On holography and anomalies of D-type 6D (2,0) theories

with F. Bonetti & M. Del Zotto (in (never-ending) progress)

Can the anomaly formula for D-type (2,0) 6D theories - which Piljin derived in 2001 - be re-derived holographically?

D_Q anomaly: $$I^{(2,0)}_{D_N} = N I_8^{(2,0)\,\text{tm}}(TW, SO(5)) + N(2N-1)(2N-2)\frac{p_2(SO(5))}{24}$$

Plan:

- A-type heories
  - ◇ M5 anomalies and holographic derivation of A-type anomaly formula
  - ◇ *7D topological sector from M-theory*
- D-type theories
  - ◇ The set-up
  - ◇ *7D topological sector from M-theory*
  - ◇ Holographic derivation of D-type anomaly formula

M-theory (M5) holography basics:

11D theory with 32 supercharges:

$$I_{11} = \frac{1}{2\kappa_{11}^2}\left[\int d^{11}x\sqrt{-G}R_G - \frac{1}{2}G\wedge *G - \frac{1}{6}C\wedge G\wedge G\right]$$

Supersymmetric flux backgrounds ($G\neq 0$, $\mathrm{d}G=0$):

- FR: $AdS_7\times S^4$ max susy ($SO(5)$ symmetry)
- AdS_5 solutions
  - ◇ GM, BBBW:
    $$\begin{cases} \text{(squashed) } S^4 \text{ fibered over } \Sigma_g \ - \text{ at least } U(1)^2 \text{ symmetry} \\ \text{8 or 16 supercharges} \end{cases}$$
  - ◇ GMSW:
    $$\begin{cases} \text{Mink}_4\times_w \mathcal{C}(M_6) \quad \Rightarrow \quad AdS_5\times M_6 \\ M_6 - \text{compact complex} - S^2 \text{ bundle over Kähler } K_4 \end{cases}$$
- CFT side: (tacitly) A-type theories

M-theory - strong coupling limit of type IIA strings

Higher derivative terms in string theory (α' and g_s expansion) $\sim \mathcal{R}^{3l+1}$ lift to 11D

Special CP-odd couplings : $\Rightarrow$ $C \wedge X_8$

$$
\begin{aligned}
X_8 &= \frac{1}{48}\left(\frac{1}{4}p_1(TM)^2 - p_2(TM)\right) \\
&= \frac{1}{(2\pi)^4}\left(-\frac{1}{768}(\text{tr}\, R^2)^2 + \frac{1}{192}\text{tr}\, R^4\right)
\end{aligned}
$$

▷ 5pt function at one-loop

◇ $\hat{\chi} = \frac{1}{4!(4\pi)^2} \cdot \frac{1}{2^4}$
$\epsilon^{i_1 \cdots i_8} R_{a_1 a_2} \left(\Gamma^{a_1 a_2}\right)^{i_1 i_2} R_{a_3 a_4} \left(\Gamma^{a_3 a_4}\right)^{i_3 i_4} R_{a_5 a_6} \left(\Gamma^{a_5 a_6}\right)^{i_5 i_6} R_{a_7 a_8} \left(\Gamma^{a_7 a_8}\right)^{i_7 i_8}$

◇ $\hat{\chi} = \frac{1}{16}\left(8\chi + p_1(TM)^2 - 4p_2(TM)\right)$

▷ Needed for **string dualities**

$\Rightarrow$ solutions of M-theory $\neq$ solutions of D=11 sugra

M5-branes

Classical soliton of 11D sugra

- the metric: $ds^2_{10} = e^{N_1 u(r)} \mathrm{d}s^2_6(W_{||}) + e^{N_2 u(r)} (\mathrm{d}s^2_5)_\perp$ (r - distance away from M5)
- the four-form : $G_4 \sim \star_\perp du(r)$

$$\mathrm{d}G_4 = \delta_5(r)$$

- zero-mode expansion $G_4 \to G^{(0)} + h_3 \wedge \mathrm{d}u(r) + \cdots$

$$\mathrm{d} * G_4 \sim G_4 \wedge G_4 \qquad \Rightarrow \qquad h_3 = - *_{||} h_3$$

Theory on M5 $\Leftrightarrow$

- $(2,0)$ tensor multiplet
- $(\beta^-, \psi^\alpha, x^a)$ $\alpha = 1, ..., 4;\ a = 1, ..., 5$
- $SO(5)$ R-symmetry
- ADE classification - non-Abelian M5

Symmetries of the theory without M5 $\left[\ \frac{1}{6} C \wedge G \wedge G + C \wedge X_8\ \right]$

- shift: $C_3 \to C_3 + \mathrm{d}\Lambda$
- diffeomorphisms

With M5 $\quad i : W_6 \hookrightarrow M_{11}$

- $\delta(\int_{M_{11}} C \wedge G \wedge G) \quad \to \quad \int_{W_6} i^*(\Lambda \wedge G)$
  - ⋆ M5 coupling $\int_{W_6} h_3 \wedge i^* C$
  - ⋆ $\delta h_3 = i^*(\mathrm{d}\Lambda)$ .... relative cohomology ($\mathrm{d}h = i^* G$)

