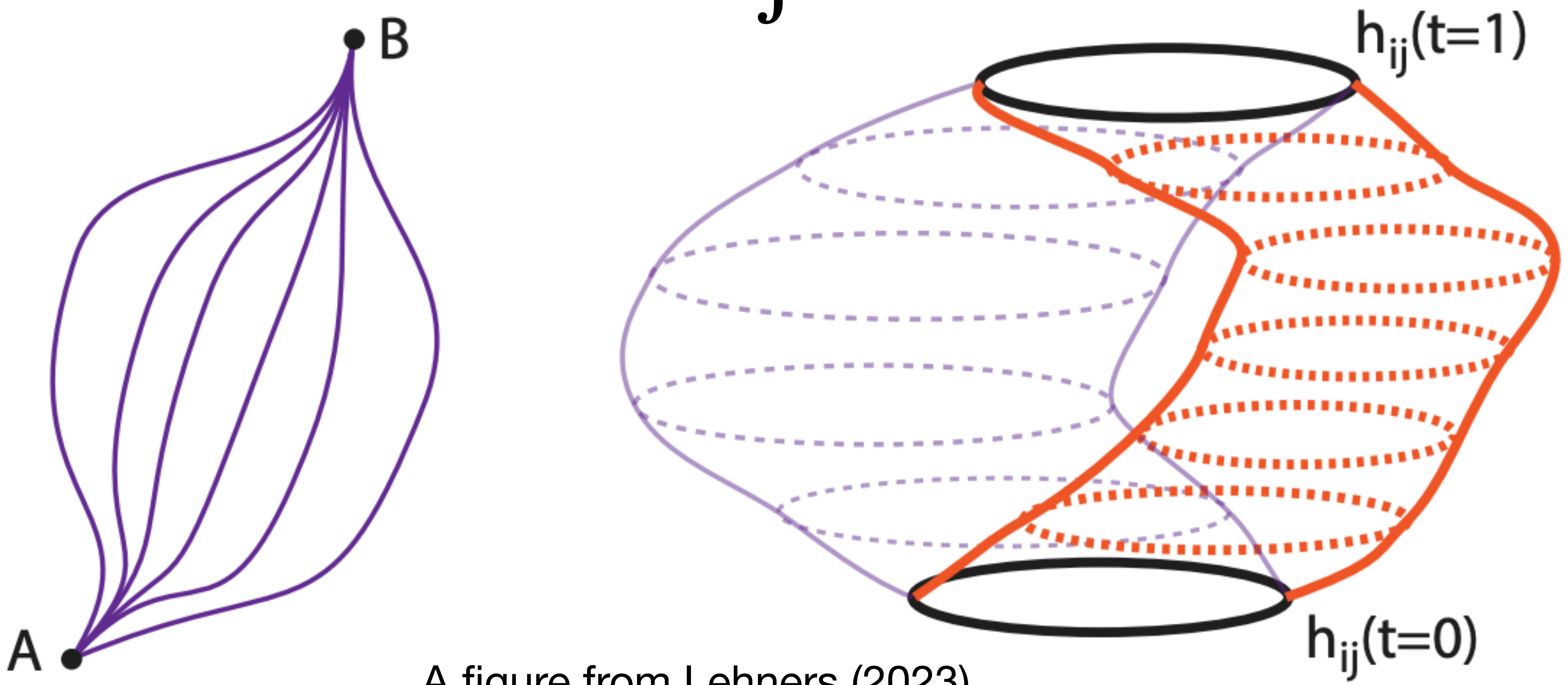
path integral as a propagator

$$= \int_{i\to f^j} DgD\phi\, e^{-S_E}$$

Euclidean analytic continuation

$$\Psi\left[h_{\mathrm{f}}, \phi_{\mathrm{f}}; h_{\mathrm{i}}, \phi_{\mathrm{i}}\right] = \int \mathscr{D} g_{\mu\nu} \mathscr{D}\phi\, e^{-S_{\mathrm{E}}\left[g_{\mu\nu}, \phi\right]}$$





A figure from Lehners (2023)

Euclidean path integral approach

(Hartle and Hawking, 1983)

$$|f\rangle = \sum_j a_j |f^j\rangle$$

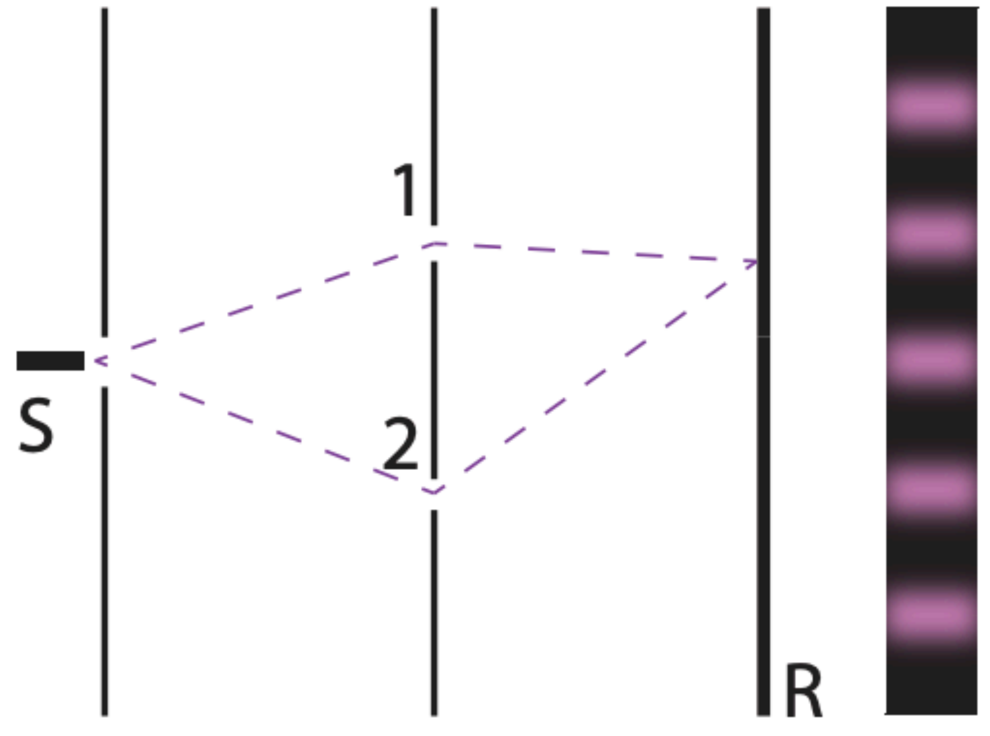
$$\langle f|i\rangle = \int_{i\to f^j} DgD\phi\, e^{-S_E}$$

$$\cong \sum_{i\to f^j} e^{-S_E^{\text{on-shell}}}$$

steepest−descent approximation

$$\Psi\left[h_{\rm f}, \phi_{\rm f}; h_{\rm i}, \phi_{\rm i}\right] \simeq \sum e^{-S_{\rm E}^{\rm instanton}}$$

need to find/sum instantons



A figure from Lehners (2023)

Background solutions (Euclidean regime)

$$S_E = \int \sqrt{+g}\, d^4x \left[\frac{R}{16\pi} + \frac{1}{2}\left(\partial_\mu \Phi\right)^2 + U(\Phi) \right],$$

$$t = -i\tau$$

$$ds_E^2 = \sigma^2 \left(d\tau^2 + a^2(\tau) d\Omega_3^2 \right), \quad \sigma^2 = 8\pi U_0/3 \text{ with a constant } U_0$$

$$d\Omega_3^2 = d\chi^2 + \sin^2\chi (d\theta^2 + \sin^2\theta d\varphi^2)$$

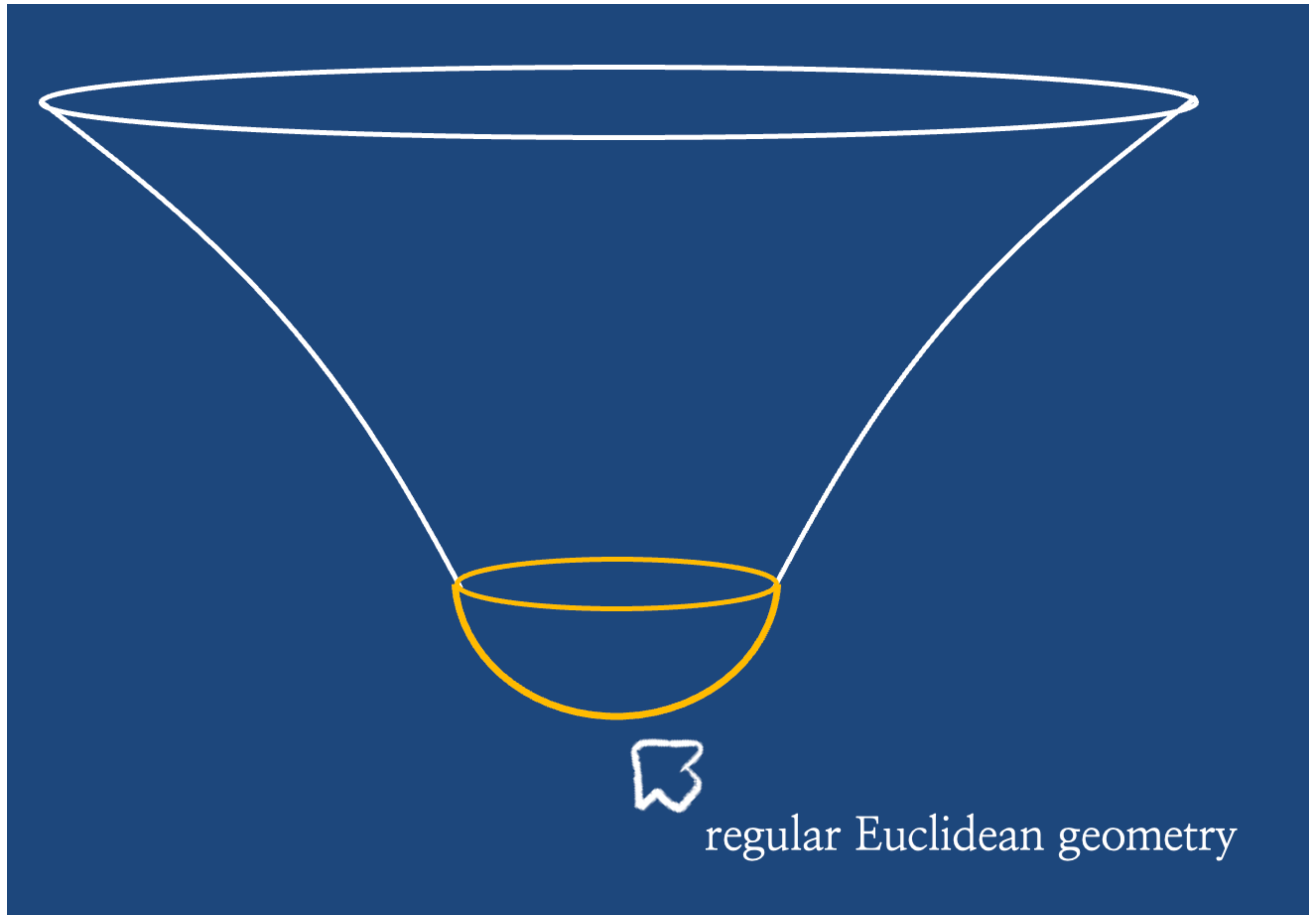
After redefining $\phi = \sqrt{4\pi/3}\,\Phi$ and $V = U/U_0$,

