Indices of supersymmetric theories in various dimensions

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High Energy Physics – Theory

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Witten Index and Wall Crossing

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We compute the Witten index of one-dimensional gauged linear sigma models with at least □ = 2 supersymmetry. In the phase where the gauge group is broken to a finite group, the index is expressed as a certain residue integral. It is subject to a change as the Fayet–Iliopoulos parameter is varied through the phase boundaries. The wall crossing formula is expressed as an integral at infinity of the Coulomb branch. The result is applied to many examples, including quiver quantum mechanics that is relevant for BPS states in d = 4 □ = 2 theories.

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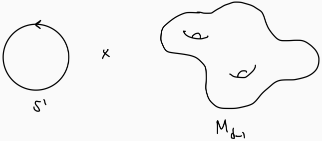
In 2014, we wrote a paper on calculating the Witten index of 1D GLSM with $\mathcal{N} = 2$ supersymmetry.

I will discuss some developments during the last 10 years since then.

Indices in various dimensions

Witten index can be defined for d-dimensional supersymmetric QFTs on compact manifolds $S^1 \times M_{d-1}$:



$$I[M_{d-1}] = \mathrm{Tr}_{\mathcal{H}_{M_{d-1}}} (-1)^F$$

Why do we care?

- Strong coupling dynamics of quantum field theories.
- Counting problems in gravity, string theory.
- Enumerative geometry, topological invariants, etc,.

What's understood so far

The best understood backgrounds are the one that preserves two real supercharges with $U(1)_R$ symmetry. Computations in these backgrounds can be thought of as direct generalizations of [HKY 2014].

$$I[M_{d-1}] = \mathrm{Tr}_{\mathcal{H}_{M_{d-1}}}(-1)^F$$

What's understood so far:

1D: Witten index of $\mathcal{N} = 2$ GLSMs

2D: $M_1 = S^1$ Elliptic genus of $\mathcal{N} = (0,2)$ GLSMs

3D: $M_2 = \Sigma_g$ an arbitrary Riemann surface for $\mathcal{N} = 2$

4D: M_3 an arbitrary three-manifolds for $\mathcal{N} = 2$ (*),

M_3 an Seifert-manifold for $\mathcal{N} = 1$.

5D: M_4 an arbitrary four-manifolds for $\mathcal{N} = 1$ (*)

(*): one susy preserving bg.

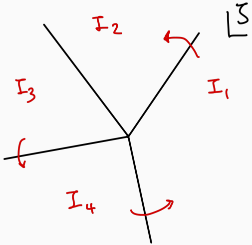
1D

Witten index of 1D $\mathcal{N} = 2$ GLSM

Consider 1d $\mathcal{N} = 2$ Gauge Linear Sigma Models (GLSM). For each $U(1)$ factor of the gauge group G, one can assign a Fayet-Iliopoulos (FI) parameter

$$\zeta(D) \ .$$

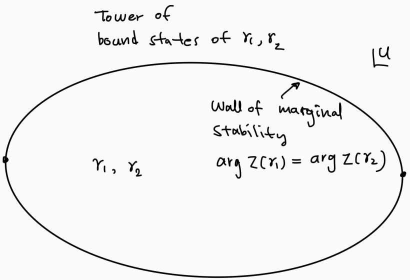
The Witten index enjoys integrality and deformation invariance. But the Q-exactness can be spoiled by the existence of a non-compact direction. There may exists a codimension-one locus in the space of ζ, across which the Witten index jump discontinuously.



Original motivation

When a GLSM describes the effective quantum mechanics of BPS particles in a 4d $\mathcal{N} = 2$ theory, the discontinuity in the Witten index explains the wall-crossing phenomena therein. The relevant quantity in 4d is the protected spin character defined in [Gaiotto-Moore-Neitzke 2010][Manschot-Pioline-Sen 2010]:

$$\overline{\Omega}(\gamma : y) = \mathrm{Tr}_{H_\gamma^{\mathrm{BPS}}} y^{2J_3}(-y)^{2I_3} \ .$$



Coulomb branch integral of 1d GLSM

Main idea:

(1) Go to the localization scheme which reduces to the path integral down to a finite-dimensional integral over the classical Coulomb branch parametrized by $u \in \mathcal{M}_{\text{Coulomb}} = (T \times \mathfrak{t})/W_G$

(2) The u, $\bar{u}$- integral can be written as a total derivative. Schematically,

$$\int \frac{dD}{D} \int_{\mathcal{M}_{\text{Coulomb}}} du d\bar{u}\, \partial_{\bar{u}}(\cdots) \ ,$$

where $(\cdots)$ is a meromorphic function.

