Majorana Phase and Matter Effects in Neutrino Chiral Oscillation

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1. Neutrino Chiral Oscillation

Equation of motion for a neutrino in free space

$$(i\partial\!\!\!/ - m)\psi = 0\,, \quad i\partial\!\!\!/\psi_L - m\psi_R = 0\,, \quad i\partial\!\!\!/\psi_R - m\psi_L = 0\,, \qquad \psi(t,\mathbf{x}) = U(t)\psi(0)e^{i\mathbf{p}\cdot\mathbf{x}}$$

$$\psi_L = \tfrac{1-\gamma_5}{2}\psi\,, \quad \psi_R = \tfrac{1+\gamma_5}{2}\psi\,, \quad \psi = \psi_L + \psi_R\,. \qquad U = e^{-iHt}\,, \quad H = \gamma^0\boldsymbol{\gamma}\cdot\mathbf{p} + m\gamma^0 = \boldsymbol{\alpha}\cdot\mathbf{p} + m\beta$$

How left-handed and right-handed are entangeled in free space?

In chiral representation: $\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},\ \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},\ \gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$

$$U(t) = e^{-iHt} = \cos(Et) - i\frac{\boldsymbol{\alpha}\cdot\mathbf{p} + m\beta}{E}\sin(Et)$$

$$\psi^h(t,\mathbf{x}) = U(t)\psi^h(0)e^{i\mathbf{p}\cdot\mathbf{x}} = \psi^h(0)e^{-i(Et-\mathbf{p}\cdot\mathbf{x})}\,, \quad \psi^h(0) = \frac{1}{\sqrt{2E}}\begin{pmatrix} \sqrt{E-h\cdot p}\;u^h \\ \sqrt{E+h\cdot p}\;u^h \end{pmatrix}$$

$\psi(0)^\dagger\psi(0) = 1.$ $h = \pm 1$ - helicity, $\mathbf{p}\cdot\boldsymbol{\sigma}u^h = (h\cdot p)u^h$, $\mathbf{p} = (p_x, p_y, p_z) = p(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$.

$$u^{h=+1} = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\phi} \end{pmatrix}, \quad u^{h=-1} = \begin{pmatrix} -\sin(\theta/2)e^{-i\phi} \\ \cos(\theta/2) \end{pmatrix},$$

Oscilation probability from i to k for Dirac neutrinos:

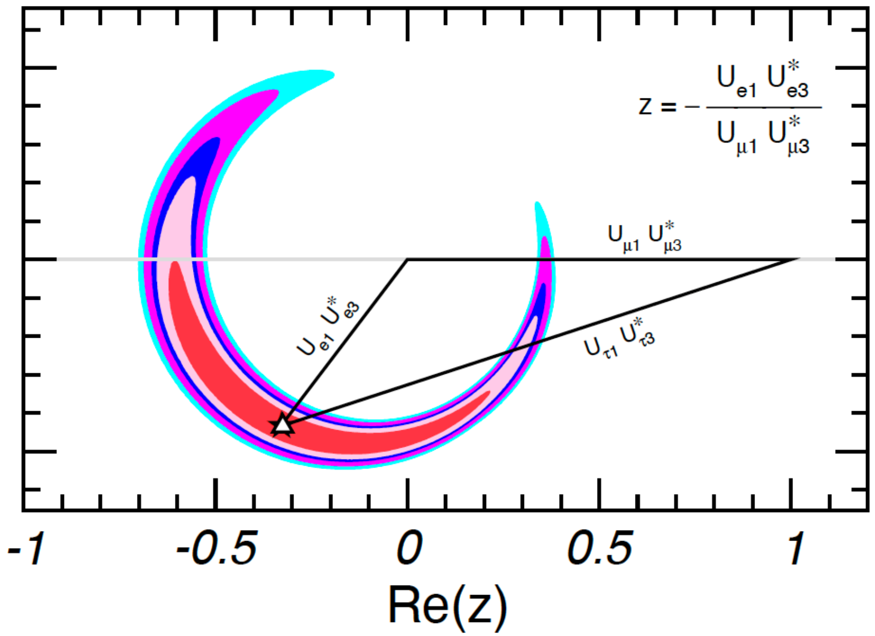
$$P(\psi_i \to \psi_k) = |<\psi(0)_k|\psi(t)_i>|^2 = |\sum V_{ij}V^*_{kj}e^{-i(E_jt-\mathbf{p}\cdot\mathbf{x})}|^2$$

Neutrion oscillations with 3 generations

$$\begin{pmatrix} c_{12}\, c_{13} & s_{12}\, c_{13} & s_{13}\, e^{-i\delta_{\rm CP}} \\ -s_{12}\, c_{23} - c_{12}\, s_{13}\, s_{23}\, e^{i\delta_{\rm CP}} & c_{12}\, c_{23} - s_{12}\, s_{13}\, s_{23}\, e^{i\delta_{\rm CP}} & c_{13}\, s_{23} \\ s_{12}\, s_{23} - c_{12}\, s_{13}\, c_{23}\, e^{i\delta_{\rm CP}} & -c_{12}\, s_{23} - s_{12}\, s_{13}\, c_{23}\, e^{i\delta_{\rm CP}} & c_{13}\, c_{23} \end{pmatrix}$$

Table 14.7: 3ν oscillation parameters obtained from different global analyses of neutrino data. In all cases, the numbers labeled as NO (IO) are obtained assuming NO (IO), *i.e.*, relative to the respective local minimum. SK-ATM makes reference to the tabulated χ^2 map from the Super-Kamiokande analysis of their data in Ref. [97].