one can derive the equations of motion for the background geometry:

$$\dot{a}^2 - 1 + a^2\left(-\dot{\phi}^2 + V(\phi)\right) = 0,$$

$$\ddot{a} + 2a\dot{\phi}^2 + aV = 0,$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - \frac{1}{2}\frac{dV}{d\phi} = 0$$



dS instanton / Wick−rotated

$$\dot{a}^2 - 1 + a^2\left(-\dot{\phi}^2 + V(\phi)\right) = 0,$$

$$\ddot{a} + 2a\dot{\phi}^2 + aV = 0,$$

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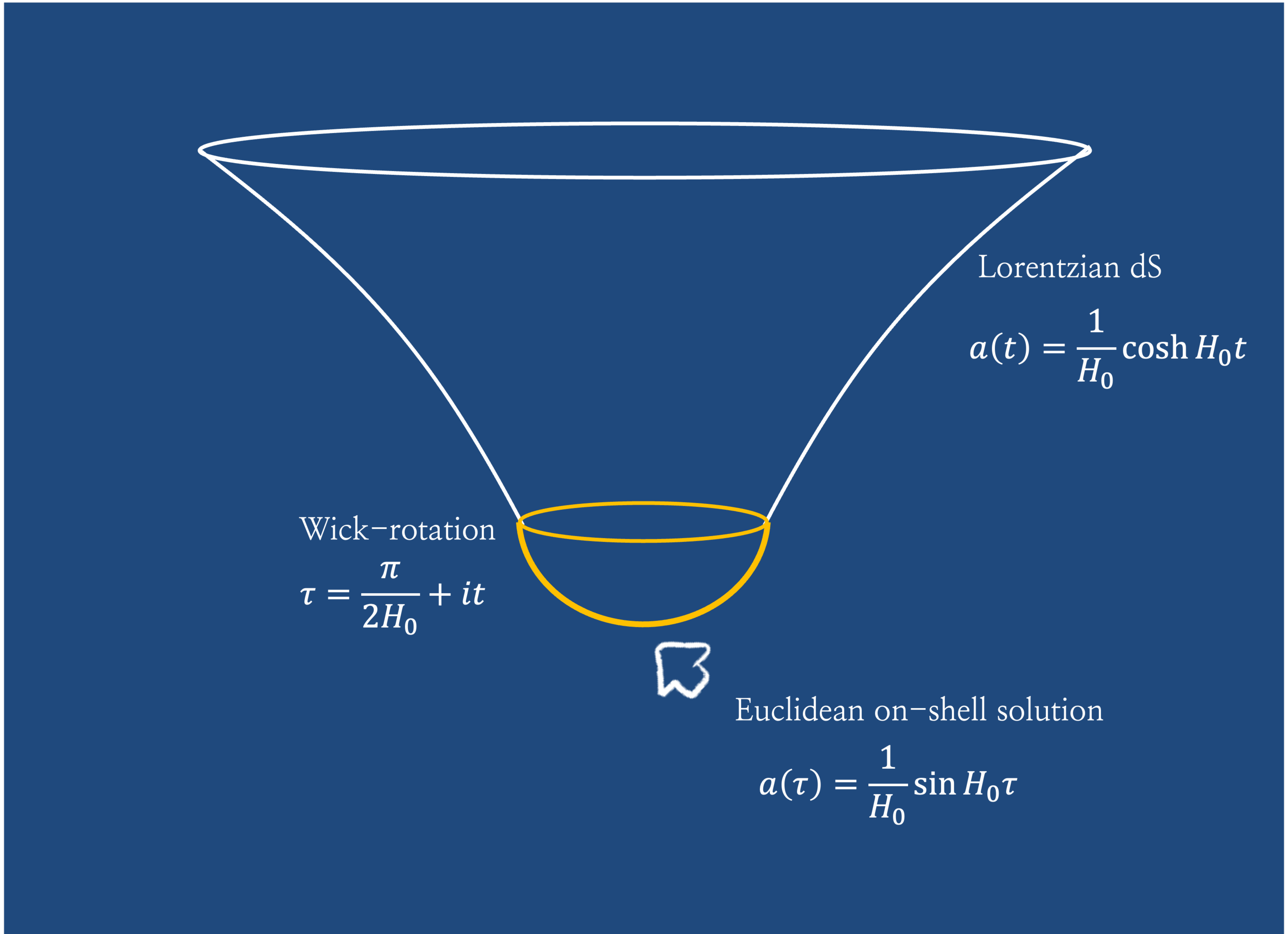
Regularity at $a(0) = 0$ requires

$$\dot{a}(0) = 1 \quad \textit{and} \quad \dot{\phi}(0) = 0$$



Turn on perturbations

Halliwell and Hawking (1985)

Laflamme (1987)

$$\Psi\left[a,\phi,\delta\phi\right]=\Psi_{\rm bg}\left[a,\phi\right]\prod_{nlm}\psi_{nlm}\left[f_{nlm};a,\phi\right],\quad f_{nlm}\text{ : matter perturbations}$$

Ignoring the details, $S_E(f_{nlm})=\int_{\tau_{\rm i}}^{\tau_{\rm f}} d\tau L_{nlm}(f_{nlm})$

$$\ddot{f}_{nlm}+3\frac{\dot{a}}{a}\dot{f}_{nlm}+\left(\frac{1}{2}\frac{d^2V}{d\phi^2}+\frac{n^2-1}{a^2}\right)f_{nlm}\simeq 0$$

$$\psi_{nlm}\left[\hat{f}_{nlm}\right]=C_{nlm}\exp\left[-\left(\frac{a^3\dot{f}_{nlm}}{2f_{nlm}}\right)\Bigg|_{\tau_{\rm f}}\hat{f}^2_{nlm}\right]$$

The wave function is Gaussian-like. This is a kind of vacua, but this is not guaranteed whether this vacuum is the Euclidean vacuum.

Turn on perturbations

$$\Psi\left[a,\phi,\delta\phi\right] = \Psi_{\rm bg}\left[a,\phi\right]\prod_{nlm}\psi_{nlm}\left[f_{nlm};a,\phi\right],\quad f_{nlm}: \text{matter perturbations}$$

$$\psi_{nlm}\left[\hat{f}_{nlm}\right] = C_{nlm}\exp\left[-\left(\frac{a^3\dot{f}_{nlm}}{2f_{nlm}}\right)\bigg|_{\tau_{\rm f}}\hat{f}^2_{nlm}\right],$$

Wick-rotate the wave function to the Lorentzian time t

$$\psi_{nlm}\left[\hat{f}_{nlm}\right] = C_{nlm}\exp\left[i\left(\frac{a^3\dot{f}_{nlm}}{2f_{nlm}}\right)\bigg|_{t_{\rm f}}\hat{f}^2_{nlm}\right].$$

It is of the form of the usual Quantization in the Schrödinger picture

$$\ddot{f}_{nlm} + 3\frac{\dot{a}}{a}\dot{f}_{nlm} + \left(\frac{1}{2}\frac{d^2V}{d\phi^2} + \frac{n^2-1}{a^2}\right)f_{nlm} \simeq 0.$$

By defining $f_{nlm} = v_{nlm}/a$ and $d\eta = dt/a$

$$v''_{nlm} = -\left[n^2 - 1 - \frac{a''}{a} + \frac{a^2}{2}\frac{d^2V}{d\phi^2}\right]v_{nlm},$$

$$\mathscr{P}(n) = n\left(n^2-1\right)\left\langle\left|\hat{f}_n\right|^2\right\rangle = \frac{n\left(n^2-1\right)}{2a^2{\rm Re}\left[-i\frac{v'_n}{v_n}\right]},$$