- diffeomorphisms and $\delta \int C \wedge X_8$?
  - ⋆ X_8 - (made of) characteristic class(es)... $X_8 = \mathrm{d}X_7^{(0)}$ and $\delta X_7^{(0)} = \mathrm{d}X_6^{(1)}$
  - ⋆ assume trivial normal bundle ($p_i(TM)|_W = p_i(TW)$)
  - ⋆ $\delta \int C \wedge X_8 \quad \to \quad \int_{W_6} X_6^{(1)}$
  - ⋆ anomaly inflow

M5 ANOMALY

(2,0) tensor multiplet:

- Woldvolume chiral 2-form
  - $I_\beta = \frac{1}{5760}\left(16p_1(TW)^2 - 112p_2(TW)\right) \sim L(TW)$
- Worldvolume fermions
  - $I_D = \frac{1}{2}\widehat{A}(TW)$
  - for trivial normal bundle:

$$I_D = 4 \times \frac{1}{2}\widehat{A}(TW) = \frac{1}{5760}\left(14p_1(TW)^2 - 8p_2(TW)\right)$$

- Total anomaly: $I_{\text{M5}} = \frac{1}{48}\left(\frac{1}{4}p_1(TW)^2 - p_2(TW)\right)$
- Cancelled via inflow from a bulk coupling $\sim C_3 \wedge X_8(TM)$

$$G_4 \wedge \delta X_7^{(0)} \to \delta_5(M5) \wedge X_6^{(1)}(TM) \quad \leftrightarrow \quad \mathrm{d}^{-1}\delta\mathrm{d}^{-1} I_{(2,0)}$$

- Nontrivial normal bundle... $C \wedge G \wedge G$ is NOT ... diff invariant!

Non-trivial normal bundle

(single) M5 worldvolume :

- Chiral 2-form
  - ◇ $I_\beta = \frac{1}{5760}\left(16p_1(TW)^2 - 112p_2(TW)\right) \sim L(TW)$
- Fermions
  - ◇ $I_D = \frac{1}{2}\widehat{A}(TW)\,\text{ch}S(\mathcal{N})$
  - ◇ $\text{ch}S(\mathcal{N}) = 4 + \frac{1}{2}p_1(\mathcal{N}) + \frac{1}{96}p_1(\mathcal{N})^2 + \frac{1}{24}p_2(\mathcal{N}) + \ldots$

$$I_D = 4 \times \frac{1}{2}\widehat{A}(TW) + \ldots = \frac{1}{5760}\left(14p_1(TW)^2 - 8p_2(TW)\right) + \ldots$$

- Total anomaly:
  $I_{\text{M5}} = \frac{1}{48}\left(\frac{1}{4}(p_1(TW)^2 + p_1(\mathcal{N})^2 - 2p_1(TW)p_1(\mathcal{N})) - p_2(TW) + p_2(\mathcal{N})\right)$
- Anomaly from the bulk (using $p_1(TM|_W) = p_1(TW) + p_1(\mathcal{N})$, ...)

$$I_{\text{bulk}} = -\frac{1}{48}\left(\frac{1}{4}(p_1(TW)^2 + p_1(\mathcal{N})^2 - 2p_1(TW)p_1(\mathcal{N})) - p_2(TW) - p_2(\mathcal{N})\right)$$

- The result: $I_{\text{M5}} + I_{\text{bulk}} = \frac{p_2(\mathcal{N})}{24}$!!!

Non-singular p-branes

Brane worldvolume W_d ($d = p + 1$) located at $y^a = 0$ ($a = 1, ..., D - d$) in M_D

$S_\epsilon(W_d)$ - S^{D-d-1} sphere bundle - boundary of tubular neighbourhood of rad ϵ, $D_\epsilon(W_d)$

- Magnetic source:
  - ◇ $\mathrm{d}G_{D-d-1} = 2\pi\delta(y^1)...\delta(y^{D-d})\mathrm{d}y^1 \wedge ... \wedge \mathrm{d}y^{D-y} \quad \Rightarrow \quad 2\pi\Phi_{D-d}$

Thom class of N $\Phi_{D-d} = \begin{cases} \bullet \quad \mathrm{d}(\rho(r)e_{2n-1}/2) \qquad 2n-1 = D-d-1 \\ \bullet \quad \mathrm{d}\rho(r)e_{2n}/2 \qquad 2n = D-d-1 \end{cases}$

e_{D-d-1} - global angular form

- rank$(N) = 2n$ - sphere bundle has fibers S^{2n-1}
  - ◇ $\mathrm{d}e_{2n-1} = -\pi^*(\chi(\mathcal{N})) \qquad \chi(\mathcal{N}) \in H^{2n}(M, \mathbb{Z})$
- rank$(N) = 2n+1$ - sphere bundle has fibers S^{2n}
  - ◇ $\mathrm{d}e_{2n} = 0 \qquad (e_{2n} = \mathrm{d}e^{(0)}_{2n-1}, \;\; \delta e^{(0)}_{2n-1} = e^{(1)}_{2n-2})$
  - ◇ cohomology class e_{2n} : $\quad [e^2_{2n}] = \pi^*(p_n(\mathcal{N}))$
  - ◇ at the level of differential forms : $\quad \pi_*(e^3_{2n}) = \pi_*(e_{2n}\pi^*p_n(\mathcal{N})) = 2p_n(\mathcal{N})$