	Ref. [185] w/o SK-ATM		Ref. [185] w SK-ATM		Ref. [186] w SK-ATM		Ref. [187] w SK-ATM	
NO	Best Fit Ordering		Best Fit Ordering		Best Fit Ordering		Best Fit Ordering	
Param	bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
$\frac{\sin^2\theta_{12}}{10^{-1}}$	$3.10^{+0.13}_{-0.12}$	$2.75 \to 3.50$	$3.10^{+0.13}_{-0.12}$	$2.75 \to 3.50$	$3.04^{+0.14}_{-0.13}$	$2.65 \to 3.46$	$3.20^{+0.20}_{-0.16}$	$2.73 \to 3.79$
$\theta_{12}/^\circ$	$33.82^{+0.78}_{-0.76}$	$31.61 \to 36.27$	$33.82^{+0.78}_{-0.76}$	$31.61 \to 36.27$	$33.46^{+0.87}_{-0.88}$	$30.98 \to 36.03$	$34.5^{+1.2}_{-1.0}$	$31.5 \to 38.0$
$\frac{\sin^2\theta_{23}}{10^{-1}}$	$5.58^{+0.20}_{-0.33}$	$4.27 \to 6.09$	$5.63^{+0.18}_{-0.24}$	$4.33 \to 6.09$	$5.51^{+0.19}_{-0.80}$	$4.30 \to 6.02$	$5.47^{+0.20}_{-0.30}$	$4.45 \to 5.99$
$\theta_{23}/^\circ$	$48.3^{+1.2}_{-1.9}$	$40.8 \to 51.3$	$48.6^{+1.0}_{-1.4}$	$41.1 \to 51.3$	$47.9^{+1.1}_{-4.0}$	$41.0 \to 50.9$	$47.7^{+1.2}_{-1.7}$	$41.8 \to 50.7$
$\frac{\sin^2\theta_{13}}{10^{-2}}$	$2.241^{+0.066}_{-0.065}$	$2.046 \to 2.440$	$2.237^{+0.066}_{-0.065}$	$2.044 \to 2.435$	$2.14^{+0.09}_{-0.07}$	$1.90 \to 2.39$	$2.160^{+0.083}_{-0.069}$	$1.96 \to 2.41$
$\theta_{13}/^\circ$	$8.61^{+0.13}_{-0.13}$	$8.22 \to 8.99$	$8.60^{+0.13}_{-0.13}$	$8.22 \to 8.98$	$8.41^{+0.18}_{-0.14}$	$7.9 \to 8.9$	$8.45^{+0.16}_{-0.14}$	$8.0 \to 8.9$
$\delta_{\rm CP}/^\circ$	222^{+38}_{-28}	$141 \to 370$	221^{+39}_{-28}	$144 \to 357$	238^{+41}_{-33}	$149 \to 358$	218^{+38}_{-27}	$157 \to 349$
$\frac{\Delta m^2_{21}}{10^{-5}\ {\rm eV}^2}$	$7.39^{+0.21}_{-0.20}$	$6.79 \to 8.01$	$7.39^{+0.21}_{-0.20}$	$6.79 \to 8.01$	$7.34^{+0.17}_{-0.14}$	$6.92 \to 7.91$	$7.55^{+0.20}_{-0.16}$	$7.05 \to 8.24$
$\frac{\Delta m^2_{32}}{10^{-3}\ {\rm eV}^2}$	$2.449^{+0.032}_{-0.030}$	$2.358 \to 2.544$	$2.454^{+0.029}_{-0.031}$	$2.362 \to 2.544$	$2.419^{+0.035}_{-0.032}$	$2.319 \to 2.521$	2.424 ± 0.03	$2.334 \to 2.524$
IO	$\Delta\chi^2 = 6.2$		$\Delta\chi^2 = 10.4$		$\Delta\chi^2 = 9.5$		$\Delta\chi^2 = 11.7$	
$\frac{\sin^2\theta_{12}}{10^{-1}}$	$3.10^{+0.13}_{-0.12}$	$2.75 \to 3.50$	$3.10^{+0.13}_{-0.12}$	$2.75 \to 3.50$	$3.03^{+0.14}_{-0.13}$	$2.64 \to 3.45$	$3.20^{+0.20}_{-0.16}$	$2.73 \to 3.79$
$\theta_{12}/^\circ$	$33.82^{+0.78}_{-0.76}$	$31.61 \to 36.27$	$33.82^{+0.78}_{-0.75}$	$31.62 \to 36.27$	$33.40^{+0.87}_{-0.81}$	$30.92 \to 35.97$	$34.5^{+1.2}_{-1.0}$	$31.5 \to 38.0$
$\frac{\sin^2\theta_{23}}{10^{-1}}$	$5.63^{+0.19}_{-0.26}$	$4.30 \to 6.12$	$5.65^{+0.17}_{-0.22}$	$4.36 \to 6.10$	$5.57^{+0.17}_{-0.24}$	$4.44 \to 6.03$	$5.51^{+0.18}_{-0.30}$	$4.53 \to 5.98$
$\theta_{23}/^\circ$	$48.6^{+1.1}_{-1.5}$	$41.0 \to 51.5$	$48.8^{+1.0}_{-1.2}$	$41.4 \to 51.3$	$48.2^{+1.0}_{-1.4}$	$41.8 \to 50.9$	$47.9^{+1.0}_{-1.7}$	$42.3 \to 50.7$
$\frac{\sin^2\theta_{13}}{10^{-2}}$	$2.261^{+0.067}_{-0.064}$	$2.066 \to 2.461$	$2.259^{+0.065}_{-0.065}$	$2.064 \to 2.457$	$2.18^{+0.08}_{-0.07}$	$1.95 \to 2.43$	$2.220^{+0.074}_{-0.076}$	$1.99 \to 2.44$
$\theta_{13}/^\circ$	$8.65^{+0.13}_{-0.12}$	$8.26 \to 9.02$	$8.64^{+0.12}_{-0.13}$	$8.26 \to 9.02$	$8.49^{+0.15}_{-0.14}$	$8.0 \to 9.0$	$8.53^{+0.14}_{-0.15}$	$8.1 \to 9.0$
$\delta_{\rm CP}/^\circ$	285^{+24}_{-26}	$205 \to 354$	282^{+23}_{-25}	$205 \to 348$	247^{+26}_{-27}	$193 \to 346$	281^{+23}_{-27}	$202 \to 349$
$\frac{\Delta m^2_{21}}{10^{-5}\ {\rm eV}^2}$	$7.39^{+0.21}_{-0.20}$	$6.79 \to 8.01$	$7.39^{+0.21}_{-0.20}$	$6.79 \to 8.01$	$7.34^{+0.17}_{-0.14}$	$6.92 \to 7.91$	$7.55^{+0.20}_{-0.16}$	$7.05 \to 8.24$
$\frac{\Delta m^2_{32}}{10^{-3}\ {\rm eV}^2}$	$-2.509^{+0.032}_{-0.032}$	$-2.603 \to -2.416$	$-2.510^{+0.030}_{-0.031}$	$-2.601 \to -2.419$	$-2.478^{+0.035}_{-0.033}$	$-2.577 \to -2.375$	$-2.50\pm^{+0.04}_{-0.03}$	$-2.59 \to -2.39$



PDG 2023

Weak interaction produced neutrinos

What neutrino state is produced in weak interaction? Let us use $\pi^- \to \mu^- + \bar{\nu}$ as example

$$M(\pi^- \to \mu^- \bar{\nu}) \sim < 0|\bar{u}\gamma^\mu \frac{1-\gamma_5}{2} d > \bar{\mu}\gamma_\mu \frac{1-\gamma_5}{2}\nu$$

$$\sim i f_\pi P^\mu_\pi \bar{\mu}\gamma_\mu (1-\gamma_5)\nu = i f_\pi \bar{\mu}(m_\nu \frac{1+\gamma_5}{2} + m_\mu \frac{1-\gamma_5}{2})\nu \; .$$

The neutrino participating the weak interaction and produced is in a state of $\psi(0)$

$$\psi(0) = (m_\nu \frac{1+\gamma_5}{2} + m_\mu \frac{1-\gamma_5}{2})\nu$$

$$= \frac{1}{\sqrt{2E(m_\nu^2 + m_\mu^2)}} \left(m_\nu \begin{pmatrix} 0 \\ \sqrt{E-P}u_+ + \sqrt{E+P}u_- \end{pmatrix} + m_\mu \begin{pmatrix} \sqrt{E+P}u_+ + \sqrt{E-P}u_- \\ 0 \end{pmatrix} \right) \; .$$

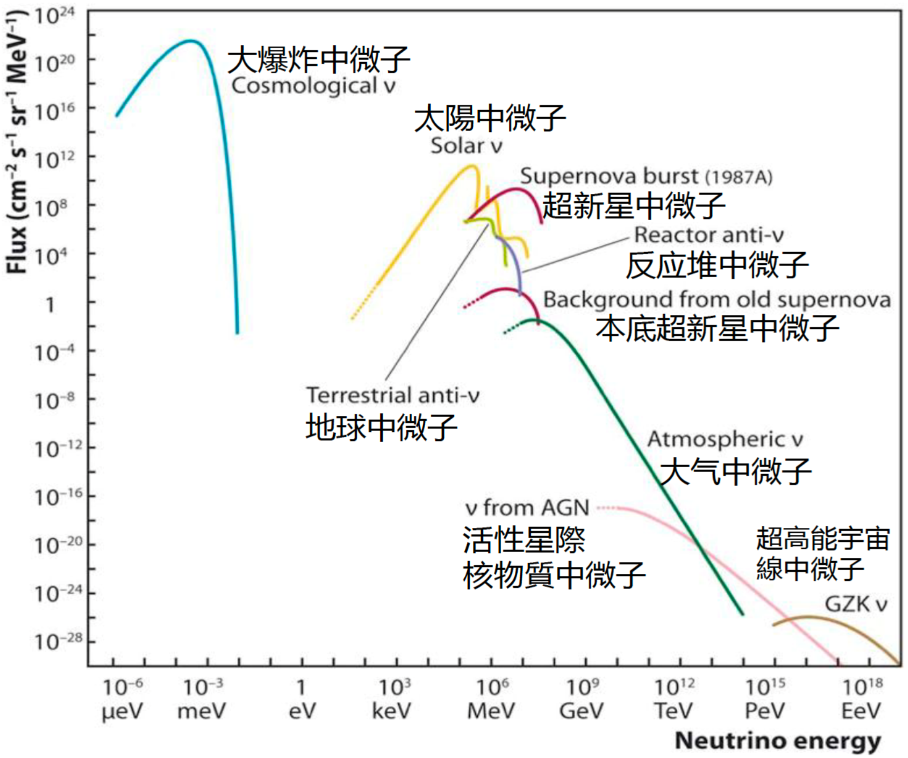
Relativistic case, $E >> m_\nu : \psi(0) \approx \begin{pmatrix} u_+ \\ 0 \end{pmatrix}$, Non-relativistic case, $E \sim m_\nu : \psi(0) \approx \begin{pmatrix} (u_- + u_+)/\sqrt{2} \\ 0 \end{pmatrix}$

Chiral oscillation probability extremely small for relativistic neutrinos because the suppresion factor: m^2/E^2.