(3) Dependence on the continuous parameter ζ is encoded in the D-integral. By carefully performing the integration by parts with this dependence, we arrive at a universal formula

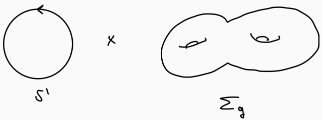
$$I(\zeta) = \frac{1}{|W_G|} \sum_{p} \text{JK-Res}(Q(p), \zeta)\ g(u, z) d^r u \ .$$

Some applications

- BPS dynamics of 4d $\mathcal{N} = 2$ theories and Quivers.
- Instanton counting problems in higher-dimensional SCFTs.
- Enumerative geometry and higher-dimensional gauge theories. (Today)

3D

This idea can be used to construct more general framework that computes observables in higher-dimensional gauge theories. In 3D, we consider $\mathcal{N} = 2$ theories on a closed Riemann surface:



- The supersymmetric background preserves $\mathcal{N} = 2$ in 1d after the topological twist.
- The effective quantum mechanics on S^1 is described by fluctuations of $n_c = h^0(E)$ chiral and $n_f = h^1(E)$ fermi multiplets which satisfies

$$n_c - n_f = \deg(E) - g + 1 \ ,$$

  by Riemann-Roch theorem.
- We integrate over the space of flat connections on Σ_g.

Integral formula and the Bethe vacua formalism

Similarly to the 1d case, the path integral can be reduced down to a finite-dimensional integral over the Coulomb branch. This gives an integral formula of the 3d twisted index. [Closset-HK 16] [Benini-Zaffaroni 16]

Alternatively, one can reduce the theory on S^1 and obtain an effective 2d $\mathcal{N}=(2,2)$ theory on Σ_g, which includes towers of KK modes. The twisted index can be written as [Nekrasov-Shatashvilli 09, 14]

$$I = \sum_{P(x)=0} \mathcal{H}^{g-1}(x) \ .$$

where the summation is over the solution to a polynomial equation, which are in one-to-one correspondence with the 2d massive Coulomb branch vacua.

Partition functions on Seifert three-manifolds

This framework can be expanded to the cases where S^1 non-trivially fibers over Σ_g [Closset-HK-Willett 17] ,

$$Z_{M_{g,p}} = \sum_{P(x)=0} \mathcal{H}^{g-1}(x) F^p(x) \ .$$

and to an arbitrary closed oriented Seifert manifolds [Closset-HK-Willett 18] .

This includes $M_3 = S^3$ or Lens spaces, which have special applications in SCFTs at the infrared fixed points.

Some applications to geometry

(1) The effective quantum mechanics on S^1 is the sigma model into the moduli space of (generalized) vortices on Σ_g, which has applications in the quantum K-theory. The index computes [Bullimore-Ferrari-HK 18]

$$\chi_{\mathrm{vir}}(M, E)\ .$$

where M is the moduli space of quasi-maps into the Higgs branch.

(2) In general, the 3d twisted index exhibits wall-crossing with respect to "1d FI parameter". This allows us to understand the wall-crossing phenomena associated to the stability condition, e.g., in the moduli space of stable pairs

$$(E\ ,\phi)\ ,$$

where E is a holomorphic vector bundle and $\phi \in H^0(E)$ on Σ_g. [Bullimore-Ferrari-HK 19]

Partition functions on Seifert spaces - Modular data of VOAs

More recently, there has been progress in understanding structure of the vertex operator algebra associated with 3d $\mathcal{N}=2$ theories with some twist. There exists a class of 3d supersymmetric theories which support rational vertex algebra on the boundary. [Gang-Kim-Lee-Shim-Yamazaki 21]...[Gang-HK-Spencer 23]