M5 ($W_6 \hookrightarrow M_{11}$) - $SO(5)$ N bundle

$\mathbf{so(5)} \cong \Lambda^2\mathbb{R}^5$ - connection on N: $\quad \Theta^{ab} = -\Theta^{ba}$

Define $\hat{y}^a = y^a/r\,; \quad (D\hat{y})^a = \mathrm{d}\hat{y}^a - \Theta^{ab}\hat{y}^b\,; \quad F^{ab} = \mathrm{d}\Theta^{ab} - \Theta^{ac}\wedge\Theta^{ca}$

$$\begin{cases} \bullet \quad e_4(\Theta) = \frac{1}{64\pi^2}\epsilon_{a_1\ldots a_5}\left((D\hat{y})^{a_1}\ldots(D\hat{y})^{a_4} - 2F^{a_1a_2}(D\hat{y})^{a_3}(D\hat{y})^{a_4} + F^{a_1a_2}F^{a_3a_4}\right)\hat{y}^{a_5} \\ \bullet \quad e_3^{(0)}(\Theta) = \frac{1}{32\pi^2}\epsilon_{a_1\ldots a_5}\left(\Theta^{a_1a_2}\mathrm{d}\Theta^{a_3a_4}\hat{y}^{a_5} - \frac{1}{2}\Theta^{a_1a_2}\Theta^{a_3a_4}\mathrm{d}\hat{y}^{a_5} - 2\Theta^{a_1a_2}\mathrm{d}\hat{y}^{a_3}\mathrm{d}\hat{y}^{a_4}\hat{y}^{a_5}\right) \\ \bullet \quad e_2^{(1)}(\varepsilon,\,\Theta) = \frac{1}{16\pi^2}\epsilon_{a_1\ldots a_5}\varepsilon^{a_1a_2}\left(\mathrm{d}\hat{y}^{a_3}\mathrm{d}\hat{y}^{a_4}\hat{y}^{a_5} - \Theta^{a_3a_4}\mathrm{d}\hat{y}^{a_5}\right) \end{cases}$$

Under gauge transformations: $\delta\Theta^{ab} = (D\varepsilon)^{ab}, \quad \delta\hat{y}^a = \varepsilon^{ab}\hat{y}^b$

In the presence of M5:

- $G = \mathrm{d}C \quad \longrightarrow \quad \mathrm{d}C - 2\pi\mathrm{d}\rho\wedge e_3^{(0)}/2$
  - ◇ $\quad \delta C = -2\pi\mathrm{d}\rho\wedge e_2^{(1)}/2$
- Introduce σ_3: $\quad G_4 - 2\pi\rho e_4/2 = \mathrm{d}(C_3 - 2\pi\rho e_3^{(0)}/2) \equiv \mathrm{d}(C_3 - 2\pi\sigma_3)$
- CS couplings in presence of M5:

$$S_{\text{CS}} = \lim_{\epsilon\to 0} -\frac{1}{6(2\pi)^2}\int_{M_{11}-D_\epsilon(W_6)}(C_3 - 2\pi\sigma_3)\wedge\mathrm{d}(C_3 - 2\pi\sigma_3)\wedge\mathrm{d}(C_3 - 2\pi\sigma_3)$$

Remember $\pi_*(e_4^3) = \pi^* p_n(\mathcal{N})) = 2p_2(\mathcal{N})$

Under diffs ($SO(5)$ transformations), S_{CS} varies....

- $\delta((C_3 - 2\pi\sigma_3) = -2\pi \mathrm{d}(\rho e_2^{(1)}/2)$

$$\begin{aligned} \delta S_{\text{CS}} &= \lim_{\epsilon\to 0} \frac{1}{12\pi} \int_{M_{11}-D_\epsilon(W_6)} \mathrm{d}(\rho e_2^{(1)}/2) \wedge \mathrm{d}(C_3 - 2\pi\sigma_3) \wedge \mathrm{d}(C_3 - 2\pi\sigma_3) \\ &= -\frac{2\pi}{6} \int_{S_\epsilon(W_6)} \frac{e_4}{2} \wedge \frac{e_4}{2} \wedge \frac{e_2^{(1)}}{2} \end{aligned}$$

Anomaly cancellation!... and a key to non-Abelian $(2,0)$ theories

- $I_{\text{M5}} + I_{\text{bulk}} + \delta S_{\text{CS}} = 0$
- Q coincident M5 - symmetry enhancement to $SU(Q)$
  - ◇ $\mathrm{d}G_4 = 2\pi Q \mathrm{d}\rho e_4/2$
  - ◇ no new ingredients in the anomaly cancellation