Numerically vailid to use Dirac neutrinos to describe neutrino oscillations. Oscillation period can of order ps. Similar for charged lepton, bu the oscillation period is Et = 2pi,

t <2pi.me = 1.3×10^{-21}s too short. Always time averaged effect.

However the effect can be large for non-relativistic netrinos, such as background cosmic nutrinos where p is small compared with m.



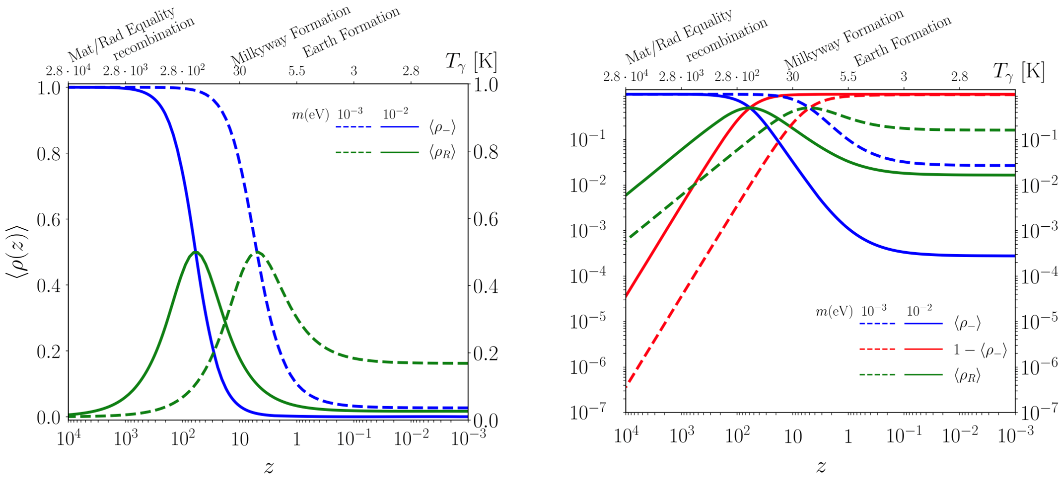
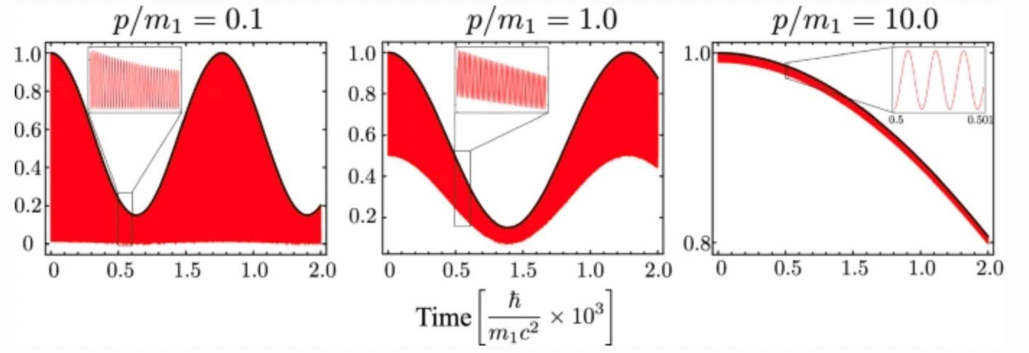


Fig. 1. Evolution of the chiral density matrix elements $\langle\rho_-\rangle$ (blue) or $1-\langle\rho_-\rangle$ (red), and $\rho_R = \mathbb{R}[c_L c_R^*]$ (green) as functions of the redshift z for $m = 10^{-2}$ eV (solid) and $m = 10^{-3}$ eV (dashed) with $|\mathbf{p}_d| = 1$ MeV. The redshift at neutrino decoupling is taken to be $z_d \sim 6 \times 10^9$.

Fig. 2



Survival probability $\mathcal{P}_{e\to e}(t)$ as a function of time. The black curves indicate the standard survival probability formula (16) and the red curves depict the full formula including the fast chiral oscillations (depicted in the insets). Parameters: $\sin^2\theta = 0.306$, $m_2^2 = \Delta_{21}^2 + m_1^2$, with $\Delta_{21}^2/m_1^2 = 0.01$

S-F Ge & P Pasquini, PLB811(2020)135961; V Bittencourt, A. Bernardini & M. Blasone, EPJC81 (2021)411.

Neutrino Chiral Oscillation

In the SM neutrinos are produced by W and/or Z interactions.

At production $t=0$ point, they are left-handed and normalized, $\psi_L^h(0)=\sqrt{\frac{2E}{E-h\cdot p}}\frac{1-\gamma_5}{2}\psi^h(0)$.

$$\psi_L^h(t)=\sqrt{\frac{2E}{E-h\cdot p}}e^{-iHt}\frac{1-\gamma^5}{2}\psi^h(0)=\psi_L^h(t)=\sqrt{\frac{2E}{E-h\cdot p}}\left(e^{-iEt}\frac{1-\gamma_5}{2}\psi^h(0)-i\frac{m}{E}\sin(Et)\left[\beta,\frac{1-\gamma_5}{2}\right]\psi^h(0)\right).$$

Used

$$U(t)=e^{-iHt}=\cos(Et)-i\frac{\boldsymbol{\alpha}\cdot\mathbf{p}+m\beta}{E}\sin(Et)$$

$$\psi_L^{h\dagger}\psi_L^h(t)=\left(\cos(Et)+i\frac{h\cdot p}{E}\sin(Et)\right)e^{i\mathbf{p}\cdot\mathbf{L}}\,,\quad \psi_R^{h\dagger}\psi_L^h(t)=\left(-i\frac{m}{E}\sin(Et)\right)e^{i\mathbf{p}\cdot\mathbf{L}}$$

$$P(\nu_L^h\to\nu_L^h)=|\psi_L^{h\dagger}\psi_L^h(t)|^2=1-\frac{m^2}{E^2}\sin^2(Et)\,,\quad P(\nu_L^h\to\nu_R^h)=|\psi_R^{h\dagger}\psi_L^h(t)|^2=\frac{m^2}{E^2}\sin^2(Et)$$

Left-handed neutrinos oscillated into right-handed ones!

$$P(\nu_{Li}^h\to\nu_{Lk}^h)=|V_{ij}V_{kj}^*(\cos(E_jt)+i\frac{h\cdot p}{E_j}\sin(E_jt))|^2\,,\quad P(\nu_{Li}^h\to\nu_{Rk}^h)=|-iV_{ij}V_{kj}^*\frac{m_j}{E_j}\sin(E_jt)|^2\psi_R^{h\dagger}\psi_L^h(t)|^2$$

S-F Ge & P Pasquini, PLB811(2020)135961; V Bittencourt, A. Bernardini & M. Blasone, EPJC81 (2021)411.