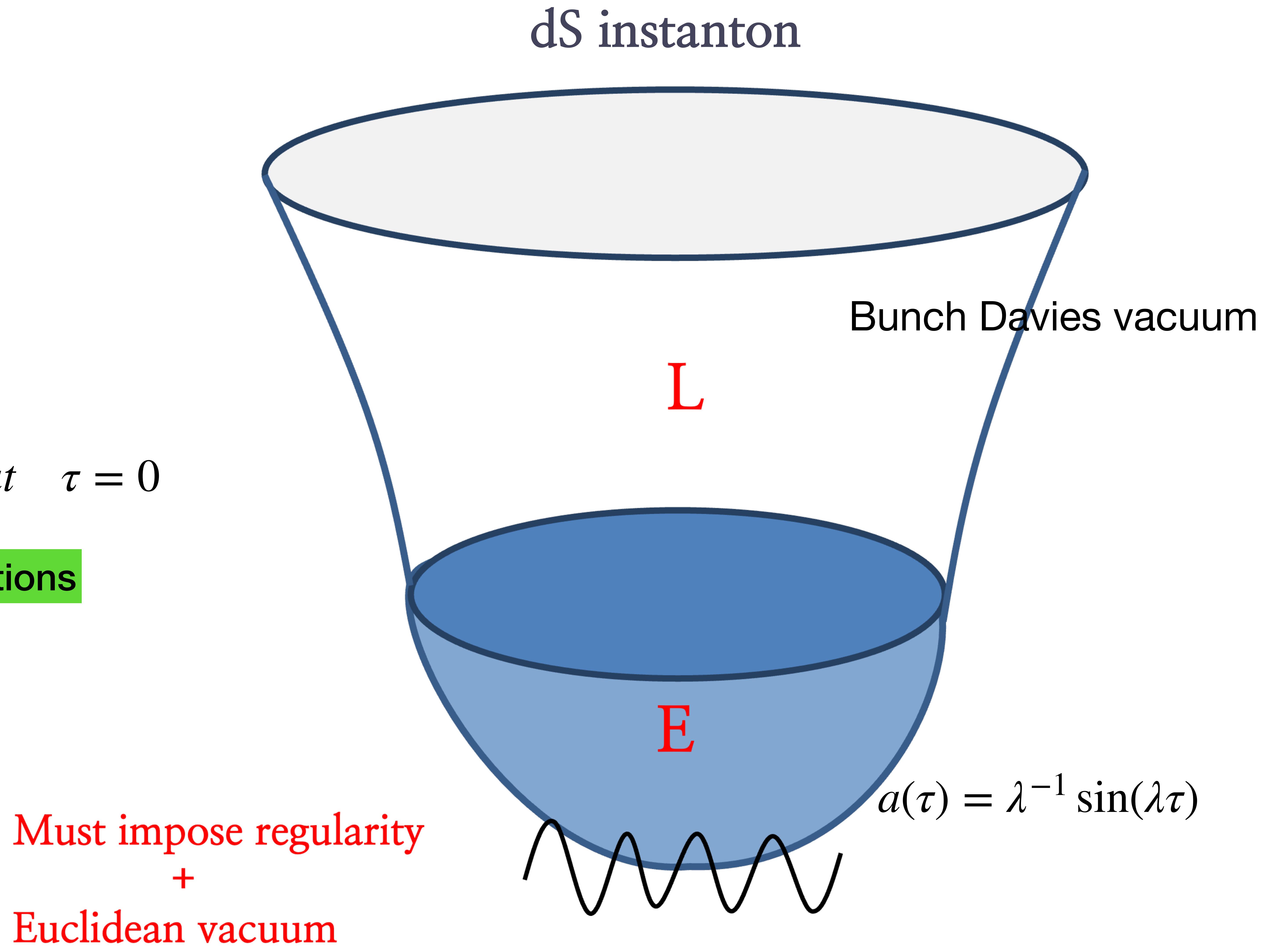
where $'$ denotes differentiation with respect to η

No Boundary proposal and the Euclidean vacuum

$$\ddot{f}_n = -3\frac{\dot{a}}{a}\dot{f}_n + \left(\frac{n^2-1}{a^2} + \mu^2\right) f_n$$

$$\dot{f}_1(\tau = 0) = 0 \quad and \quad f_{n\geq 2}(\tau = 0) = 0 \quad at \quad \tau = 0$$

The regularity condition of the mode functions



Regularity at $a(0) = 0$

$$\ddot{f}_n = -3\frac{\dot{a}}{a}\dot{f}_n + \left(\frac{n^2-1}{a^2} + \mu^2\right) f_n \qquad a(\tau) = \lambda^{-1}\sin(\lambda\tau)$$

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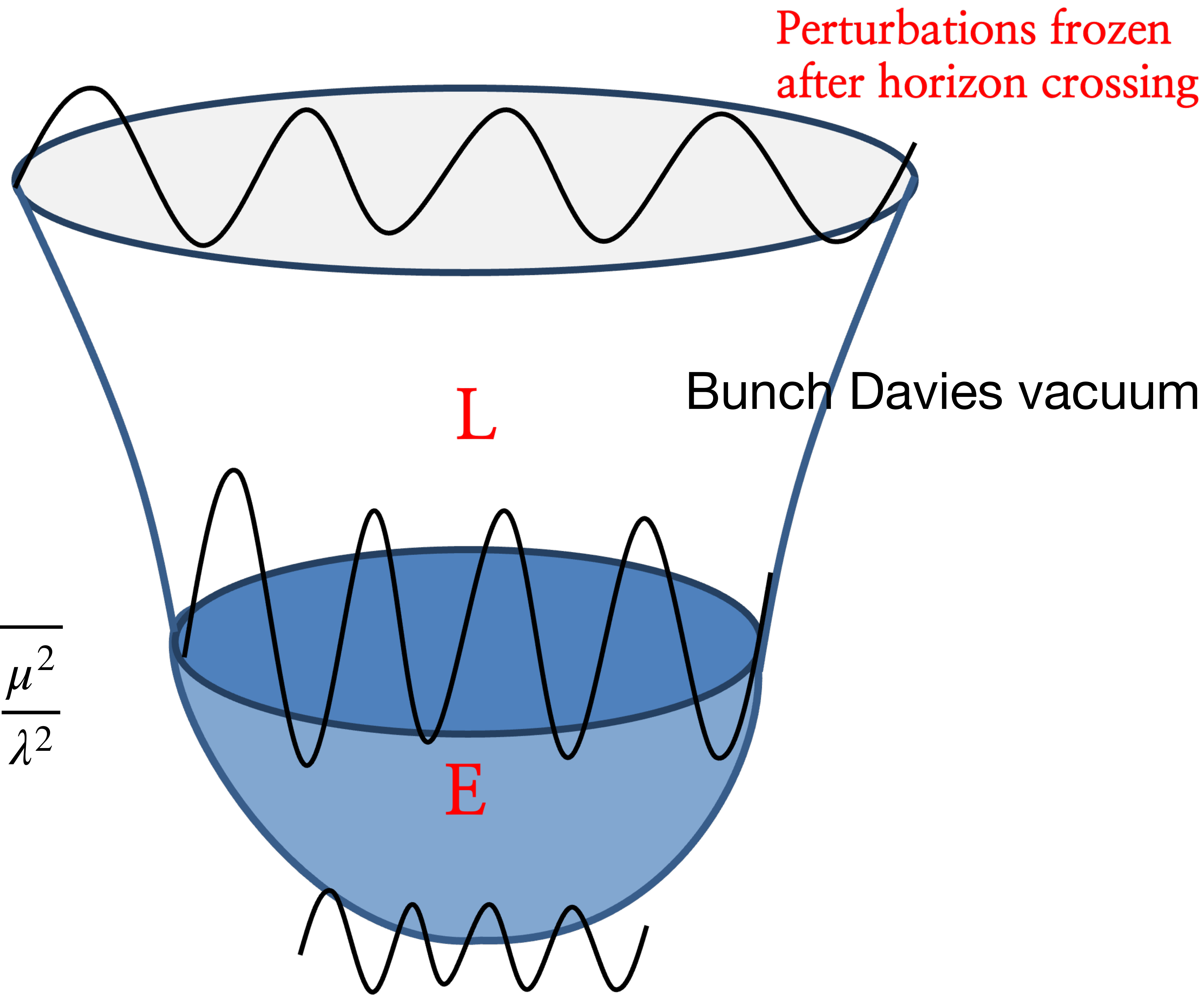


The mode solutions of the Euclidean vacuum

$$f_n(\tau) = D_n(\zeta - \zeta^2)^{(n-1)/2}{}_2F_1\left(n-\nu, n+\nu+1; n+1; \zeta\right), \ \ \nu = -\frac{1}{2} + \sqrt{\frac{9}{4} - \frac{\mu^2}{\lambda^2}}$$