⋆ $I_{\text{M5}}(Q) = Q I_{\text{M5}}(Q=1) + \frac{Q^3 - Q}{24} p_2(\mathcal{N})$

$(2,0)$ theories have ADE symmetry enhancement

- A_{Q-1}
  - ◇ remove a centre of mass (one free (2,0) multiplet) anomaly
  - ◇ $$I^{(2,0)}_{A_{Q-1}} = (Q-1) I_{\text{M5}}(Q=1) + \frac{Q^3-Q}{24} p_2(\mathcal{N})$$
- D_Q
  - ◇ $\mathbb{R}^5/\mathbb{Z}_2$ fixed points
  - ◇ $$I^{(2,0)}_{D_Q} = Q I_{\text{M5}}(Q=1) + \frac{Q(2Q-1)(2Q-2)}{24} p_2(\mathcal{N})$$
- E
  - ◇ no direct calculation (brane picture), but ... using 5D gauge CS confirm
  - ◇ $$I^{(2,0)}_{G} = r_G I_{\text{M5}}(Q=1) + \frac{d_G \cdot h^\vee_G}{24} p_2(\mathcal{N})$$
  - ◇ $r_G, d_G, h^\vee_G$ - rank, dimension, dual Coxeter

E-strings

- ◇ M5 inflow + HW
- ◇ HW: $G|_{x^{11}=0} = I_4(1) \sim \alpha'/4(\text{tr}\, R^2/2 - \text{tr}\, F_1^2)$
- ◇ $I_{12}(i) = I_4(i) \wedge X_8 + \frac{1}{6} I_4(i) \wedge I_4(i) \wedge I_4(i) \quad (i=1,2); \quad I_{12} = I_{12}(1) + I_{12}(2)$

Holographic derivation (M-theory on S^4)

- ◇ $\mathcal{I}_{12} = -\frac{1}{6}\, G_4 \wedge G_4 \wedge G_4 - G_4 \wedge X_8$
- ◇ $G_4 = Q e_4/2 \qquad \Leftrightarrow \qquad \frac{1}{2\pi}\int_{S^4} G_4 = Q$
  - ○ vacuum $G^\circ = 2\pi Q\,$Vol(S^4) $\quad \Rightarrow \quad$ fluctuations, $SO(5)$ invariance

M-theory on a spacetime with non-trivial boundary:

- ◇ $S_\epsilon(W_6) \sim M_{11} - D_\epsilon(W_6):\qquad S^4 \hookrightarrow S_\epsilon(W_6) \to W_6$

Non-invariance of the action under diffeomorphisms:

- ◇ $\delta S_{\rm M} = 2\pi \int_{S_\epsilon(W_6)} \mathcal{I}_{10}^{(1)}$
  - $*$ $\mathrm{d}\mathcal{I}_{10}^{(1)} = \delta \mathcal{I}_{11}^{(0)}\,, \qquad \mathrm{d}\mathcal{I}_{11}^{(0)} = \mathcal{I}_{12}$

$$I_8^{\rm inflow} = \int_{S^4} \mathcal{I}_{12}$$

$$I_8^{\rm inflow} + I_8^{\rm CFT} + I_8^{\rm decoup} = 0$$

7D TQFT captures anomalies and encoded global symmetries of 6D QFT

M-theory branes wrapping compact horizon geometry ("cycles at infinity") $\Rightarrow$ topological operators in 7D TQFT

M2 on [pt(S^4)$\times\Sigma_3$] $\Rightarrow$ $\boldsymbol{Q}_3(\Sigma_3) = e^{2\pi i \int_{\Sigma_3} c_3}$

- $dG_4 = 0\,, \qquad dG_7 = \frac{1}{2}G_4^2 + X_8 + \delta_8(\Sigma_3 \times \mathrm{pt} \subset M_{11})$
  - ◇ $\delta_8(\Sigma_3 \times \mathrm{pt} \subset M_{11}) = \delta_4(\Sigma_3 \subset M_7) \wedge v_4$ (v_4 - volume form on S^4)
- $G_4 = Qv_4 + g_4$
  - ◇ g_4 - closed 4-form with integral periods in external spacetime $g_4 = \mathrm{d}c_3$
  - ◇ $g_4 = -\frac{1}{Q}\delta_4(\Sigma_3) + \frac{1}{Q}d\int_{S^4} G_7$
- Effect of M2/[pt(S^4)$\times\Sigma_3$] on M2/[pt(S^4)$\times\Sigma_3'$]
  - ◇ $\boldsymbol{Q}_3(\Sigma_3') \approx e^{2\pi i \int_{\Sigma_3'} C_3} = e^{2\pi i \int_{B_4'} G_4} = e^{2\pi i \int_{B_4'} g_4} = e^{2\pi i \frac{-1}{Q} \int_{B_4'} \delta_4(\Sigma_3)}(\dots)$
  - ◇ $\int_{B_4'} \delta_4(\Sigma_3) = L_{M_7}(\Sigma_3, \Sigma_3')$ linking number of Σ_3 and Σ_3' in M_7

Correlators of 7D topological operators

$$\langle \boldsymbol{Q}_3(\Sigma_3)\boldsymbol{Q}_3(\Sigma_3')\rangle = \exp\left[2\pi i \frac{1}{Q} L_{M_7}(\Sigma_3,\Sigma_3')\right]$$

Can be derived from

$$S = -2\pi \frac{Q}{2}\int_{M_7} c_3 \wedge dc_3$$

- ◇ known 7D TQFT for 6D (2,0) theory of tyme A_{Q-1}
- ◇ CS theory originates from M-theory $C_3G_4G_4$ coupling
- ◇ c_3 is not in AdS_7 massless spectrum!