2. Neutrino Oscillation in Matter

When neutrinos travel in matter, due to interaction of neutrions with matter mediated by W and Z, the Lagrangian is modified

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - j^\mu \bar{\psi}\gamma_\mu \frac{1-\gamma_5}{2}\psi \ .$$

j^μ is the matter current which neutrino can interact.

In the rest frame of the homogeneous, isotropic, unpolarized electrical neutrality medium, $j^\mu = (\rho, \vec{0})$

$\rho = \sqrt{2}G_F\left(N_e\delta_{\alpha e} - \frac{1}{2}N_n\right)$. N_e, N_n number density of electron and neutron, $\delta_{\alpha e}$ are zero for ν_μ and ν_τ.

$$\left(i\partial\!\!\!/ - m - \rho\gamma_0\frac{1-\gamma_5}{2}\right)\psi = 0 \qquad H = \mathbf{p}\cdot\boldsymbol{\alpha} + m\beta + \rho\frac{1-\gamma_5}{2} = \begin{pmatrix} \rho - \mathbf{p}\cdot\boldsymbol{\sigma} & m \\ m & \mathbf{p}\cdot\boldsymbol{\sigma} \end{pmatrix} = \begin{pmatrix} \rho - h\cdot p & m \\ m & h\cdot p \end{pmatrix}$$

The eigenvalues of H are: $E_1 = \frac{\rho}{2} + E_h$, $E_2 = \frac{\rho}{2} - E_h$, $E_h = \sqrt{m^2 + (h\cdot p - \rho/2)^2}$,

$$\psi_1 = \frac{1}{\sqrt{2E_h}}\begin{pmatrix} \sqrt{E_h - (h\cdot p - \frac{\rho}{2})}\, u^h \\ \sqrt{E_h + (h\cdot p - \frac{\rho}{2})}\, u^h \end{pmatrix} , \quad \psi_2 = \frac{1}{\sqrt{2E_h}}\begin{pmatrix} \sqrt{E_h + (h\cdot p - \frac{\rho}{2})}\, u^h \\ -\sqrt{E_h - (h\cdot p - \frac{\rho}{2})}\, u^h \end{pmatrix}$$

The evolution for a given wave function entering the media with momentum $\vec{p}$ at $t = 0$ is

$$\psi_L^h(t) = e^{-iHt}\psi_L^h(0) = e^{-\frac{i}{2}\rho t}\begin{pmatrix} \cos(E_h t) + i\frac{h\cdot p - \frac{\rho}{2}}{E_h}\sin(E_h t) & -i\frac{m}{E_h}\sin(E_h t) \\ -i\frac{m}{E_h}\sin(E_h t) & \cos(E_h t) - i\frac{h\cdot p-\frac{\rho}{2}}{E_h}\sin(E_h t) \end{pmatrix}\begin{pmatrix} u^h \\ 0 \end{pmatrix},$$

$$P(\psi_L^h \to \psi_L^h) = 1 - \frac{m^2}{E_h^2}\sin^2(E_h t)\ , \quad P(\psi_L^h \to \psi_R^h) = \frac{m^2}{E_h^2}\sin^2(E_h t).$$

Similar as that for free space, but dependent on matter density via $E_h = \sqrt{m^2 + (h\cdot p - \rho/2)^2}$.

There is a resonant enhanced chiral oscillation at $h \cdot p - \rho/2 = 0$!

From the above, one can easily recover the usual matter oscillation formalism with $J_L^\mu = (\rho, \vec{0})$ in the relativistic case $p \gg M > m \gg \rho$. Keeping the leading effect in this limit, one obtains,

$$H_{\text{eff}} = p + \frac{M^\dagger M}{2p} - h\cdot\rho\ . \tag{13}$$

Then we can find that matter effect would influence the contribution from mixing angle and mass square. Note that for $h = -1$ helicity, it is the usual leading order neutrino oscillation in matter effective Hamiltonian which can cause matter induced MSW resonant effect. But for $h = +1$ or $\rho < 0$, the matter effects are different.

3. Majorana and Seesaw Neutrinos in Free Space

Pure left-handed neutrinos, having Majorana mass like in Type II seesaw model

$$\mathcal{L} = \bar{\nu}_L i\partial\!\!\!/ \nu_L - \frac{1}{2} m \left(\bar{\nu}_L^c \nu_L + \text{h.c.}\right) - \bar{\nu}_L j_L^\mu \gamma_\mu \nu_L = \frac{1}{2} \bar{\psi}^m \left(i\partial\!\!\!/ - m\right) \psi^m - \bar{\psi}^m j^\mu \gamma_\mu \tfrac{1-\gamma_5}{2} \psi^m \quad , \quad \psi^m = \nu_L + \nu_L^c \, .$$

$$(i\partial\!\!\!/ - \widehat{M})\psi^m - j_L^\mu \gamma_\mu \tfrac{1-\gamma_5}{2} \psi^m + (j_L^\mu)^* \gamma_\mu \tfrac{1+\gamma_5}{2} \psi^m = 0 \, ,$$

$$\Rightarrow H = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m - \beta j_L^\mu \gamma_\mu (1 - \gamma_5) = \begin{pmatrix} \rho - \mathbf{p} \cdot \boldsymbol{\sigma} & m \\ m & \mathbf{p} \cdot \boldsymbol{\sigma} - \rho \end{pmatrix} \, ,$$

ρ in the Dirac neutrino case is replaced by 2ρ.

$$U(t) = \begin{pmatrix} \cos(E_h^m t) + i \frac{h \cdot p - \rho}{E_h^m} \sin(E_h^m t) & -i \frac{m}{E_h^m} \sin(E_h^m t) \\ -i \frac{m}{E_h^m} \sin(E_h^m t) & \cos(E_h^m t) - i \frac{h \cdot p - \rho}{E_h^m} \sin(E_h^m t) \end{pmatrix}$$

where $E_h^m = \sqrt{m^2 + (h \cdot p - \rho)^2}$. Resonant point shifted to: $p - \rho = 0$.

$$P(\nu_L^h \to \nu_L^h) = \cos^2(E_h^m t) + \frac{(h \cdot p - \rho)^2}{(E_h^m)^2} \sin^2(E_h^m t) \, , \quad P(\nu_L^h \to (\nu_L^h)^c) = \frac{m^2}{(E_h^m)^2} \sin^2(E_h^m t)$$

One also obtains the same $H_{eff} = p + \frac{M^\dagger M}{2p} - h \cdot \rho$.

Seesaw Neutrino Oscillation

$$\mathcal{L} = \bar{\nu}_L i\partial\!\!\!/ \nu_L + \bar{N}_R i\partial\!\!\!/ N_R - \frac{1}{2}\left(\begin{pmatrix}\bar{\nu}_L^c & \bar{N}_R\end{pmatrix}\begin{pmatrix}M_L & M_D^T \\ M_D & M_R\end{pmatrix}\begin{pmatrix}\nu_L \\ N_R^c\end{pmatrix} + \text{h.c.}\right) = \bar{\psi}_L i\partial\!\!\!/ \psi_L - \frac{1}{2}\left(\bar{\psi}_L^c \mathcal{M}\psi_L + \text{h.c.}\right) \ ,$$

$$\psi_L = \begin{pmatrix}\nu_L \\ N_R^c\end{pmatrix} \ , \quad \mathcal{M} = \begin{pmatrix}M_L & M_D^T \\ M_D & M_R\end{pmatrix} \ ,$$

$$V^T \mathcal{M} V = \widehat{M} = \text{diag}\{m_1, m_2, m_3, M_1, M_2, M_3\} \ , \ \ \psi_L^m = V^\dagger \psi_L \ , \ \ \psi^m = \psi_L^m + (\psi_L^m)^c$$

$$\mathcal{L} = \bar{\psi}_L^m i\partial\!\!\!/ \psi_L^m - \frac{1}{2}\left((\bar{\psi}_L^m)^c \widehat{M} \psi_L^m + \text{h.c.}\right) = \frac{1}{2}\left(\bar{\psi}^m (i\partial\!\!\!/ - \widehat{M})\psi^m\right) \ ,$$

$$H = \begin{pmatrix}\boldsymbol{\alpha}\cdot\mathbf{p} + \widehat{M}_l \beta & 0 \\ 0 & \boldsymbol{\alpha}\cdot\mathbf{p} + \widehat{M}_h\end{pmatrix} \ ,$$

$$\widehat{M}_l = \text{diag}\{m_1, m_2, m_3\} \ , \ \widehat{M}_h = \text{diag}\{M_1, M_2, M_3\} \ .$$