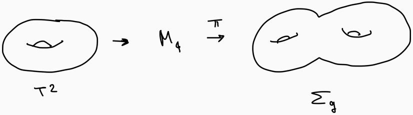
The partition functions on Seifert spaces can be used to extract the modular data:

$$Z_{M_{g,p}} = \sum_{P(x)=0} \mathcal{H}^{g-1}(x) F^p(x) = \sum_{\alpha:\text{simple modules}} S_{0\alpha}^{2-2g} T_{\alpha\alpha}^{-p} \ .$$

A 3d mirror symmetry predicts various interesting conjectures about the modular forms (which are characters of these VOAs) and novel dualities among them. [Ferrari-Garner-HK 23],[Cruetzig-Garner-HK to appear]

4D

This framework (at least formally) generalizes to 4d $\mathcal{N} = 1$ theory. The relevant four-dimensional geometry is an elliptic fibration over a Riemann surface.



This preserves two real supercharges with $U(1)_R$ symmetry. This includes most of the known observables of 4d $\mathcal{N} = 1$ theories, including superconformal index, and lens space indices. They can be universally written in the form of

$$Z_{M_{g,p}} = \sum_{P(x)=0} \mathcal{H}^{g-1}(x) F^p(x) \ .$$

now $P(x)$ is an elliptic equation. [Closset-HK-Willett 18]

This formula has been recently applied to understand the black hole microstate counting problem with $\mathcal{N} = 1$ symmetry.

4D $\mathcal{N} = 2$ on an arbitrary four-manifolds

In fact, 4d $\mathcal{N} = 2$ theory can be put on an arbitrary four-manifold M_4 with full topological twist. This background preserves one supercharge. The partition function on such a background for pure $SU(2)$ theory has been computed in late 90's by [Moore-Witten], and can be identified with the Donaldson's invariant.

- When $b_2^+(M_4) \neq 1$, the partition function reduces to the computation of the Seiberg-Witten invariant, which is much easier to compute.
- For $b_2^+(M_4) = 1$, one should also integrate over the Coulomb branch, also called the "u-plane". This computation involves interesting theory of (mock-)modular functions. [Korpas-Manscho-Moore-Nidaiev 2019]...
- Can be dimensionally reduced to compute the fully twisted partition function of 3d $\mathcal{N} = 4$ theory, which is (for some reason) more difficult to compute directly.

5D

5D N=1 on an arbitrary four-manifolds

Supersymmetric correlation functions defined on a compact five-manifold X_5 are one of the key tools in studying 5d SQFTs.

$$\langle O(x)...\rangle_{X_5} = \int [dV] \;\; O(x)...e^{-S_{X_5}[V]}$$

Compared to lower dimensional SQFTs, very little is known about these objects.

When $M_5 = X_4 \times S^1$, it is expected to compute the K-theoretic Donaldson invariants, which is essentially the Dirac index of the instanton moduli space:

$$\langle O(x)...\rangle_{X_4 \times S^1} = \sum_k \mathcal{R}^{d(k)} \int_{\mathcal{M}_k} \hat{A}(\mathcal{M}_k) \wedge \mathrm{ch}(\mathcal{L}^n) \ ,$$

where

$$\mathcal{M} = \bigsqcup_k \mathcal{M}_k \ ,$$

is the moduli space of G-instantons on X_4. This has been computed recently in mathematical literature for a large class of X_4, e.g., in [Göttsche, Nakajima, Yoshioka 06] [Göttsche, Kool, Williams 19] [Göttsche, Kool 20]

Can we make this relation more precise and provide path integral derivations in 5d supersymmetric gauge theories? [HK-Manschot-Moore-Tao-Zhang, to appear]

Two approaches

Two approaches for computing the observables in 5d $\mathcal{N} = 1$ theories on $X_4 \times S^1$:

(1) U-plane integral

- Reduction to 4d $\mathcal{N} = 2$ effective theories
- Applicable for general closed smooth X_4

(2) Localization in G-gauge theories

- Reduction to effective quantum mechanics on S^1
- Restricted to toric X_4
- useful in understanding geometric interpretation of the partition functions.