Non-commutativity of fluxes in M-theory (HQ on $M_{11} = \mathbb{R}_t \times X_{10}$)

- ◇ Page charges or electric fluxes - characters of "magnetic translation group"
- ◇ Generators $\mathcal{W}(\phi_3) = \exp 2\pi i \int_{X_{10}} \phi_3 \wedge P_7$ $\quad$ ($\phi_3 \in \Omega^3(X_{10})$; $P_7 = \Pi_7 + \cdots$ - P.C.)
- ◇ $$\mathcal{W}(\phi_3)\mathcal{W}(\phi_3') = \mathcal{W}(\phi_3')\mathcal{W}(\phi_3)\exp(-2\pi i \int_{X_{10}} \phi_3 \wedge \phi_3' \wedge G_4)$$
- ◇ $X_{10} = M_6 \times S^4$: $\quad \phi_3 = \frac{1}{Q}\Phi_3\ ,\quad \Phi_3 \in \Omega^3_{\mathbb{Z}}(M_6)$ and $\Sigma_3 = \mathrm{PD}_{M_6}[\Phi_3]$

$\Rightarrow$ Phase: $\quad \exp\left[-2\pi i Q \cdot \frac{1}{Q}\cdot\frac{1}{Q}\int_{M_6}\Phi_3 \wedge \Phi_3'\right] = \exp\left[-2\pi i \frac{1}{Q}\Sigma_3 \cdot_{M_6} \Sigma_3'\right]$

$AdS_7 \times \mathbb{RP}^4$ is maximally supersymmetriic

- S^n: $\delta_{AB} y^A y^B = 1 \qquad (A, B = 1, ..., n+1)$
- Stereographic coordinates x^i $(i = 1, \ldots, n)$: $y^i = \frac{2x^i}{1+x\cdot x}$ and $y^{n+1} = \frac{x\cdot x - 1}{1+x\cdot x}$
- Antipodal involution σ: $\sigma : y^A \mapsto -y^A$ and $\sigma : x^i \mapsto -\frac{x^i}{x\cdot x}$
- Action of σ on sections of the spinor bundle on S^n $\psi(x) \quad \Rightarrow$

$$n = 0 \mod 4 : \quad w_1(T\mathbb{RP}^n) = \alpha \,, \quad w_2(T\mathbb{RP}^n) = 0 \,, \quad (\text{Pin}^+)$$
$$n = 1 \mod 4 : \quad w_1(T\mathbb{RP}^n) = 0 \,, \quad w_2(T\mathbb{RP}^n) = \alpha^2 \,, \quad (\text{Spin}^c)$$
$$n = 2 \mod 4 : \quad w_1(T\mathbb{RP}^n) = \alpha \,, \quad w_2(T\mathbb{RP}^n) = \alpha^2 \,, \quad (\text{Pin}^-)$$
$$n = 3 \mod 4 : \quad w_1(T\mathbb{RP}^n) = 0 \,, \quad w_2(T\mathbb{RP}^n) = 0 \,. \quad (\text{Spin})$$

α - generator of $H^1(\mathbb{RP}^n; \mathbb{Z}_2) \cong \mathbb{Z}_2$:

- $\psi(x)$ descends to a section of Pin$^+$ bundle on $\mathbb{RP}^4$ if $(\sigma\psi)(x) = \pm\psi(x)$
- Killing spinors on S^n $(D_i\epsilon = \frac{i}{2}\gamma_i\epsilon)$: $\epsilon(x) = \frac{1}{\sqrt{1+x\cdot x}}(\mathbb{I} + i x\cdot\gamma)\epsilon_0$

$$(\sigma\epsilon)(x) = \epsilon(x) \qquad \Rightarrow \qquad \text{Killing } (\mathrm{Pin}^+)\text{-pinors on } \mathbb{RP}^4$$

M-theory on non-orientable spaces and m_c structures

Pin^+ structure $\quad\Rightarrow\quad$ suitable tangential structure needed for gravitino

$m_{\rm c}$ structure on a Pin^+ manifold - choice of *twisted* integer lift of $w_4(TM) \in H^4(M;\mathbb{Z}_2)$

