Correlated Resetting Gas

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Collaborators

- Marco Biroli (LPTMS, Univ. Paris Saclay)
- Manas Kulkarni (ICTS, Bangalore)
- Hernan Larralde (UNAM, Mexico)
- Gregory Schehr (LPTHE, Univ. Sorbonne)
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References:

M. Biroli, H. Larralde, S. N. Majumdar, G. Schehr, "Extreme Statistics and Spacing Distribution in a Brownian Gas Correlated by Resetting", Phys. Rev. Lett., **130**, 207101 (2023)

M. Biroli, H. Larralde, S. N. Majumdar, G. Schehr, "Exact extreme, order and sum statistics in a class of strongly correlated system", Phys. Rev. E **109**, 014101 (2024).

M. Biroli, M. Kulkarni, S. N. Majumdar, G. Schehr, "Dynamically emergent correlations between particles in a switching harmonic trap ", Phys. Rev. E **109**, L032106 (2024).

S. Sabhapandit & S. N. Majumdar, "Noninteracting particles in a harmonic trap with a stochastically driven center ", arXiv: 2404.02480 (to appear in J. Phys. A: Math. Theor.)

Plan

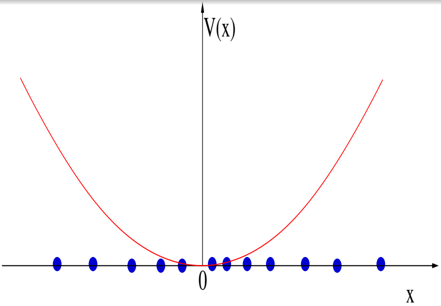
- Correlated gas in thermal equilibrium: examples and observables
- Correlated gas in **nonequilibrium** stationary state created by **resetting**
- Exact results for various observables: average density, extreme and order statistics, gap statistics, full counting statistics
- Summary and Conclusions

One dimensional Correlated Gas

In

Thermal Equilibrium

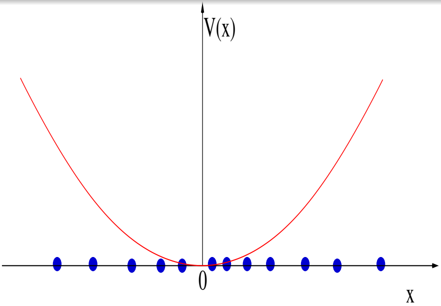
Correlated gas in thermal equilibrium



N particles on a line with coordinates
$\Longrightarrow \{x_1, x_2, \ldots, x_N\}$

$V(x) \rightarrow$ external confining potential

Correlated gas in thermal equilibrium



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Energy of the gas:

$$E[\{x_i\}] = \sum_i V(x_i) + \sum_{i \neq j} V_2(x_i, x_j) + \sum_{i \neq j \neq k} V_3(x_i, x_j, x_k) + \ldots$$

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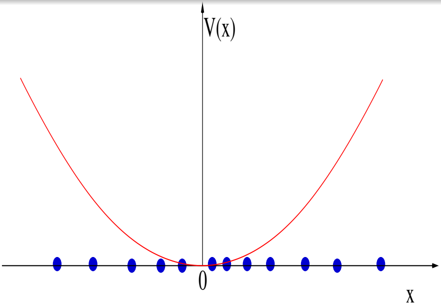

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$$P(x_1, x_2, \ldots, x_N) = \frac{1}{Z} e^{-\beta E[\{x_i\}]}$$

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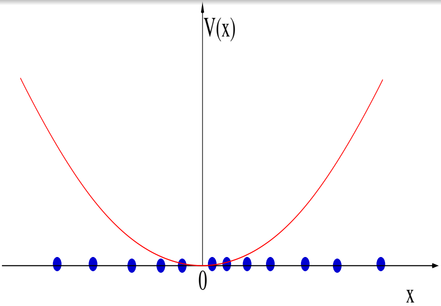

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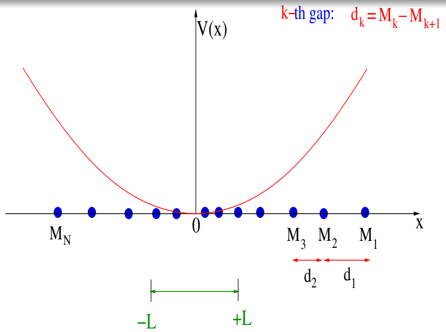
$$P(x_1, x_2, \ldots, x_N) = \frac{1}{Z}\, e^{-\beta\, E[\{x_i\}]} \neq p(x_1)p(x_2)\ldots p(x_N)$$