$$\zeta \equiv (1 - \cos(\lambda\tau))/2$$

Euclidean Wormholes

Background

$$\dot{a}^2 - 1 + a^2\left(-\dot{\phi}^2 + V(\phi)\right) = 0,$$

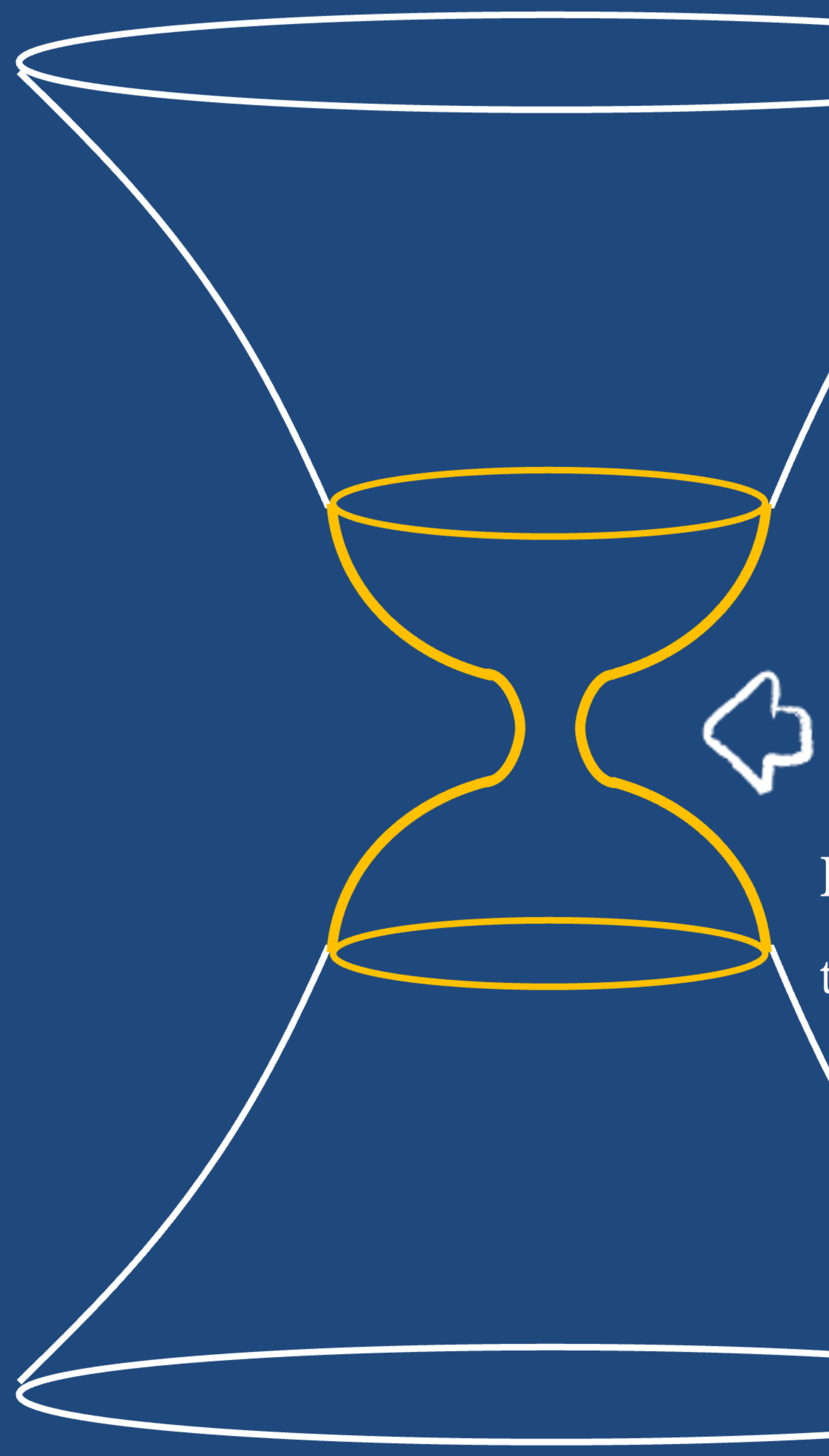
$$\ddot{a} + 2a\dot{\phi}^2 + aV = 0,$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - \frac{1}{2}\frac{dV}{d\phi} = 0$$

e.g.

$$V(\phi) = V_0$$

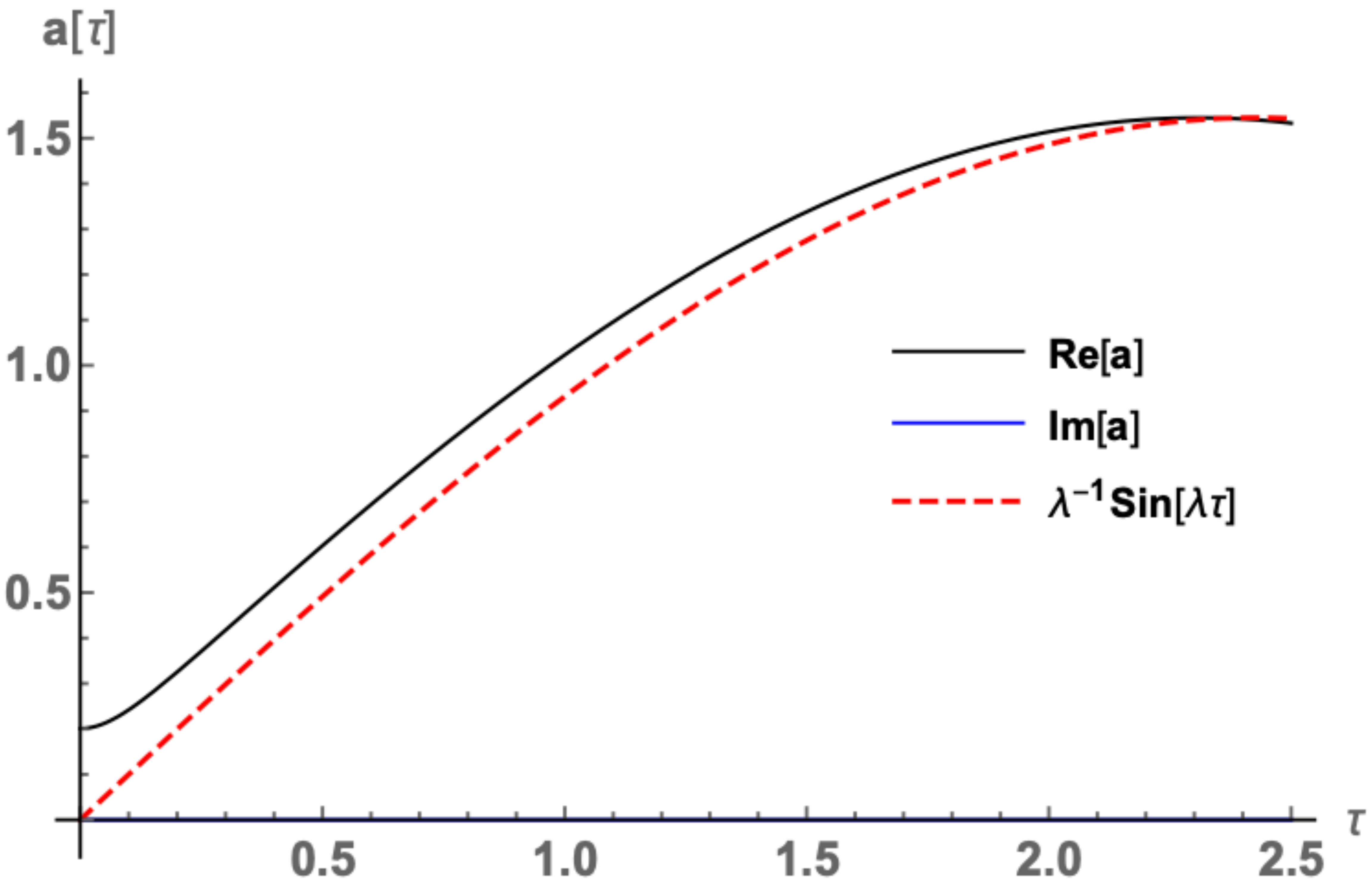
$$\frac{d\phi}{d\tau} = i\frac{A}{a^3}, \qquad \dot{a}^2 = 1 - a^2 - \frac{A^2}{a^4}$$

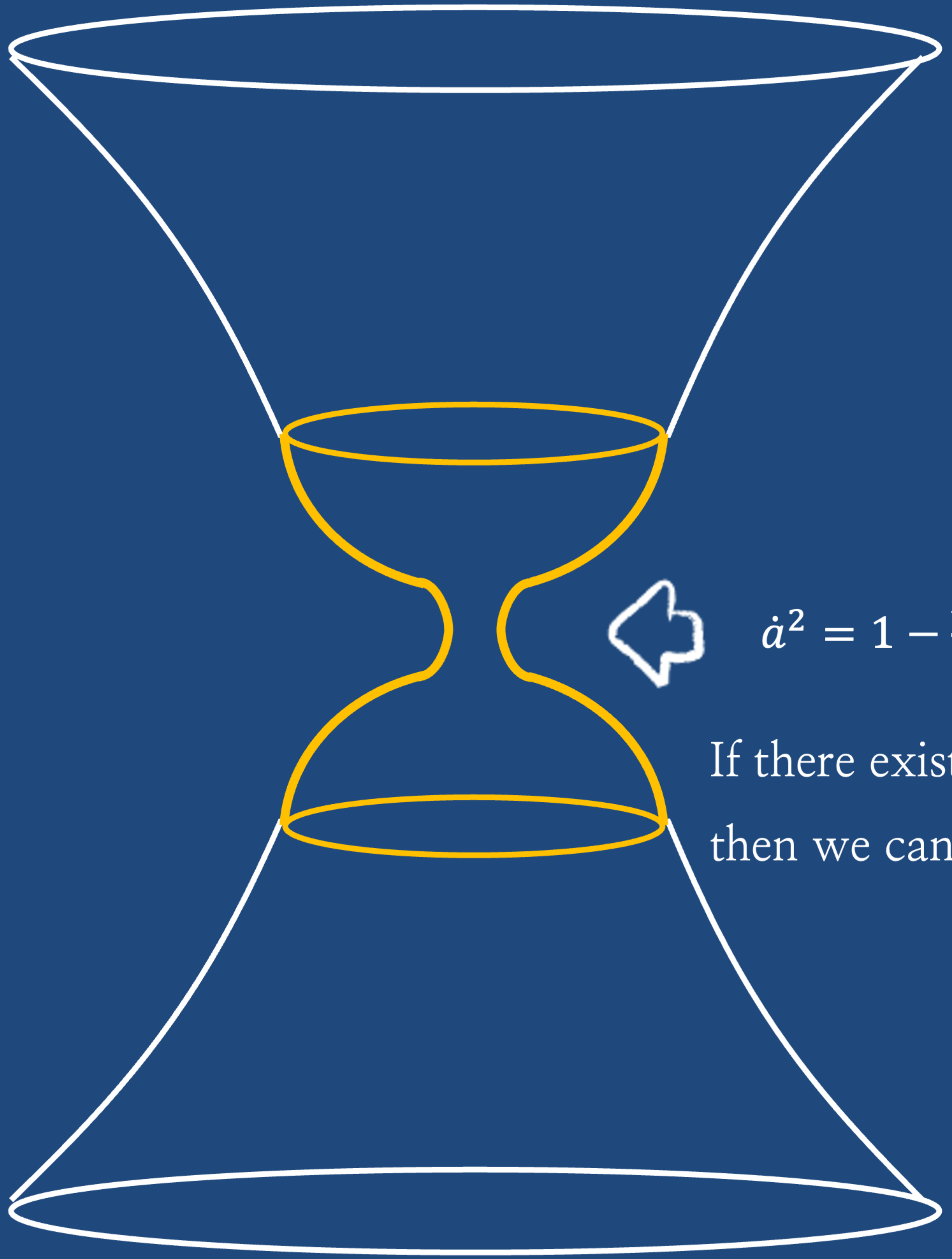


$$\dot{a}^2 = 1 - \frac{\Lambda}{3}a^2 - \frac{C}{a^n}$$

If there exists this correction term, then we can obtain a wormhole.