- twisted integer cohomology class - element of $H^\bullet(M;\widetilde{\mathbb{Z}})$ (or $H^\bullet(M;\widetilde{\mathbb{R}})$)
- $\widetilde{\mathbb{Z}}$ - constant sheaf of integers on M twisted by the orientation bundle of M
- Pin^+ manifold M admits $m_{\rm c}$ structure iff $\tilde{\beta} w_4(TM) = 0$,
  - ⋆ $\tilde{\beta}$ - Bockstein homomorphism $H^4(M;\mathbb{Z}_2) \to H^5(M;\tilde{\mathbb{Z}})$ associated to short exact sequence $\quad 0 \to \widetilde{\mathbb{Z}} \xrightarrow{2} \widetilde{\mathbb{Z}} \to \mathbb{Z}_2 \to 0$
- G_4 - closed twisted 4-form on M_{11} $\quad$ ($G_4 \in \widetilde{\Omega}^4_d(M_{11})$ and $[G_4]_{\rm dR} \in H^4(M_{11};\widetilde{\mathbb{R}})$)
  - ◇ $\exists\, a_4$ s.t. $\quad 2[G_4]_{\rm dR} = \widetilde{\varrho}(a_4)$
  - ◇ $\widetilde{\varrho} : H^\bullet(M_{11};\widetilde{\mathbb{Z}}) \to H^\bullet(M_{11};\widetilde{\mathbb{R}})$ - natural map induced by $\widetilde{\mathbb{Z}} \to \widetilde{\mathbb{R}}$
  - ◇ $a_4 \in H^4(M_{11};\widetilde{\mathbb{Z}})$ - twisted integral class - twisted lift of $w_4(TM_{11})$
- $$\int_{\mathcal{C}_4} G_4 = \int_{\mathcal{C}_4} \tfrac{1}{2} a_4 \overset{\text{mod } \mathbb{Z}}{=} \tfrac{1}{2}\int_{\mathcal{C}_4} w_4(TM_{11})$$
  - ◇ Spin M_{11}: $\quad$ familiar $\int_{\mathcal{C}_4} G_4 \overset{\text{mod } \mathbb{Z}}{=} \frac{1}{2}\int_{\mathcal{C}_4} \lambda(TM_{11})$ (for $2\lambda(TM_{11}) = p_1(TM_{11})$)

$M_{11} = M_7 \times \mathbb{RP}^4$ (M_7 - orientable and Spin)

$$H^\bullet(\mathbb{RP}^4;\mathbb{Z}) = \{\mathbb{Z},0,\mathbb{Z}_2,0,\mathbb{Z}_2\}\ ,$$
$$H^\bullet(\mathbb{RP}^4;\widetilde{\mathbb{Z}}) = \{0,\mathbb{Z}_2,0,\mathbb{Z}_2,\mathbb{Z}\}\ ,$$
$$H^\bullet(\mathbb{RP}^4;\mathbb{Z}_2) = \{\mathbb{Z}_2,\mathbb{Z}_2,\mathbb{Z}_2,\mathbb{Z}_2,\mathbb{Z}_2\}\ nn \qquad (1)$$

- t_1 - generator of $H^1(\mathbb{RP}^4;\widetilde{\mathbb{Z}})$
- $t_2 := t_1^2$, $t_3 := t_1^3\,, t_4 := t_1^4$ - generators of $H^2(\mathbb{RP}^4;\mathbb{Z})$, $H^3(\mathbb{RP}^4;\mathbb{Z})$, $H^4(\mathbb{RP}^4;\mathbb{Z})$
- Poincaré duality exchanges twisted and untwisted coefficients):

$$H_p(\mathbb{RP}^4;\mathbb{Z}) \cong H^{4-p}(\mathbb{RP}^4;\widetilde{\mathbb{Z}})\ , \quad H_p(\mathbb{RP}^4;\widetilde{\mathbb{Z}}) \cong H^{4-p}(\mathbb{RP}^4;\mathbb{Z})\ , \quad p=0,\dots,4$$

- v_4 - generator of $H^4(\mathbb{RP}^4;\widetilde{\mathbb{Z}})$ ($\int_{\mathbb{RP}^4} v_4 = 1$)
- total SW class: $w(T\mathbb{RP}^4) = (1+\alpha)^5 = 1+\alpha+\alpha^4$ ($\alpha \in H^1(\mathbb{RP}^4;\mathbb{Z}_2) \cong \mathbb{Z}_2$)
- v_4 - twisted integral lift of $w_4(T\mathbb{RP}^4)$ (fixing m_c structure) : $v_4 = w_4(T\mathbb{RP}^4) \mod 2$
- background flux: $G_4 = \left(Q - \frac{1}{2}\right) v_4$

Topological operators from branes

Field	Supergravity origin
c_3	G_4 on $t_1 = \mathrm{PD}_{\mathbb{RP}^4}[\mathbb{RP}^3]$
b_3	G_7 on $t_4 = \mathrm{PD}_{\mathbb{RP}^4}[\widetilde{\mathrm{pt}}]$
a_1	G_4 on $t_3 = \mathrm{PD}_{\mathbb{RP}^4}[\mathbb{RP}^1]$
a_5	G_7 on $t_2 = \mathrm{PD}_{\mathbb{RP}^4}[\mathbb{RP}^2]$