No **factorization** in the presence of **interactions**

Observables of interest



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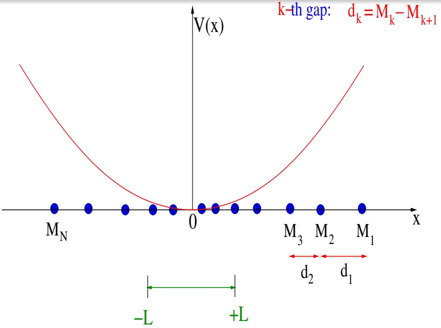

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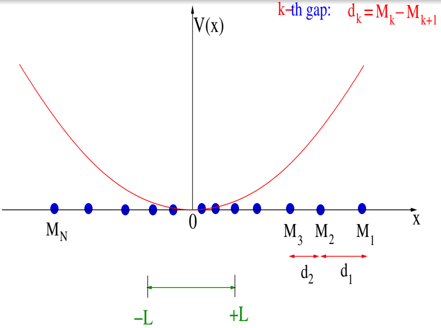

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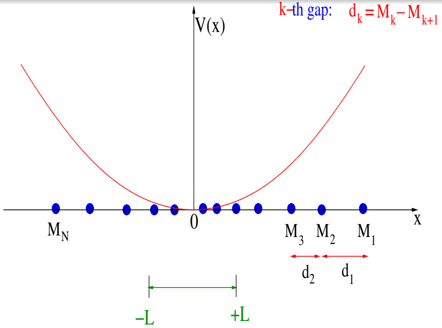

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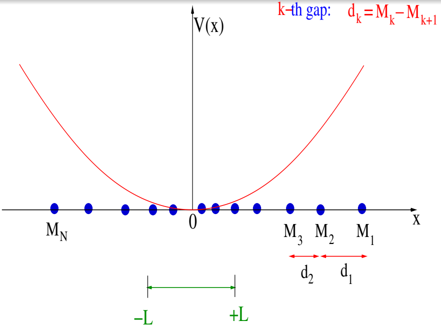

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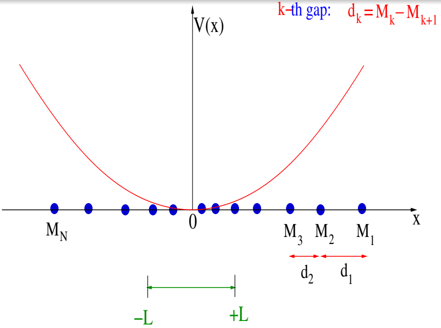

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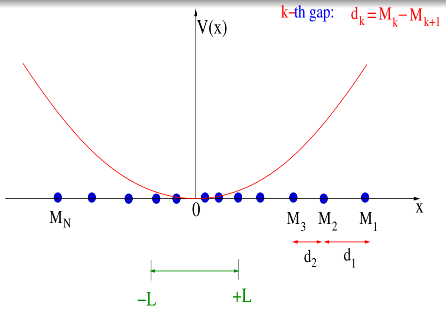
Generally hard to compute for a **correlated/interacting** gas !

Ideal gas: no interaction



In the absence of interactions

Energy: $E[\{x_i\}] = \sum_{i=1}^{N} V(x_i)$

Joint distribution **factorises** (i.i.d)

$$P(\{x_i\}) = p(x_1)p(x_2)\ldots p(x_N)$$

where $p(x) = \frac{e^{-\beta\, V(x)}}{\int dx'\, e^{-\beta\, V(x')}}$

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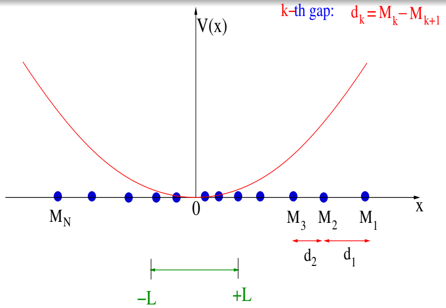

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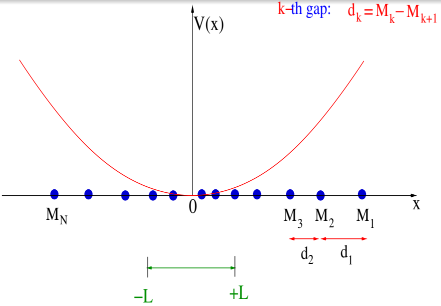