The initial state would be

$$\psi_{Li}^h = \begin{pmatrix} V_{i1}^* \begin{pmatrix} u^h \\ 0 \end{pmatrix} \\ \ldots \\ V_{i6}^* \begin{pmatrix} u^h \\ 0 \end{pmatrix} \end{pmatrix}, \quad (\psi_{Li}^h)^c = \begin{pmatrix} V_{i1} \begin{pmatrix} 0 \\ u^h \end{pmatrix} \\ \ldots \\ V_{i6} \begin{pmatrix} 0 \\ u^h \end{pmatrix} \end{pmatrix},$$

$$U(t) = e^{-iHt}$$

$$= \begin{pmatrix} \begin{pmatrix} \cos(E_{m_1}t)+i\frac{h\cdot p}{E_{m_1}}\sin(E_{m_1}t) & -i\frac{m_1}{E_{m_1}}\sin(E_{m_1}t) \\ -i\frac{m_1}{E_{m_1}}\sin(E_{m_1}t) & \cos(E_{m_1}t)-i\frac{h\cdot p}{E_{m_1}}\sin(E_{m_1}t) \end{pmatrix} & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & \begin{pmatrix} \cos(E_{M_3}t)+i\frac{h\cdot p}{E_{M_3}}\sin(E_{M_3}t) & -i\frac{M_3}{E_{M_3}}\sin(E_{M_3}t) \\ -i\frac{M_3}{E_{M_3}}\sin(E_{M_3}t) & \cos(E_{M_3}t)-i\frac{h\cdot p}{E_{M_3}}\sin(E_3^M t) \end{pmatrix} \end{pmatrix}$$

$$E_{m_i} = \sqrt{p^2 + m_i^2}\,, \quad E_{M_i} = \sqrt{p^2 + M_i^2}$$

The oscillation probability

$$P(\psi_{Li}^h \to \psi_{Lj}^h) = \left|\psi_{Lj}^\dagger e^{-iHt}\psi_{Li}|^2\right|^2 = \left|\sum_k V_{ik}V_{jk}^* \left(\cos(E_k t) + i\frac{h\cdot p}{E_k}\sin(E_k t)\right)\right|^2$$

$$P(\psi_{Li}^h \to (\psi_{Lj}^h)^c) = \left|(\psi_{Lj}^c)^\dagger e^{-iHt}\psi_{Li}\right|^2 = \left|\sum_k V_{ik}^* V_{jk}^* \left(-i\frac{m_k}{E_k}\sin(E_k t)\right)\right|^2$$

$$P((\psi_{Li}^h)^c \to (\psi_{Lj}^h)^c) = \left|(\psi_{Lj}^c)^\dagger e^{-iHt}\psi_{Lj}^c\right|^2 = \left|\sum_k V_{ik}^* V_{jk} \left(\cos(E_k t) - i\frac{h\cdot p}{E_k}\sin(E_k t)\right)\right|^2$$

$$P((\psi_{Li}^h)^c \to \psi_{Lj}^h) = \left|\psi_{Lj}^\dagger e^{-iHt}\psi_{Lj}^c\right|^2 = \left|\sum_k V_{ik}V_{jk} \left(-i\frac{m_k}{E_k}\sin(E_k t)\right)\right|^2$$

Majorana phase effects on chiral oscillation

One light and one heavy example: $\widehat{M} = \begin{pmatrix} m & 0 \\ 0 & M \end{pmatrix}$, $V = \begin{pmatrix} V_{a1} & V_{a2} \\ V_{s1} & V_{s2} \end{pmatrix} = \begin{pmatrix} \cos\theta & e^{i\eta}\sin\theta \\ -\sin\theta & e^{i\eta}\cos\theta \end{pmatrix}$.

$$\nu_L^h = \begin{pmatrix} V_{a1}^* \begin{pmatrix} u^h \\ 0 \end{pmatrix} \\ V_{a2}^* \begin{pmatrix} u^h \\ 0 \end{pmatrix} \end{pmatrix}, \quad N_R^h = \begin{pmatrix} V_{s1} \begin{pmatrix} 0 \\ u^h \end{pmatrix} \\ V_{s2} \begin{pmatrix} 0 \\ u^h \end{pmatrix} \end{pmatrix}, \quad (\nu_L^h)^c = \begin{pmatrix} V_{a1} \begin{pmatrix} 0 \\ u^h \end{pmatrix} \\ V_{a2} \begin{pmatrix} 0 \\ u^h \end{pmatrix} \end{pmatrix}, \quad (N_R^h)^c = \begin{pmatrix} V_{s1}^* \begin{pmatrix} u^h \\ 0 \end{pmatrix} \\ V_{s2}^* \begin{pmatrix} u^h \\ 0 \end{pmatrix} \end{pmatrix}. \qquad (2$$

$$P(\nu_L^h \to \nu_L^h) = \left(\cos^2\theta\cos(E_m t) + \sin^2\theta\cos(E_M t)\right)^2 + p^2\left(\frac{\cos^2\theta}{E_m}\sin(E_m t) + \frac{\sin^2\theta}{E_M}\sin(E_M t)\right)^2$$

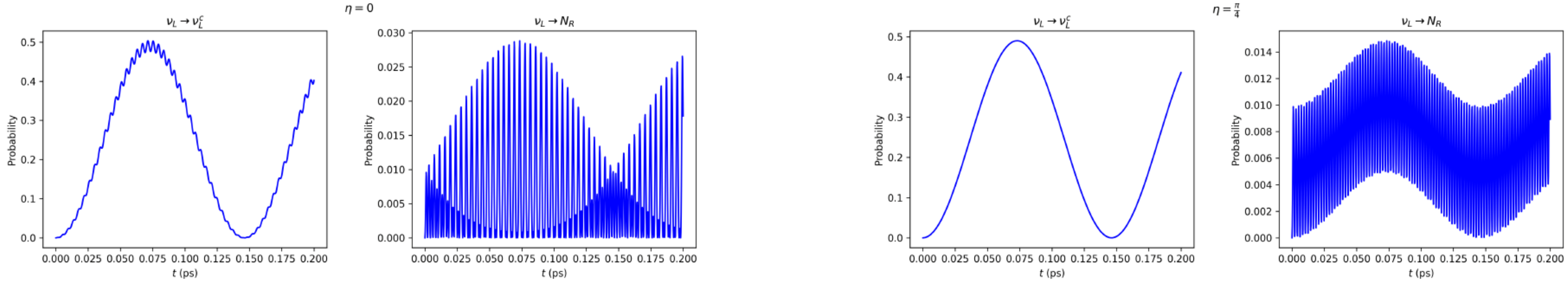
$$P(\nu_L^h \to (N_R^h)^c) = \frac{\sin^2 2\theta}{4}\left(\left(\cos(E_m t) - \cos(E_M t)\right)^2 + p^2\left(\frac{\sin(E_m t)}{E_m} - \frac{\sin(E_M t)}{E_M}\right)^2\right)$$

$$P(\nu_L^h \to (\nu_L^h)^c) = \frac{m^2}{(E_m)^2}\cos^4\theta\sin^2(E_m t) + \frac{mM}{2E_m E_M}\sin^2 2\theta\cos 2\eta\sin(E_m t)\sin(E_M t) + \frac{M^2}{(E_M)^2}\sin^4\theta\sin^2(E_M t)$$

$$P(\nu_L^h \to N_R^h) = \frac{\sin^2 2\theta}{4}\left(\frac{m^2}{(E_m)^2}\sin^2(E_m t) - \frac{2mM}{E_m E_M}\cos 2\eta\sin(E_m t)\sin(E_M t) + \frac{M^2}{(E_M)^2}\sin^2(E_M t)\right).$$

Majorana phase show up in the chiral oscillation explicitly!