Topological operator	Brane origin
$\boldsymbol{Q}_3(\Sigma_3) = e^{i\pi \int_{\Sigma_3} \mathsf{c}_3}$	M2 on $\Sigma_3 \times \widetilde{\mathrm{pt}}$
$\widehat{\boldsymbol{Q}}_3(\Sigma_3) = e^{i\pi \int_{\Sigma_3} \mathsf{b}_3} T_3(\Sigma_3; \mathsf{a}_1, \mathsf{c}_3)$	M5 on $\Sigma_3 \times \mathbb{RP}^3$
$\boldsymbol{Q}_1(\Sigma_1) = e^{i\pi \int_{\Sigma_1} \mathsf{a}_1}$	M2 on $\Sigma_1 \times \mathbb{RP}^2$
$\widehat{\boldsymbol{Q}}_5(\Sigma_5) = e^{i\pi \int_{\Sigma_5} \mathsf{a}_5} T_5(\Sigma_5; \mathsf{c}_3)$	M5 on $\Sigma_5 \times \mathbb{RP}^1$

- ⋆ M2 wraps twisted cycles (elements of $H_\bullet(\mathbb{RP}^4; \widetilde{\mathbb{Z}})$) - couples to (twisted) C_3
- ⋆ M5 wraps elements of $H_\bullet(\mathbb{RP}^4; \mathbb{Z})$ - couples to C_6 ($\star G_4$ - untwisted)
- ⋆ $T_p(\Sigma_p)$ - dressing by a p-dimensional TQFT localized on Σ_p
  - ◇ naive holonomies $e^{i\pi \int_{\Sigma_3} \mathsf{b}_3}$ and $e^{i\pi \int_{\Sigma_5} \mathsf{a}_5}$ - not gauge invariant
  - ◇ T_3 & T_5 $\Rightarrow$ topological operators $\widehat{\boldsymbol{Q}}_3(\Sigma_3)$ & $\widehat{\boldsymbol{Q}}_5(\Sigma_5)$ - non-invertible
  - ◇ Origin: $h_3 \wedge i^* C_3$ coupling on M5

7D topological operators and TQFT couplings

$\boldsymbol{Q}_3(\Sigma_3)$ = M2-brane on $\Sigma_3 \times \widetilde{\mathrm{pt}}$ & $\widehat{\boldsymbol{Q}}_3(\widehat{\Sigma}_3)$ = M5-brane on $\widehat{\Sigma}_3 \times \mathbb{RP}^3$

- ◇ $\widetilde{\mathrm{pt}} = \mathrm{PD}_{\mathbb{RP}^4}[t_1^4]$ - generator of $H_0(\mathbb{RP}^4; \widetilde{\mathbb{Z}})$
- ◇ $\mathbb{RP}^3 = \mathrm{PD}_{\mathbb{RP}^4}[t_1]$ - generator of $H_3(\mathbb{RP}^4; \mathbb{Z})$

$$\boldsymbol{Q}_3(\Sigma_3)\widehat{\boldsymbol{Q}}_3(\widehat{\Sigma}_3) = \widehat{\boldsymbol{Q}}_3(\widehat{\Sigma}_3)\boldsymbol{Q}_3(\Sigma_3)\exp\left[-2\pi i\frac{1}{2}(\Sigma_3 \cdot_{M_6} \widehat{\Sigma}_3)\right]$$

$$\widehat{\boldsymbol{Q}}_3(\widehat{\Sigma}_3')\widehat{\boldsymbol{Q}}_3(\widehat{\Sigma}_3) = \widehat{\boldsymbol{Q}}_3(\widehat{\Sigma}_3)\widehat{\boldsymbol{Q}}_3(\widehat{\Sigma}_3')\exp 2\pi i\frac{\mp Q}{4}(\widehat{\Sigma}_3' \cdot_{M_6} \widehat{\Sigma}_3)$$

$$\Rightarrow \qquad S_{\text{7d TQFT}}^{\text{quadratic}} = 2\pi\int_{M_7}\left[-\frac{1}{2}Qc_3 \wedge dc_3 + 2c_3 \wedge db_3 + 2a_1 \wedge da_5\right]$$

- ◇ $G_4 * G_4 \sim G_4 G_7 \quad \Rightarrow \quad (2a_1 \wedge da_5)$

From the continuum to the discrete formulation:

- ◇ $c_3 \mapsto \frac{1}{2}\mathsf{c}_3,\ b_3 \mapsto \frac{1}{2}\mathsf{b}_3,\ a_1 \mapsto \frac{1}{2}\mathsf{a}_1,\ a_5 \mapsto \frac{1}{2}\mathsf{a}_5,$
- ◇ $\mathsf{c}_3, \mathsf{b}_3, \mathsf{a}_1, \mathsf{a}_5$ - cochains in $C^\bullet(M_7; \mathbb{Z}_2)$

$$S_{\text{7d TQFT}} = 2\pi\int_{M_7}\left[-\frac{N}{8}\mathsf{c}_3 \cup \delta\mathsf{c}_3 + \frac{1}{2}\mathsf{c}_3 \cup \delta\mathsf{b}_3 + \frac{1}{2}\mathsf{a}_1 \cup \delta\mathsf{a}_5 + \frac{1}{4}\mathsf{a}_1 \cup \mathsf{c}_3 \cup \mathsf{c}_3\right]$$

(back to) ANOMALIES!