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- Average density: $\rho(x,N) = \frac{1}{N}\sum_{i=1}^{N}\langle\delta(x_i - x)\rangle = p(x)$
- Distribution of the k-th maximum $M_k \Longrightarrow$ Order statistics
- Distribution of the k-th gap $d_k = M_k - M_{k+1}$
- Full counting statistics (FCS): $\mathrm{Prob.}[N_L, N]$

Exact results for observables in the Ideal gas

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- **Order Statistics:** Distribution of the k-th maximum M_k

$$\mathrm{Prob.}[M_k = w] = \frac{N!}{(k-1)!\,(N-k)!}\, p(w) \left[\int_w^{\infty} p(y) dy\right]^{k-1} \left[\int_{-\infty}^{w} p(y) dy\right]^{N-k}$$

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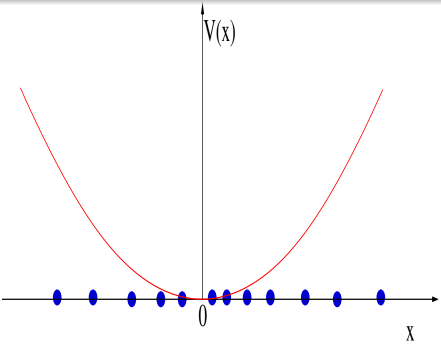
- **Full Counting Statistics:**

$$\mathrm{Prob.}[N_L, N] = \binom{N}{N_L} {q_L}^{N_l} (1 - q_L)^{N-N_L} \quad \text{where} \quad q_L = \int_{-L}^{L} p(y)\, dy$$

$N_L \Rightarrow$ no. of particles in the interval $[-L, L]$

Example 1 of a correlated gas: Dyson's log-gas



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$$E[\{x_i\}] = \frac{N}{2}\sum_{i=1}^{N} x_i^2 - \frac{1}{2}\sum_{i\neq j} \log|x_i - x_j|$$

pairwise logarithmic repulsion Dyson, 1962

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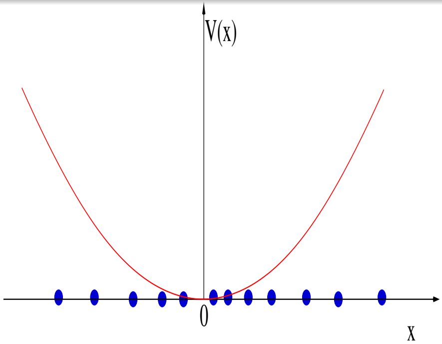

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Consider an $(N \times N)$ Gaussian Hermitian random matrix H_{ij} whose entries are distributed via:

$$\mathrm{Prob.}[H] \propto \exp\left[-N \sum_{i,j} |H_{ij}|^2\right] \propto \exp\left[-N\, \mathrm{Tr}\, \left(H^\dagger H\right)\right]$$

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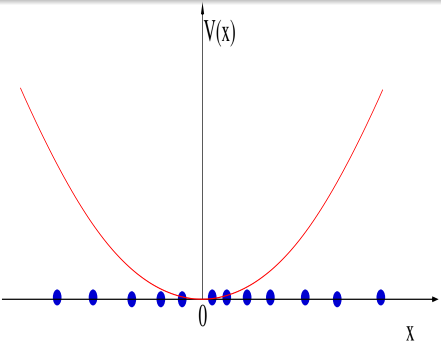

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N real eigenvalues: $\{\lambda_1, \lambda_2, \ldots, \lambda_N\} \longrightarrow$ **strongly correlated**

Dyson's log-gas

Joint distribution of eigenvalues of an $(N \times N)$ Gaussian Hermitian random matrix (Wigner, 1951):

$$P(\{\lambda_i\}) = \frac{1}{Z_N} \exp\left[-N\sum_{i=1}^{N}\lambda_i^2\right] \prod_{i<j}|\lambda_i - \lambda_j|^2$$

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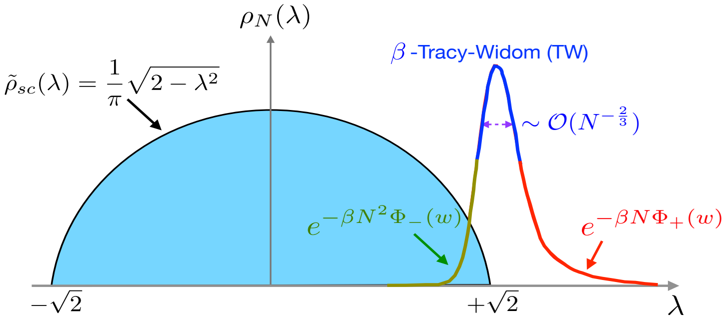
Hence one can identify the eigenvalues $\{\lambda_1, \lambda_2, \ldots \lambda_N\} \equiv \{x_1, x_2, \ldots, x_N\}$ as the positions of a 1-d gas of N particles with pairwise log-repulsion with $\beta = 2$ (Dyson, 1962)