5d $\mathcal{N} = 1$ G-gauge theory

- Supersymmetric G Yang-Mills theory

$$S_{YM} = \frac{1}{g_{\mathrm{YM}}^2} \int d^5x \ \mathrm{tr} \left[\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + |D_\mu \sigma|^2 + \frac{1}{2} D^{AB} D_{AB} + (\text{fermionic}) \right]$$

- Global symmetry group is $SU(2)_R \times U(1)_I$, where $U(1)_I$ is a global symmetry associated to the current

$$j = * \ \mathrm{tr}\,(F \wedge F) \ ,$$

whose charged particles are instanton particles.

- The theory can be put on a smooth $X_4 \times S^1$ with the Donaldson twist. The topological reduction gives 1d $\mathcal{N} = 1$ quantum mechanics into the moduli space of instantons

$$S^1 \to \mathcal{M} = \bigsqcup_k \mathcal{M}_k \ .$$

Chern-Simons observable

One can turn on the following coupling:

$$S_{\text{mixed CS}} = \int_{X_4 \times S^1} F_{(I)} \wedge \mathrm{Tr}\left(A \wedge dA + \frac{2}{3}A^3\right) + (\text{SUSY completion}) \ ,$$

In the presence of the background flux,

$$\left[\frac{F_{(I)}}{2\pi}\right] = \mathbf{n}$$

the above coupling induces a line bundle $\mathcal{L}$ on the moduli space $\mathcal{M}$.

The Hilbert space of the effective QM becomes the section of $S \otimes \mathcal{L}$ over $\mathcal{M}$, and its Witten index computes

$$\sum_k \mathcal{R}^{d(k)} \int_{\mathcal{M}_{k,\nu}} \hat{A}(\mathcal{M}_{k,\nu}) \wedge e^{c_1(\mathcal{L})} \ ,$$

where $\nu = w_2(P)$, where P is the principal G-bundle.

Anomalies

If the target space of SQM is not spin, there can be global anomalies, due to the fact that

$$Pfaff(\not{D})$$

is not well-defined on the loop space of moduli space.

The instanton moduli spaces are not always spin. For $G = SO(3)$ on an almost complex X_4 (together with some technical assumption), $\mathcal{M}_k$ is not spin if

$$w_2(X_4) \cdot w_2(P) \neq 0 \ ,$$

[Freed, Hopkins, Moore, Witten, unpublished]

But this anomaly can be canceled by the Chern-Simons observable introduced in the last slide. This coupling provides extra factor in the path integral, so that

$$Pfaff(\not{D}) \exp\left(-\oint_{S^1} \mathcal{A}_{(I)}\right) \ ,$$

is well-defined.

U-plane integral

U-plane integral

Following [Moore, Witten 97], the partition function of effective 4d theory for $b_2^+(X) > 0$ can be written as

$$Z_{J,\mu}[\mathcal{R}, \mathbf{n}] = \Phi_{J,\mu}[\mathcal{R}, \mathbf{n}] + \sum_{i=1}^{4} Z^{SW}_{J,\mu,i}[\mathcal{R}, \mathbf{n}] \ ,$$

where Φ is the integral over the Coulomb branch, so-called "U-plane integral" contribution. Z^{SW} is the Seiberg-Witten contribution at the four singular points in the U-plane, where a BPS particle becomes massless.

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- For $b_2^+ > 1$, $Z_{J,\mu}$ is independent of metric on X.
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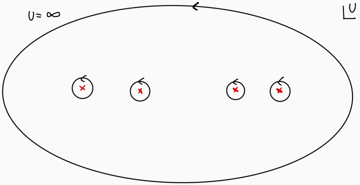
- For $b_2^+ > 1$, $Z_{J,\mu}$ is independent of metric on X.
- For $b_2^+ > 1$, $\Phi_{J,\mu}$ identically vanishes.
- For $b_2^+ = 1$, the partition functions are expected to jump discontinuously as a function of metric on X.
- The metric dependence comes through J, the period point. $J \in H^2(X, \mathbb{R})$ with $J = *J$ and $J^2 = 1$).