This also applies to Type II seesaw with two generations!



(a)$\eta = 0$ (b)$\eta = \frac{\pi}{4}$

FIG. 1: Neutrino chiral rotation in last two equation in Eq.(23) in non-relativistic regime. The initial state is an active left-handed neutrino. The input parameters used are: $p = 0.01$eV, $m = 0.01$eV, $M = 1$eV, and $\sin\theta = \sqrt{m/M} = 0.1$, and the Majorana phase takes two different values $\eta = 0, \pi/4$.

4. Majorana and Seesaw Neutrinos in Matter

The general seesaw neutrino Lagrangian in matter propagation

$$\mathcal{L} = \bar{\nu}_L i\partial\!\!\!/ \nu_L + \bar{N}_R i\partial\!\!\!/ N_R - \frac{1}{2}\left(\begin{pmatrix}\bar{\nu}_L^c & \bar{N}_R\end{pmatrix}\begin{pmatrix}M_L & M_D^T \\ M_D & M_R\end{pmatrix}\begin{pmatrix}\nu_L \\ N_R^c\end{pmatrix} + \text{h.c.}\right) - \begin{pmatrix}\bar{\nu}_L & \bar{N}_R^c\end{pmatrix}\begin{pmatrix} j_L^\mu & j_{RL}^\mu \\ j_{RL}^{\mu\dagger} & j_R\end{pmatrix}\gamma_\mu\begin{pmatrix}\nu_L \\ N_R^c\end{pmatrix}$$
$$= \bar{\psi}_L i\partial\!\!\!/ \psi_L - \frac{1}{2}\left(\bar{\psi}_L^c \mathcal{M}\psi_L + \text{h.c.}\right) - \bar{\psi}_L J^\mu \gamma_\mu \psi_L$$

For homogeneous, isotropic, unpolarized electrical neutrality matter medium at rest, only j_L^0 is non-zero

$$j_L^0 = \begin{pmatrix}\rho_e & 0 & 0 \\ 0 & \rho_\mu & 0 \\ 0 & 0 & \rho_\tau\end{pmatrix} = \begin{pmatrix}\sqrt{2}G_F\left(N_e - \frac{1}{2}N_n\right) & 0 & 0 \\ 0 & -\frac{G_F}{\sqrt{2}}N_n & 0 \\ 0 & 0 & -\frac{G_F}{\sqrt{2}}N_n\end{pmatrix} .$$

In terms of the mass eigenstate $\psi_L = V\psi_L^m$, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}\left(\bar{\psi}^m(i\partial\!\!\!/ - \widehat{M})\psi^m\right) - \bar{\psi}^m \widetilde{J}^\mu \gamma_\mu \tfrac{1-\gamma_5}{2}\psi^m \ , \quad \psi^m = \psi_L^m + (\psi_L^m)^c \ , \ \ \widetilde{J}^\mu = V^\dagger J^\mu V.$$

$$(i\partial\!\!\!/ - \widehat{M})\psi^m - \widetilde{J}^\mu \gamma_\mu \tfrac{1-\gamma_5}{2}\psi^m + (\widetilde{J}^\mu)^* \gamma_\mu \tfrac{1+\gamma_5}{2}\psi^m = 0 \ .$$

$$H = \begin{pmatrix} \boldsymbol{\alpha}\cdot\mathbf{p} & 0 \\ 0 & \boldsymbol{\alpha}\cdot\mathbf{p} \end{pmatrix} + \begin{pmatrix} \beta\widehat{M}_l & 0 \\ 0 & \beta\widehat{M}_h \end{pmatrix} + V^\dagger \begin{pmatrix} j_L^\mu & j_{RL}^{\mu\dagger} \\ j_{RL}^\mu & -j_R^{\mu T} \end{pmatrix} V\gamma^0\gamma_\mu \frac{1-\gamma^5}{2} - \left(V^\dagger \begin{pmatrix} j_L^\mu & j_{RL}^{\mu\dagger} \\ j_{RL}^\mu & -j_R^{\mu T} \end{pmatrix} V\right)^* \gamma^0\gamma_\mu \frac{1+\gamma^5}{2}$$

Because the off-diagonal interaction, difficulty to get U(t). For just one light and one heavy neutrinos, can get a closed analytic expression.

Let $\widehat{M}_l = m$ and $\widehat{M}_h = M$, similar to Dirac case $j^\mu = (\rho, \vec{0})$ and $j_{RL}^\mu = j_R^\mu = 0$,

$$\widetilde{J}^\mu\gamma_\mu = V^\dagger J^\mu V\gamma_\mu = \begin{pmatrix} \frac{\rho}{2}(1+\cos 2\theta) & \frac{\rho}{2}e^{i\eta}\sin 2\theta \\ \frac{\rho}{2}e^{-i\eta}\sin 2\theta & \frac{\rho}{2}(1-\cos 2\theta) \end{pmatrix}\gamma_0\ .$$

$$H = \begin{pmatrix} \frac{\rho}{2}(1+\cos 2\theta) - \mathbf{p}\cdot\boldsymbol{\sigma} & m & \frac{\rho}{2}e^{i\eta}\sin 2\theta & 0 \\ m & \mathbf{p}\cdot\boldsymbol{\sigma} - \frac{\rho}{2}(1+\cos 2\theta) & 0 & -\frac{\rho}{2}e^{-i\eta}\sin 2\theta \\ \frac{\rho}{2}e^{-i\eta}\sin 2\theta & 0 & \frac{\rho}{2}(1-\cos 2\theta) - \mathbf{p}\cdot\boldsymbol{\sigma} & M \\ 0 & -\frac{\rho}{2}e^{i\eta}\sin 2\theta & M & \mathbf{p}\cdot\boldsymbol{\sigma} - \frac{\rho}{2}(1-\cos 2\theta) \end{pmatrix}$$

Time averaged oscillation

The eigenvaluse of H are:
$$E_{1h} = -\sqrt{A_1 - A_2}\,, \quad E_{2h} = \sqrt{A_1 - A_2}\,,$$
$$E_{3h} = -\sqrt{A_1 + A_2}\,, \quad E_{4h} = \sqrt{A_1 + A_2}\,,$$

$$A_1 = \frac{m^2 + M^2}{2} + \left(h \cdot p - \frac{\rho}{2}\right)^2 + \frac{\rho^2}{4}$$

$$A_2 = \sqrt{\frac{2\left(m^2 - M^2\right)^2 + \rho^2\left(2\left(m^2 + M^2 - 2mM\cos 2\eta\right)\sin^2\theta + 8\left(h\cdot p - \frac{\rho}{2}\right)^2\right) - 8\rho(m^2 - M^2)(h\cdot p - \frac{\rho}{2})\cos 2\theta}{8}}$$

$$U(t) = e^{-iHt} = \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ U_{21} & U_{22} & U_{23} & U_{24} \\ U_{31} & U_{32} & U_{33} & U_{34} \\ U_{41} & U_{42} & U_{43} & U_{44} \end{pmatrix}$$