Euclidean Wormholes





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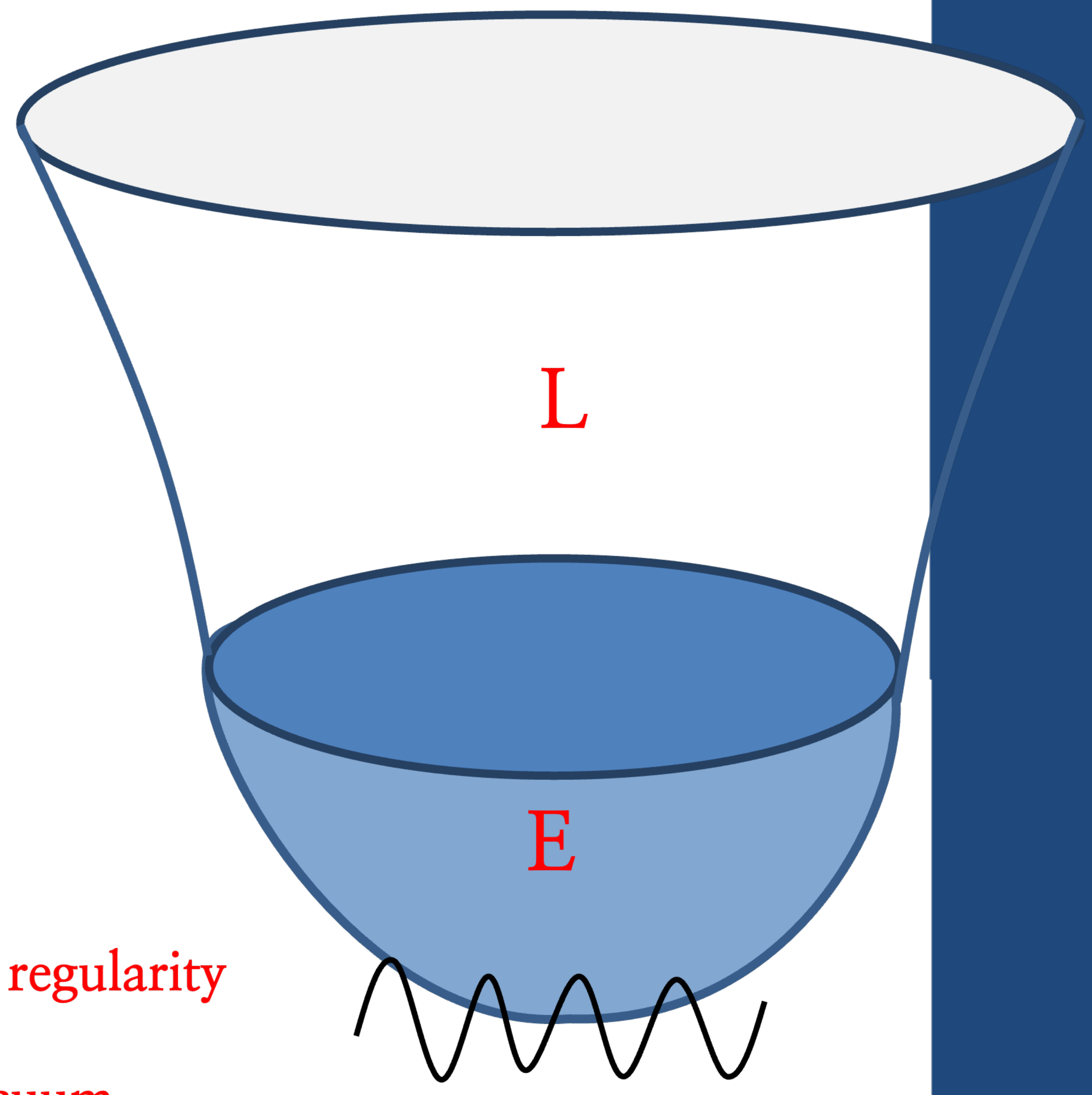
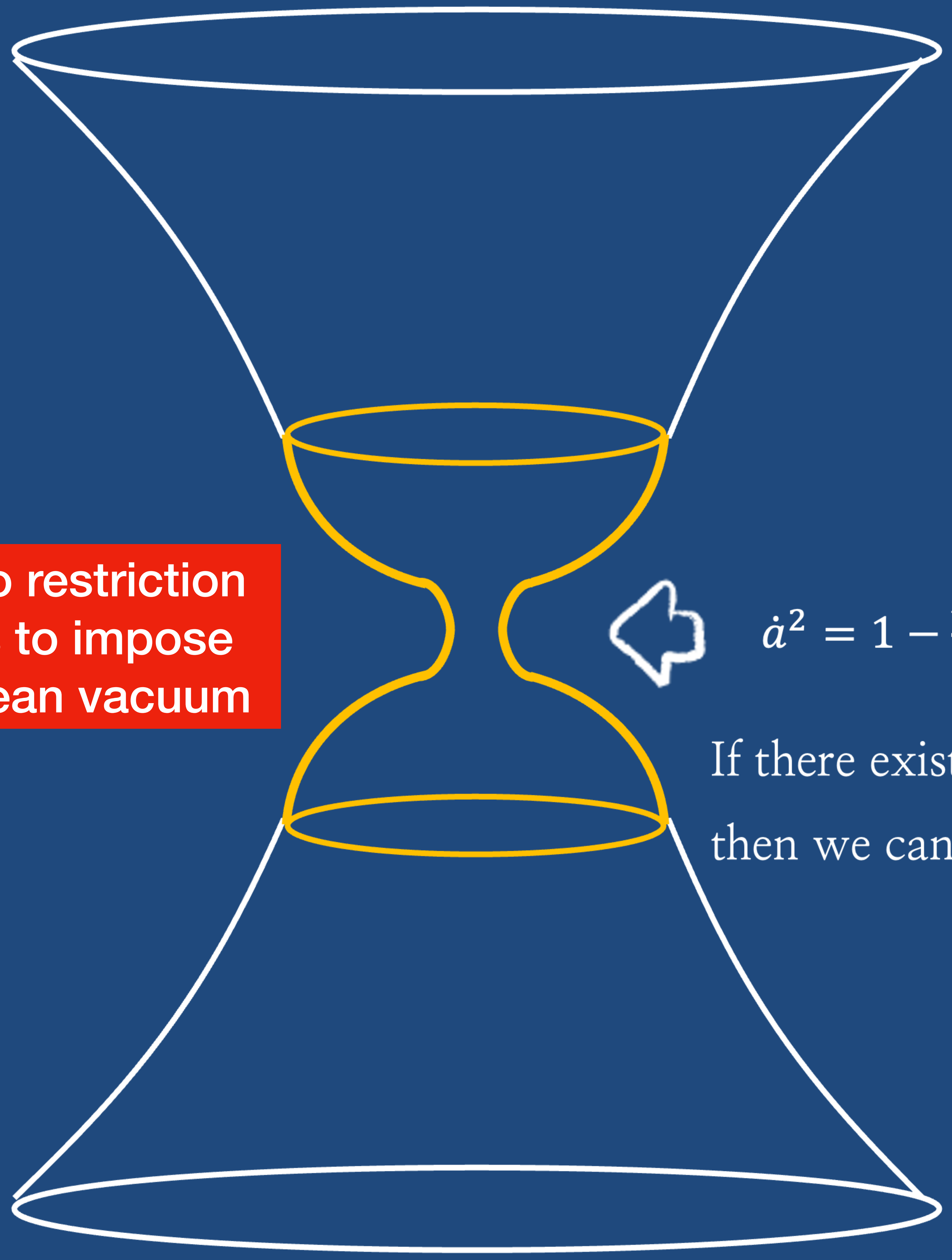


Must impose regularity
+
Euclidean vacuum

There is no restriction forcing us to impose the Euclidean vacuum



$$\dot{a}^2 = 1 - \frac{\Lambda}{3}a^2 - \frac{C}{a^n}$$

If there exists this correction term, then we can obtain a wormhole.

$$\psi_{nlm}\left[\hat{f}_{nlm}\right] = C_{nlm}\exp\left[-\left(\frac{a^3\dot{f}_{nlm}}{2f_{nlm}}\right)\Bigg|_{\tau_f}\hat{f}^2_{nlm}\right],$$