$$Z_{J,\mu} = \Phi_{J,\mu} + \sum_{i=1}^{4} Z^{SW}_{J,\mu,i}$$



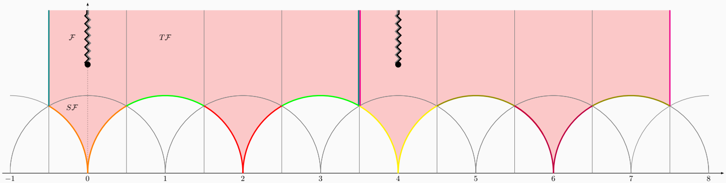
- For $b_2^+ = 1$, $Z_{J,\mu}$ is a piecewise constant function of J. The dependence on J only comes from the region $U \to \infty$.
- The J-dependence of Φ around the singularities at finite $U = U_i$ are also non-trivial, but they are canceled with the wall-crossing of $Z^{SW}_{J,\mu,i}$.
- We can utilize this fact to compute $Z^{SW}_{J,\mu,i}$ for $b_2^+ > 1$.

Fundamental domain of τ

From the Seiberg-Witten curve, we obtain the relation

$$U(\tau) = \pm \left(-4\mathcal{R}^2 \frac{\theta_2(\tau)^4 + \theta_3(\tau)^4}{\theta_2(\tau)^2\theta_3(\tau)^2} + 4\mathcal{R}^4 + 4 \right)^{1/2} .$$

The integral over the U-plane can be written as an integral over the fundamental domain of τ:



The fundamental domain is a branched double cover of $\mathbb{H}/\Gamma^0(4)$. This is related to the existence of the $\mathbb{Z}_2$ center symmetry, and also to the global anomaly of effective QM.

U-plane integral and wall-crossing

- For $b_2^+ = 1$, by analysing the contribution around $\tau \to i\infty$, we derive the wall-crossing formula.

$$Z^J[\mathcal{R}, \mathbf{n}] - Z^{J'}[\mathcal{R}, \mathbf{n}] = \sum_{k \in \mathcal{W}_{J,J'}} 8 \left[\nu_R(\tau) C^{\mathbf{n}^2} (-1)^{\langle k, K \rangle} q^{-k^2/2} e^{-2\pi i \langle k, \mathbf{n} v/2 \rangle} \right]_{q^0} ,$$

$$\text{where} \quad \mathcal{W}_{J,J'} = \left\{ k \mid \langle k - \mathbf{n}/4, J \rangle > 0 \;\text{ and }\; \langle k - \mathbf{n}/4, J' \rangle < 0 \right\} .$$

- Direct evaluation of the partition function is also possible, which involves the theory of mock modular forms.

Toric localization

When X_4 is a smooth toric four-manifold, the partition function can be written as

$$Z \sim \sum_k \int_{C_J} da \prod_{i=1}^{\chi} Z(a^i, \epsilon_1^i, \epsilon_2^i, R, \Lambda^i) \ ,$$

where $Z(a^i, \epsilon_1^i, \epsilon_2^i, R, \Lambda^i)$ is the K-theoretic Nekrasov partition functions on $S^1 \times \mathbb{C}^2_{\epsilon_1^i \epsilon_2^i}$ localized at fixed loci.

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One can argue that the contour C_J depends on the choice of metric, J. After a careful analysis of the zero mode integrals as in [HKY 14], we arrive at a wall-crossing formula,

$$Z_{J,\mu}[\mathcal{R}, \mathbf{n}] - Z_{J',\mu}[\mathcal{R}, \mathbf{n}] = \sum_{k \in \mathcal{W}_{J,J'}} \left(\underset{a=\infty}{\mathrm{res}}\, da + \underset{a=-\infty}{\mathrm{res}}\, da \right) Z_{k,\mu}(a, \mathcal{R}, \mathbf{n}) \ ,$$

It turns out that this agrees precisely with the WC formula in the U-plane integral approach.

Summary and open questions

- The partition functions of supersymmetric gauge theories on backgrounds that preserves two real supercharges have been computed in dimensions from 1 to 5.
- This technique has been applied to various counting problems in physics and mathematics.
- Some partition functions that preserve one supercharge have been computed in 4 and 5 dimensions, via powerful technique of the Seiberg-Witten theory. Lower-dimensions?
- These techniques can be generalized to the computations of six-dimensional gauge theories, or 6d SCFTs which have 5d IR descriptions.