We illustrate this with the limit $p \gg M > m \gg \rho$. In this case we have

$$P(\nu_L^h \to (\nu_L^h)^c) = \frac{(m^2 - M^2)^2}{8A_2^2} \left(\frac{(m^2 \cos^4\theta + M^2 \sin^4\theta)}{p^2} - \frac{\rho(4m^2\cos^6\theta - 4M^2\sin^6\theta + mM\cos 2\eta\cos 2\theta\sin^2 2\theta)}{p(m^2 - M^2)} \right.$$
$$\left. + \frac{2\rho^2(2m^2\cos^4\theta + 2M^2\sin^4\theta + mM\cos 2\eta\sin^2 2\theta)}{(m^2 - M^2)^2} \right),$$
$$P(\nu_L^h \to N_R^h) = \frac{(m^2 - M^2)^2 \sin^2 2\theta}{32A_2^2} \left(\frac{m^2 + M^2}{p^2} - \frac{2\rho(m^2 + M^2 - 2mM\cos 2\eta)\cos 2\theta}{p(m^2 - M^2)} + \frac{4\rho^2\left(m^2 + M^2 - 2mM\cos 2\eta\right)}{(m^2 - M^2)^2} \right) \tag{25}$$

The probabilities for the other two oscillation modes are

$$P(\nu_L^h \to \nu_L^h) = \frac{1}{2} + \frac{\cos^2 2\theta_{\rm eff}}{2} - P(\nu_L^h \to (\nu_L^h)^c)\,, \quad P(\nu_L^h \to (N_R^h)^c) = \frac{\sin^2 2\theta_{\rm eff}}{2} - P(\nu_L^h \to N_R^h)\,. \tag{26}$$

where $\cos 2\theta_{\rm eff} = \left((M^2 - m^2)\cos 2\theta + 2\rho(h \cdot p - \frac{\rho}{2})\right)/(2A_2)$. We do see the oscillation probabilities dependence one η even taking the time average. Note that the effect vanishes if any of the m, M and θ is zero.

$$P(\nu_L^h \to \nu_L^h) = \left|(\nu_L^h)^\dagger e^{-iHt}\nu_L^h\right|^2 = \left|U_{11}\frac{1+\cos 2\theta}{2} + U_{13}e^{-i\eta}\frac{\sin 2\theta}{2} + U_{31}e^{i\eta}\frac{\sin 2\theta}{2} + U_{33}\frac{1-\cos 2\theta}{2}\right|^2$$

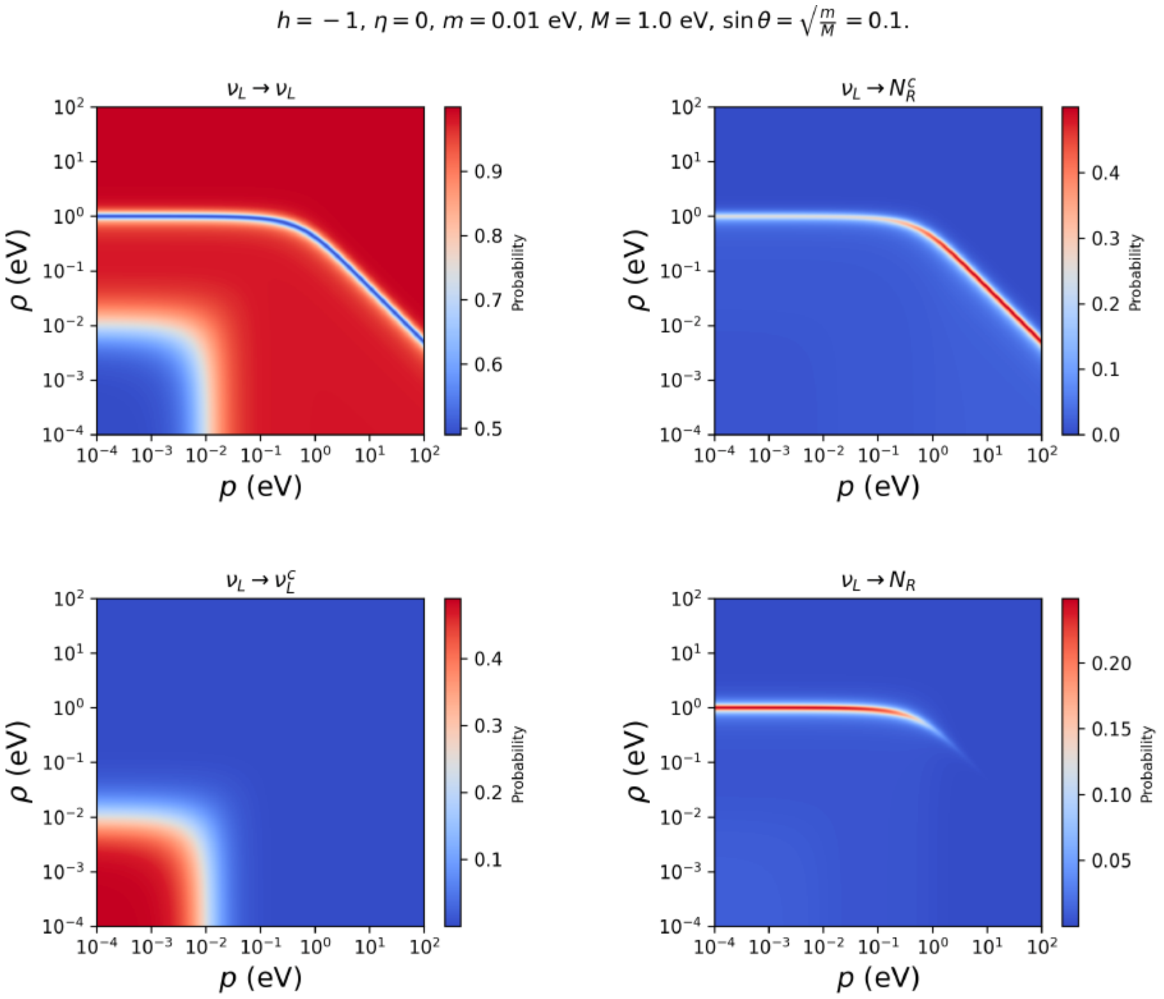
$$P(\nu_L^h \to (N_R^h)^c) = \left|((N_R^h)^c)^\dagger e^{-iHt}\nu_L^h\right|^2 = \left|-U_{11}\frac{\sin 2\theta}{2} - U_{13}e^{-i\eta}\frac{1-\cos 2\theta}{2} + U_{31}e^{i\eta}\frac{1+\cos 2\theta}{2} + U_{33}\frac{\sin 2\theta}{2}\right|^2$$