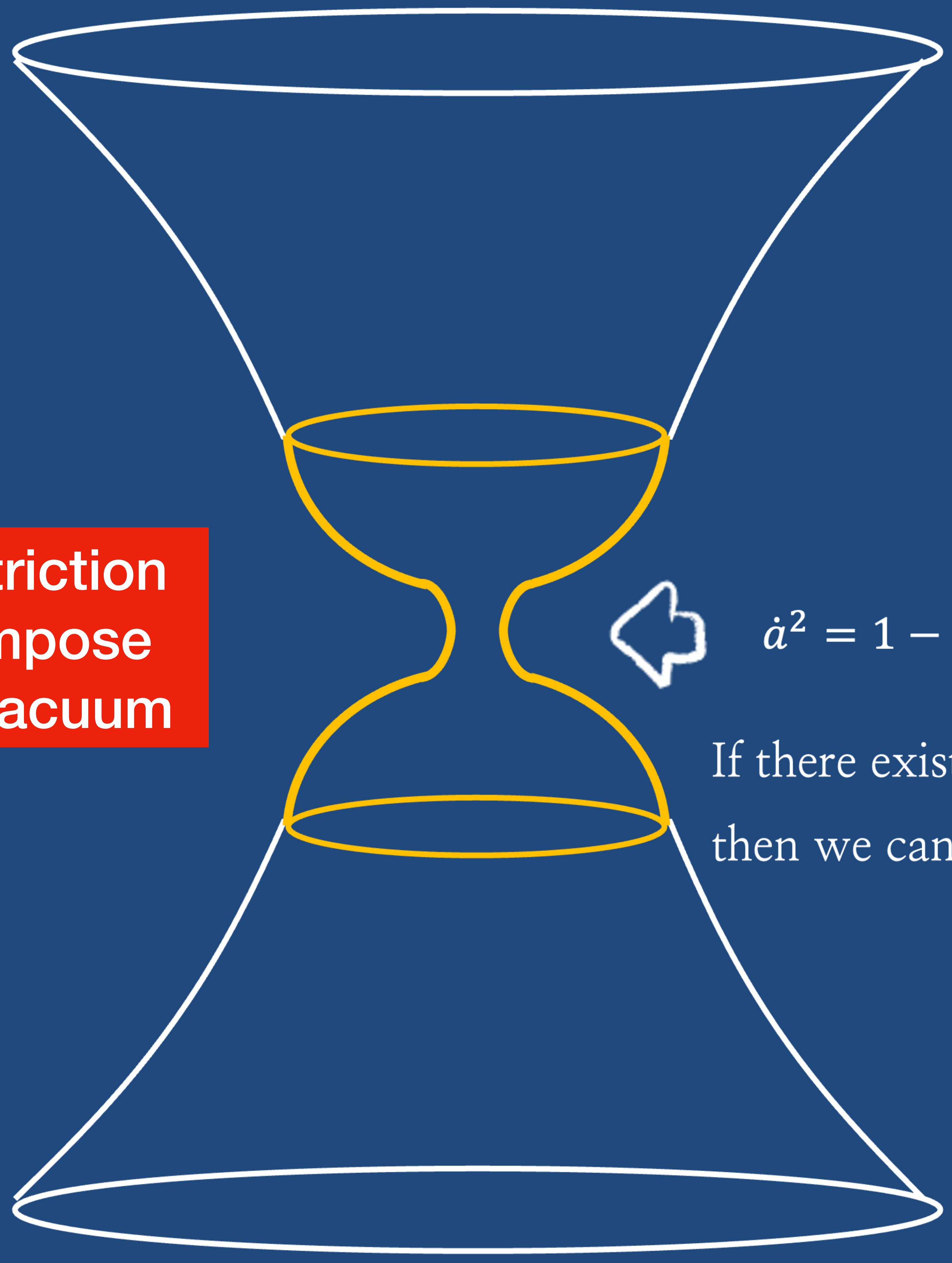
The wave function is Gaussian-like.

There was no reason to exclude the second solution of the Deq of f_{nlm}

Like the NBP.

The mixing of two solutions gives alpha vacua

There is no restriction forcing us to impose the Euclidean vacuum



$$\dot{a}^2 = 1 - \frac{\Lambda}{3}a^2 - \frac{C}{a^n}$$

If there exists this correction term, then we can obtain a wormhole.

Bunch Davies limit in closed Universe

The analytic study in the massless scalar field

A warm-up: the mode function of a massless scalar field in a flat universe (de Sitter space with flat slicing):

$$v''(\eta) + \left(k^2 - \frac{2}{\eta^2}\right)v(\eta) = 0$$

$$v(\eta) = A\frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta} + B\frac{1}{\sqrt{2k}}\left(1 + \frac{i}{k\eta}\right)e^{ik\eta}$$

$$v(\eta) = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta}$$

The mode function of BD vacuum

Bunch Davies limit in closed Universe

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$$y = \frac{k}{aH} = -k\eta \sim \frac{R_H}{l_{phy}} \qquad \Rightarrow\ v''(y) + \left(1 - \frac{2}{y^2}\right)v(y) = 0 \qquad v(\eta) = \frac{1}{\sqrt{2k}}\left(1 + \frac{i}{y}\right)e^{iy}$$

Since y is proportional to η, a plane wave in η remains as a plane wave in y.

Bunch Davies limit in closed Universe

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Since y is proportional to η, a plane wave in η remains as a plane wave in y.

So one has two special regimes

$y \gg 1$: The sub-horizon limit $\quad v''(y) + v(y) = 0. \quad v(\eta) \sim \frac{1}{\sqrt{2k}}e^{iy} = \frac{1}{\sqrt{2k}}e^{-ik\eta}.$

$y \ll 1$: The super-horizon limit

This (in terms of y) should be the precise def. of BD state that can be generalized to other topologies!

In a closed universe (de Sitter space with closed slicing), the mode function of a massless scalar field:

$$a(t) = \frac{1}{\lambda}\cosh(\lambda t)$$

$$v''(\eta) + \left[n^2 - 1 + \frac{1}{2}\big(\cos(2\eta) - 3\big)\sec^2(\eta)\right]v(\eta) = 0.$$

Defining $y = \frac{n}{aH} = \frac{n}{\tan\eta}, \quad 0 \le \eta < \pi/2$

$$\Big(1 + \frac{y^2}{n^2}\Big)\left[(n^2 + y^2)v''(y) + 2yv'(y)\right] + \left[\frac{n^2}{y^2}(y^2 - 2) - 2\right]v(y) = 0.$$

The solution can be written as

$$v_{1,RP}(\eta; n) = \frac{\sqrt{\Gamma(n-1)\Gamma(n+2)}}{(n+1)n!}n\Big(1 + \frac{i}{y}\Big)e^{-in\eta}$$

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Notice that the analytic continuation of this solution is regular in the Euclidean regime when $a = 0$.

In a closed universe (de Sitter space with closed slicing), the mode function of a massless scalar field:

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Defining $y = \frac{n}{aH} = \frac{n}{\tan\eta}$, $\quad 0 \le \eta < \pi/2$

$$\left(1 + \frac{y^2}{n^2}\right)\left[(n^2 + y^2)v''(y) + 2yv'(y)\right] + \left[\frac{n^2}{y^2}(y^2 - 2) - 2\right]v(y) = 0.$$

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Is it the correct form of the mode solution?

In the spatially flat de Sitter:

$$v(\eta) = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta}$$

$$v(\eta) = \frac{1}{\sqrt{2k}}\left(1 + \frac{i}{y}\right)e^{iy}$$

In a closed universe (de Sitter space with closed slicing), the mode function of a massless scalar field:

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Defining $y = \frac{n}{aH} = \frac{n}{\tan\eta}, \qquad 0 \le \eta < \pi/2$

$$\eta = \arctan\left(\frac{n}{y}\right)$$

A plane wave in η is NOT necessary a plane wave in y due to the non-linear transformation!

$$\left(1 + \frac{y^2}{n^2}\right)\left[(n^2 + y^2)v''(y) + 2yv'(y)\right] + \left[\frac{n^2}{y^2}(y^2 - 2) - 2\right]v(y) = 0.$$

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$$v_{1,RP}(\eta; n) = \frac{\sqrt{\Gamma(n-1)\Gamma(n+2)}}{(n+1)n!}n\left(1 + \frac{i}{y}\right)e^{-in\eta} = \frac{\sqrt{\Gamma(n-1)\Gamma(n+2)}}{(n+1)n!}n\left(1 + \frac{i}{y}\right)\exp\left[-in\arctan\left(\frac{n}{y}\right)\right].$$

In a closed universe (de Sitter space with closed slicing), the mode function of a massless scalar field:

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We can consider (n,y) in different regimes:

$y \gg n \geq 1:$ A new regime (earliest)

$n \gg y \gg 1:$ This regime gives the wave equation in y: $v''(y) + v(y) = 0.$

$n \geq 1 \gg y:$ The super-horizon limit

In a closed universe (de Sitter space with closed slicing), the mode function of a massless scalar field:

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$$\left(1 + \frac{y^2}{n^2}\right)\left[(n^2 + y^2)v''(y) + 2yv'(y)\right] + \left[\frac{n^2}{y^2}(y^2 - 2) - 2\right]v(y) = 0.$$

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We can consider (n,y) in different regimes:

$y \gg n \geq 1:$ A new regime (earliest) $\quad v_{1,RP}(\eta_1; n) \sim \frac{1}{\sqrt{n}}e^{-\frac{in^2}{y}}.$

$$\arctan\left(\frac{n}{y}\right) \simeq \frac{n}{y} + \mathcal{O}\left(\frac{n^3}{y^3}\right), \quad \frac{n}{y} \ll 1$$