Repeat the A-type holographic derivation?

- Start from

$$\mathcal{I}_{12} = -\frac{1}{6} G_4 \wedge G_4 \wedge G_4 - G_4 \wedge X_8 \ ,$$

- Consider

$$\mathbb{RP}^4 \hookrightarrow \widehat{M}_{12} \to M_8 \ .$$

- Perform fiber integration

$$\mathcal{I}_8 := \int_{\mathbb{RP}^4} \mathcal{I}_{12} \overset{??}{=} -I_{D_Q}^{(2,0)}$$

- Strategy: start from

$$S^4 \hookrightarrow M_{12} \to M_8 \ .$$

+ perform fiberwise antipodal identification on S^4

Global angular forms

On M_{12} exists a globally defined form e_4:

- e_4 is closed
  - ◇ its de Rham cohomology class is the image in $H^4(M_{12};\mathbb{R})$ of an integral cohomology class $H^4(M_{12};\mathbb{Z})$
- e_4 restricted to a generic fiber - proportional to the volume form on the fiber
  - ◇ $\int_{S^4} e_4 = 2$
  - ◇ $SO(5)$ fields turned off: $e_4 \longrightarrow 2\,\mathrm{Vol}(S^4)$
- $\int_{S^4} (e_4)^{2s+2} = 0$ & $\int_{S^4} (e_4)^{2s+1} = 2\times\big[p_2(\mathcal{N})\big]^s\,, \quad s = 0, 1, 2, \ldots$

Antipodal identification ($y^A \mapsto -y^A$): $e_4 \mapsto -e_4$

- $e_4 \longrightarrow \widehat{e}_4$ - twisted form on fibration $\widehat{M}_{12}$
  - ◇ $SO(5)$ fields turned off: $\widehat{e}_4 \longrightarrow v_4$
- $\int_{\mathbb{RP}^4} (\widehat{e}_4)^{2s+2} = 0$ & $\int_{\mathbb{RP}^4} (\widehat{e}_4)^{2s+1} = 1\times\big[p_2(\mathcal{N})\big]^s\,, \quad s = 0, 1, 2, \ldots$

Consistency of M-theory and action and cubic refinement for Pin^+ spaces

Half-integer flux quantisation

- $G_4 = a_4 - \frac{1}{2} v_4$
  - ◇ a_4 - closed 4-form with integral periods
- Consistent action for M-theory:

$$\int_{M_{12}} \left\{ \left[-\frac{1}{6} \left(a_4 - \frac{1}{2} v_4 \right)^3 - \left(a_4 - \frac{1}{2} v_4 \right) X_8 \right] - (a_4 \to 0) \right\}$$

  - ◇ v_4 - twisted integral lift of w_4 that specifies the m_c structure
  - ◇ In orientable setting $v_4 \longrightarrow \lambda$ (provides canonical integral lift of w_4)
  - ◇ In Pin^+setting - integer-values cubic form: $k_M(C) = \frac{1}{6} C^3 + C\, X_8 + \frac{1}{2} RS'(M)$
- Phase factor $\Leftrightarrow$ ambiguity in the Pfaffian of Rarita-Schwinger operator
- M-theory on $AdS_7 \times \mathbb{RP}^4$: $a_4 = Q \widehat{e}_4$
  - ◇ $(a_4 \to 0)$ subtraction $\Leftrightarrow$ $(Q \to 0)$ subtraction
  - ◇ amounts to subtracting zero for A-type but is important for D-type anomalies!

Computing the anomaly

- $G_4 = \left(N - \frac{1}{2}\right) e_4$
- $\frac{1}{6} \int_{\mathbb{RP}^4} G_4^3 = \frac{1}{6} \left(Q - \frac{1}{2}\right)^3 \int_{\mathbb{RP}^4} e_4^3 = \frac{1}{2} (2Q-1)^3 \frac{p_2(\mathcal{N})}{24}$
- $X_8|_W = I_8((2,0)\,\text{tm}) - \frac{p_2(\mathcal{N})}{24}$
- $-\mathcal{I}_8 = -\int_{\mathbb{RP}^4} \mathcal{I}_{12} = \left(Q - \frac{1}{2}\right) I_8((2,0)\,\text{tm}) + Q(2Q-1)(2Q-2)\frac{p_2(\mathcal{N})}{24}$

* * *

$$I_8^{\text{SCFT}} = \Big(-\mathcal{I}_8\Big) - \Big(-\mathcal{I}_8\Big|_{N\to 0}\Big)$$
$$= QI_8((2,0)\,\text{tm}) + Q(2Q-1)(2Q-2)\frac{p_2(\mathcal{N})}{24}$$

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