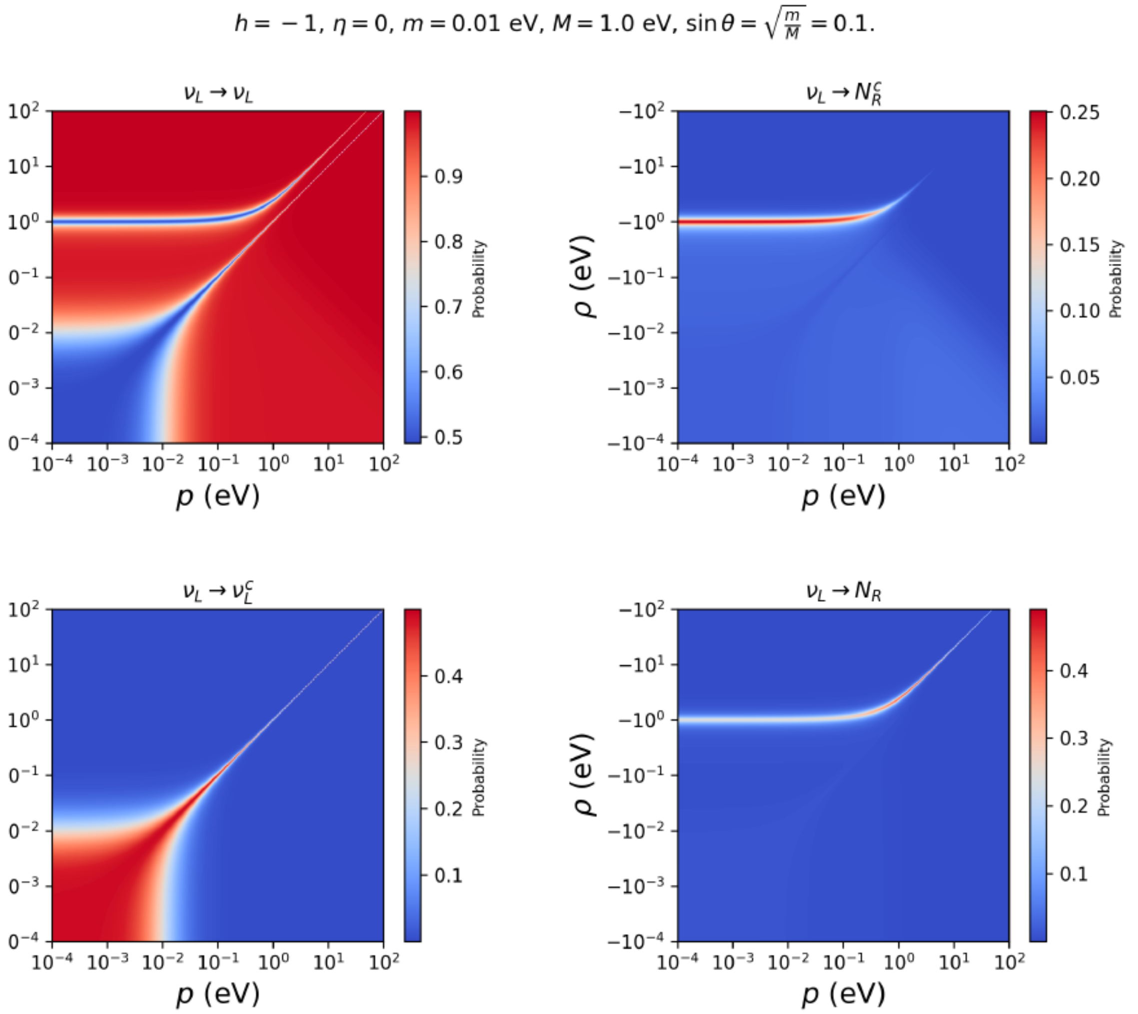
$$P(\nu_L^h \to (\nu_L^h)^c) = \left|((\nu_L^h)^c)^\dagger e^{-iHt}\nu_L^h\right|^2 = \left|U_{21}\frac{1+\cos 2\theta}{2} + U_{23}e^{-i\eta}\frac{\sin 2\theta}{2} + U_{41}e^{-i\eta}\frac{\sin 2\theta}{2} + U_{43}e^{-2i\eta}\frac{1-\cos 2\theta}{2}\right|^2$$

$$P(\nu_L^h \to N_R^h) = \left|(N_R^h)^\dagger e^{-iHt}\nu_L^h\right|^2 = \left|-U_{21}\frac{\sin 2\theta}{2} - U_{23}e^{-i\eta}\frac{1-\cos 2\theta}{2} + U_{41}e^{-i\eta}\frac{1+\cos 2\theta}{2} + U_{43}e^{-2i\eta}\frac{\sin 2\theta}{2}\right|^2$$

η enters the chiral oscillation, even when time averaged!

Detailed Numerical calculations





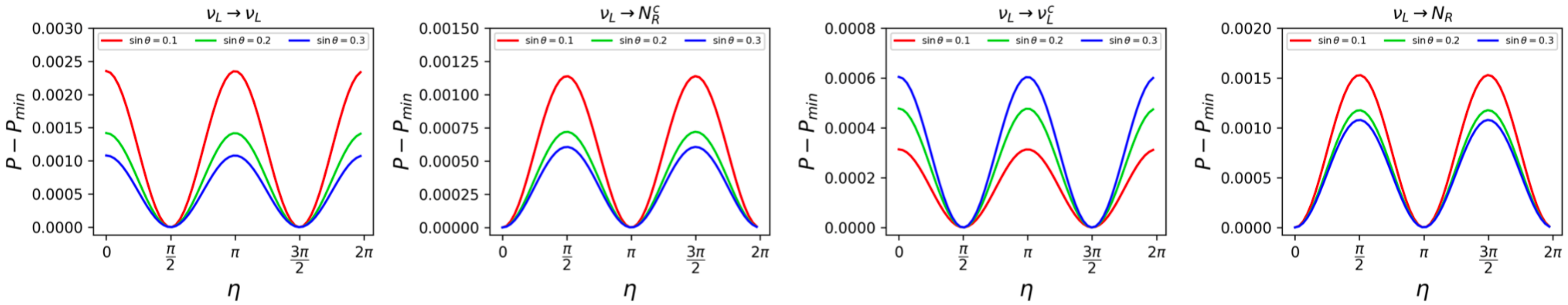


FIG. 3: The time averaged Majorana phase η effects on probabilities. The vertical axis is the difference between time averaged probabilities and their minimum values. The oscillating probabilities depend on the mixing angle θ. Other parameters are chosen from a point in matter effect significant region in Fig. 2(a).

$h = -1$, $\rho_W = 0.0026\text{eV}$, $\rho_Z = -0.0013\text{eV}$, $\theta = 33.41^\circ$, $m = 10^{-3}\text{eV}$, $M^2 - m^2 = 7.41 \times 10^{-5}\text{eV}^2$.

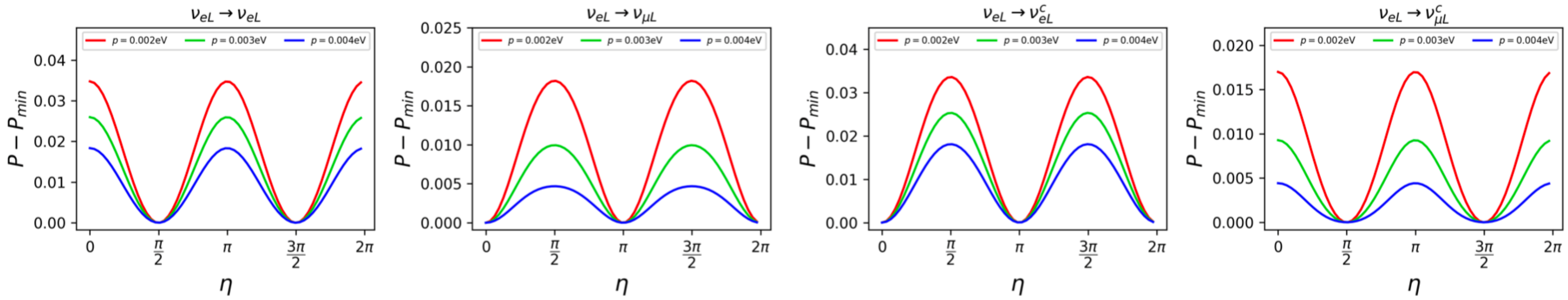


FIG. 4: The time averaged Majorana phase η effects on probabilities in two active Majorana neutrinos case. The vertical axis is the difference between time averaged probabilities and their minimum values. The red, green and blue lines are corresponding to $p = 0.002\text{eV}$, 0.003eV and 0.004eV, respectively. The mixing angle $\theta = \theta_{12}$ and mass square difference $M^2 - m^2 = \Delta m_{21}^2$, whose values are from [1]. $\rho_W = 0.0026\text{eV}$ and $\rho_Z = -0.0013\text{eV}$ are corresponding the mass density $a_p = 3.4 \times 10^{10}\text{g/cm}^3$ and $a_n = 3.4 \times 10^{10}\text{g/cm}^3$, respectively, so the total mass density $a \approx 6.8 \times 10^{10}\text{g/cm}^3$, which could be found in neutron star cluster [12].

Massive neutrinos can have chiral oscillation, supressed by m^2/E^2 for relativistic neutrinos, but the supression is lefted for non-relativisitc neutrions.

Majorana phases modify the oscillatin pattern. In principle Majorana phases can be probed in chiral oscillation.

When matter effects are included, energies are splitted into two, there is a resoant point for chiral oscillation. Also even the oscillation is time aeraged, Majorana phase still affect oscillatin pattern.

Thank you for your attentions