$n \gg y \gg 1:$ This regime gives the wave equation in y: $v''(y) + v(y) = 0.$ $\quad v_{1,RP}(\eta_1; n) \sim \frac{1}{\sqrt{n}}e^{-in\pi/2}e^{iy}.$

$n \geq 1 \gg y:$ The super-horizon limit with a extra factor $Exp[-in\pi/2]$ left $\quad v_{1,RP}(\eta; n) \sim \frac{1}{\sqrt{n}}e^{-in\pi/2}\frac{i}{y}e^{iy}.$

$$\arctan\left(\frac{n}{y}\right) \simeq \frac{\pi}{2} - \frac{y}{n} + \mathcal{O}\left(\frac{y^3}{n^3}\right), \quad \frac{n}{y} \gg 1$$

$$\mathscr{P}(n) = n\left(n^2-1\right)\left\langle\left|\hat{f}_n\right|^2\right\rangle = \frac{n\left(n^2-1\right)}{2a^2\mathrm{Re}\left[i\frac{v_n'}{v_n}\right]} = \frac{n\left(n^2-1\right)}{a^2}|v_n^2|,$$

$$2\Re e\left[\frac{iv'}{v}\right] = \frac{-i}{|v^2|}\left(v\bar{v}' - \bar{v}v'\right)$$

$$v_{1,RP}(\eta_1; n) \sim \frac{1}{\sqrt{n}} e^{-\frac{in\pi}{2}} e^{iy}$$

$$W[v,\bar{v}] \equiv \left(v\bar{v}' - \bar{v}v'\right)$$

normalization $W[v,\bar{v}] = i$

This extra phase factor $e^{-\frac{in\pi}{2}}$ does not cause any effect if we only have to consider the positive frequency mode

$$\mathscr{P}(n) = n\left(n^2-1\right)\left\langle \left|\hat{f}_n\right|^2 \right\rangle = \frac{n\left(n^2-1\right)}{2a^2 \mathrm{Re}\left[i\frac{v_n'}{v_n}\right]} = \frac{n\left(n^2-1\right)}{a^2}|v_n^2|,$$

$$2\Re e[\frac{iv'}{v}] = \frac{-i}{|v^2|}\left(v\bar{v}' - \bar{v}v'\right)$$

$$v_{1,RP}(\eta_1; n) \sim \frac{1}{\sqrt{n}} e^{-\frac{in\pi}{2}} e^{iy}$$

$$W[v, \bar{v}] \equiv \left(v\bar{v}' - \bar{v}v'\right)$$

normalization $W[v, \bar{v}] = i$

This extra phase factor $e^{-\frac{in\pi}{2}}$ does not cause any effect if we only have to consider the positive frequency mode

What if we have to consider the alpha vacuum as the general vacua in a Euclidean wormhole scenario?

An alpha vacuum can be represented as the mixing of the positive and negative frequency modes specified by an complex alpha parameter:

$$V = N\left(v + e^{\bar{\alpha}}\bar{v}\right) \quad N = \frac{1}{\sqrt{1 - e^{\alpha+\bar{\alpha}}}} \quad \Re e[\alpha] < 0$$

The convention from Bousso, Maloney, Strominger (2002)

The positive frequency mode (B.D. mode) is recovered in the limit $\Re e[\alpha] \to -\infty$

$$2\Re e[\frac{iV'}{V}] = \frac{-i}{|V|^2}\left(V\bar{V}' - \bar{V}V'\right) = \frac{-i(v\bar{v}' - \bar{v}v')}{N^2|v|^2\left[(1+e^{\alpha+\bar{\alpha}}) + \frac{2\Re e(e^{\bar{\alpha}}v^2)}{|v|^2}\right]} = \frac{1}{N^2\left[(1+e^{\alpha+\bar{\alpha}}) + \frac{2\Re e(e^{\bar{\alpha}}v^2)}{|v|^2}\right]} 2\Re e[\frac{iv'}{v}],$$

$$V\bar{V}' - \bar{V}V' = v\bar{v}' - \bar{v}v'$$

$$|V|^2 = N^2\left[(1+e^{\alpha+\bar{\alpha}})|v|^2 + 2\Re e(e^{\bar{\alpha}}v^2)\right]$$

$$v_{1,RP}(\eta;n) = \frac{\sqrt{\Gamma(n-1)\Gamma(n+2)}}{(n+1)n!} n\left(1+\frac{i}{y}\right)e^{-in\eta} = \frac{\sqrt{\Gamma(n-1)\Gamma(n+2)}}{(n+1)n!} n\left(1+\frac{i}{y}\right)e^{-in\,\arctan(\frac{n}{y})}.$$

$$2\Re e\left[\frac{iV'}{V}\right] = \frac{1}{N^2\left[(1+e^{\alpha+\bar{\alpha}})+\frac{2\Re e(e^{\bar{\alpha}}v^2)}{|v|^2}\right]} 2\Re e\left[\frac{iv'}{v}\right],$$

$y \gg n \geq 1$:

$$v_{1,RP}(\eta;n) \sim \frac{1}{\sqrt{n}} e^{-\frac{in^2}{y}}$$

$n \gg y \gg 1$: $v''(y) + v(y) = 0.$

$$v_{1,RP}(\eta;n) \sim \frac{1}{\sqrt{n}} e^{-\frac{in\pi}{2}} e^{iy}$$

$n \gg 1 \gg y$:

$$v_{1,RP}(\eta;n) \sim \frac{1}{\sqrt{n}} e^{-\frac{in\pi}{2}} \frac{i}{y} e^{iy}.$$

$$\frac{\Re e(e^{\bar{\alpha}} v_{1,RP}^2)}{|v_{1,RP}|^2} \sim -e^{\bar{\alpha}} \Re e(e^{2iy} e^{-in\pi}) = -e^{\bar{\alpha}} \cos(2y)(-1)^n \simeq (-1)^{n+1} e^{\bar{\alpha}}.$$

Assuming real α for simplicity

$$v_{1,RP}(\eta;n) = \frac{\sqrt{\Gamma(n-1)\Gamma(n+2)}}{(n+1)n!}n\left(1+\frac{i}{y}\right)e^{-in\eta} = \frac{\sqrt{\Gamma(n-1)\Gamma(n+2)}}{(n+1)n!}n\left(1+\frac{i}{y}\right)e^{-in\arctan(\frac{n}{y})}.$$

$$2\Re e\left[\frac{iV'}{V}\right] = \frac{1}{N^2\left[(1+e^{\alpha+\bar{\alpha}})+\frac{2\Re e(e^{\bar{\alpha}}v^2)}{|v|^2}\right]}2\Re e\left[\frac{iv'}{v}\right],$$

$$\frac{\Re e(e^{\bar{\alpha}}v_{1,RP}^2)}{|v_{1,RP}|^2} \sim (-1)^{n+1}e^{\bar{\alpha}}.$$

Fictitious splitting of even and odd modes in the power spectrum

$y \gg n \geq 1$:

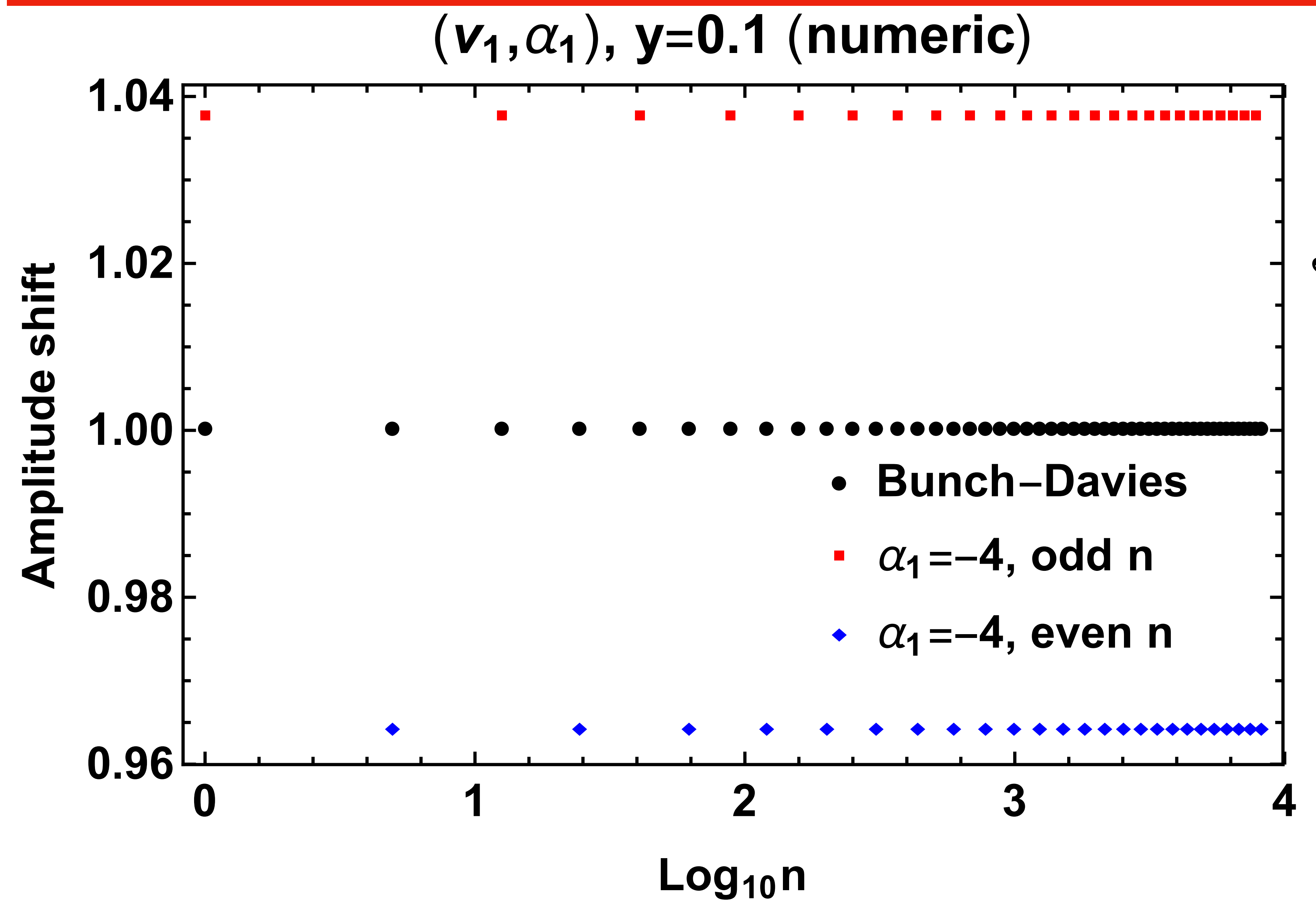
$$v_{1,RP}(\eta;n) \sim \frac{1}{\sqrt{n}}e^{-\frac{in^2}{y}}$$

$n \gg y \gg 1$: $v''(y) + v(y) = 0.$

$$v_{1,RP}(\eta;n) \sim \frac{1}{\sqrt{n}}e^{-\frac{in\pi}{2}}e^{iy}$$

$n \gg 1 \gg y$:

$$v_{1,RP}(\eta;n) \sim \frac{1}{\sqrt{n}}e^{-\frac{in\pi}{2}}\frac{i}{y}e^{iy}.$$



$$\mathscr{P}(n) = \frac{n\left(n^2-1\right)}{2a^2\mathrm{Re}\left[i\frac{V_n'}{V_n}\right]},$$

$$\frac{\Re e(e^{\bar{\alpha}}v_{1,RP}^2)}{|v_{1,RP}|^2} \sim -e^{\bar{\alpha}}\Re e(e^{2iy}e^{-in\pi}) = -e^{\bar{\alpha}}\cos(2y)(-1)^n \simeq (-1)^{n+1}e^{\bar{\alpha}}.$$

Assuming real α for simplicity

$$v_{1,RP}(\eta;n) = \frac{\sqrt{\Gamma(n-1)\Gamma(n+2)}}{(n+1)n!}n\left(1+\frac{i}{y}\right)e^{-in\eta} = \frac{\sqrt{\Gamma(n-1)\Gamma(n+2)}}{(n+1)n!}n\left(1+\frac{i}{y}\right)e^{-in\arctan(\frac{n}{y})}.$$

$$v_{2,RP}(\eta;n) = e^{in\pi/2}v_{1,RP}(\eta;n)$$

$$\arctan\left(\frac{n}{y}\right) \simeq \frac{\pi}{2} - \frac{y}{n} + \mathcal{O}\left(\frac{y^3}{n^3}\right), \frac{n}{y} \gg 1$$

$$2\Re e\left[\frac{iV'}{V}\right] = \frac{1}{N^2\left[(1+e^{\alpha+\bar{\alpha}}) + \frac{2\Re e(e^{\bar{\alpha}}v^2)}{|v|^2}\right]}2\Re e\left[\frac{iv'}{v}\right],$$

$$y = \frac{n}{aH} = \frac{n}{\tan\eta}$$

$y \gg n \geq 1:$

$$v_{1,RP}(\eta;n) \sim \frac{1}{\sqrt{n}}e^{-\frac{in^2}{y}} \qquad v_{2,RP}(\eta;n) \sim e^{\frac{in\pi}{2}}\frac{1}{\sqrt{n}}e^{-\frac{in^2}{y}}.$$

In flat universe with standard BD state:

$n \gg y \gg 1:$ $v''(y) + v(y) = 0.$

$y \gg 1:$

$$v_{1,RP}(\eta;n) \sim \frac{1}{\sqrt{n}}e^{-\frac{in\pi}{2}}e^{iy} \qquad v_{2,RP}(\eta;n) \sim \frac{1}{\sqrt{n}}e^{iy} \quad \longleftrightarrow \quad v_2(\eta) \sim \frac{1}{\sqrt{2k}}e^{iy}$$

$n \gg 1 \gg y:$

$y \ll 1:$

$$v_{1,RP}(\eta;n) \sim \frac{1}{\sqrt{n}}e^{-\frac{in\pi}{2}}\frac{i}{y}e^{iy}. \qquad v_{2,RP}(\eta;n) \sim \frac{1}{\sqrt{n}}\frac{i}{y}e^{iy}. \qquad v_2(\eta) \sim \frac{1}{\sqrt{2k}}\frac{i}{y}e^{iy}$$

$$\frac{\Re e(e^{\bar{\alpha}}v_{2,RP}^2)}{|v_{2,RP}|^2} \sim -e^{\bar{\alpha}}\Re e(e^{2iy}) = -e^{\bar{\alpha}}\cos(2y) \simeq -e^{\bar{\alpha}}. \quad \longleftrightarrow \quad \frac{\Re e(e^{\bar{\alpha}}v_2^2)}{|v_2|^2} \sim -e^{\bar{\alpha}}.$$

Assuming real α for simplicity

$$v_{1,RP}(\eta;n) = \frac{\sqrt{\Gamma(n-1)\Gamma(n+2)}}{(n+1)n!} n\left(1+\frac{i}{y}\right) e^{-in\eta} = \frac{\sqrt{\Gamma(n-1)\Gamma(n+2)}}{(n+1)n!} n\left(1+\frac{i}{y}\right) e^{-in\arctan(\frac{n}{y})}.$$

$$v_{2,RP}(\eta;n) = e^{in\pi/2} v_{1,RP}(\eta;n)$$

$$2\Re e\left[\frac{iV'}{V}\right] = \frac{1}{N^2\left[(1+e^{\alpha+\bar{\alpha}}) + \frac{2\Re e(e^{\bar{\alpha}}v^2)}{|v|^2}\right]} 2\Re e\left[\frac{iv'}{v}\right],$$

$$y \gg n \geq 1:$$

$$v_{1,RP}(\eta;n) \sim \frac{1}{\sqrt{n}} e^{-\frac{in^2}{y}} \qquad v_{2,RP}(\eta;n) \sim e^{\frac{in\pi}{2}} \frac{1}{\sqrt{n}} e^{-\frac{in^2}{y}}.$$

$$n \gg y \gg 1: \quad v''(y) + v(y) = 0.$$

$$v_{1,RP}(\eta;n) \sim \frac{1}{\sqrt{n}} e^{-\frac{in\pi}{2}} e^{iy} \qquad v_{2,RP}(\eta;n) \sim \frac{1}{\sqrt{n}} e^{iy}$$

$$n \gg 1 \gg y:$$

$$v_{1,RP}(\eta;n) \sim \frac{1}{\sqrt{n}} e^{-\frac{in\pi}{2}} \frac{i}{y} e^{iy}. \qquad v_{2,RP}(\eta;n) \sim \frac{1}{\sqrt{n}} \frac{i}{y} e^{iy}.$$

$$\frac{\Re e(e^{\bar{\alpha}} v_{1,RP}^2)}{|v_{1,RP}|^2} \sim (-1)^{n+1} e^{\bar{\alpha}}. \qquad \frac{\Re e(e^{\bar{\alpha}} v_{2,RP}^2)}{|v_{2,RP}|^2} \sim -e^{\bar{\alpha}}.$$

In the closed universe, the extra n-dependent phase factor always exists.

To have the mode solution without the phase factor when $n \gg y$, we need to throw it back to the $y \gg n \geq 1$ and the Euclidean regimes.

This means the correct initial condition for numerical evaluation for the mode solutions in a general scenario also must include this phase factor:

$$f_n(\tau_i) = e^{\frac{in\pi}{2}} \frac{1}{2} \epsilon \tau_i^2, \quad \dot{f}_n(\tau_i) = e^{\frac{in\pi}{2}} \epsilon \tau_i$$

Summary

From the quantum cosmology setting, the Euclidean wormholes provide a scenario in which the alpha vacua for matter perturbation are more general, since the regularity condition is removed.

We discuss the no-trivial phase factor in the mode functions in the closed universe whose effect appears if we consider the mixing of positive- and negative frequency modes. We identify the suitable regime for the Bunch Davies limit in a closed universe.

Summary

From the quantum cosmology setting, the Euclidean wormholes provide a scenario in which the alpha vacua for matter perturbation are more general, since the regularity condition is removed.

We discuss the no-trivial phase factor in the mode functions in the closed universe whose effect appears if we consider the mixing of positive- and negative frequency modes. We identify the suitable regime for the Bunch Davies vacuum in a closed universe.

Thank you!

Bunch Davies limit in closed Universe

The analytic study in the massless scalar field

A warm-up: the mode function of a massless scalar field in a flat universe (de Sitter space with flat slicing):

$$v''(\eta) + \left(k^2 - \frac{2}{\eta^2}\right)v(\eta) = 0 \qquad v(\eta) = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta}$$

The mixing of positive and negative frequency modes (Bogoluobov transformation) of the massless scalar field are not alpha vacua.
(since they don't preserve the full continuous symmetry of de Sitter space).
However, the issue exists generally in a closed universe scenario.

Bruce Allen (1985)

When $m^2>0$, the Euclidean vacuum is defined by $\alpha=0$. In the literature this vacuum state is also called the Bunch-Davies[16,17] or Birrell-Davies[12,18] vacuum. When $m^2=0$ the Euclidean vacuum state no longer exists. This is because (1) this state is de Sitter invariant and (2) its two-point function has only one singular point on S^4. However, the Bunch-Davies vacuum, defined by $\alpha=0$ in (4.19) and (4.20) does exist for $m^2=0$. It is simply no longer de Sitter invariant.