Most of the observables can be computed exactly $\Longrightarrow$ not that **easy !**

Observables in the log-gas model



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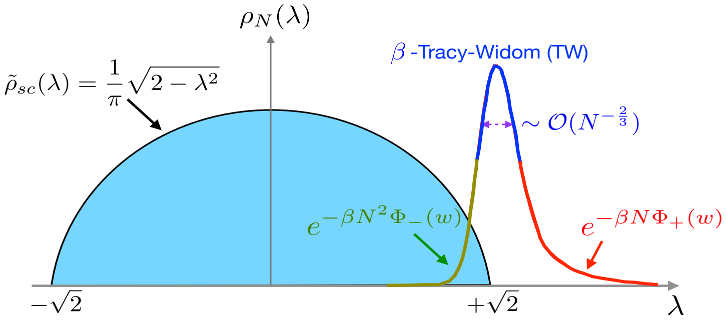

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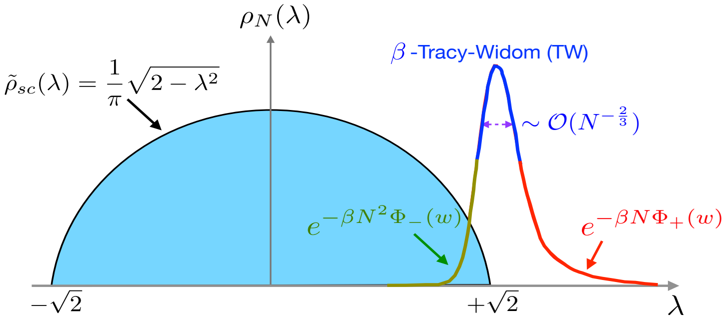

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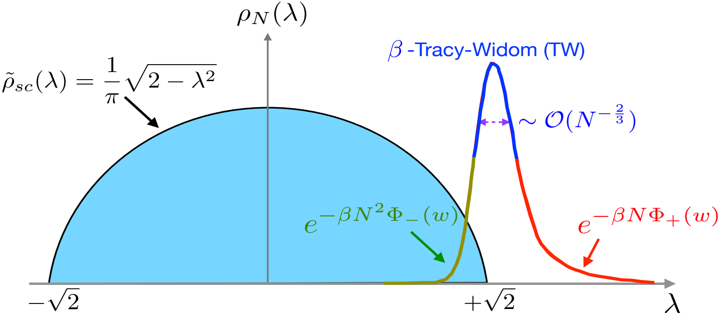

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Similarly, other observables are also known $\Longrightarrow$ huge literature

S.M. & G. Schehr, "Statistics of Extremes and Records in Random Sequences" (Oxford University Press, 2024)

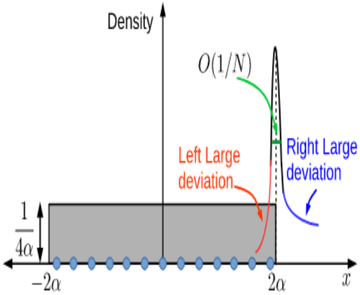
Ex 2: Jellium model in 1-d

Energy:

$$E[\{x_i\}] = \frac{N^2}{2}\sum_{i=1}^{N} x_i^2 - \alpha\, N \sum_{i\neq j} |x_i - x_j|$$

1-d Coulomb (linear) repulsion

Lenard, 1961; Prager, 1962; Baxter, 1963 ...



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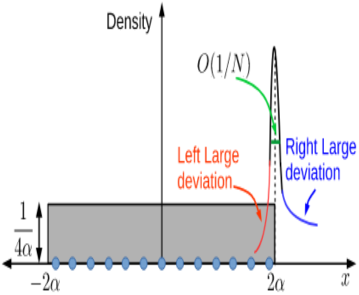

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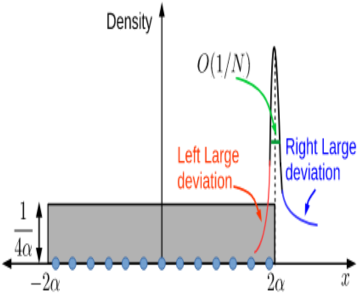

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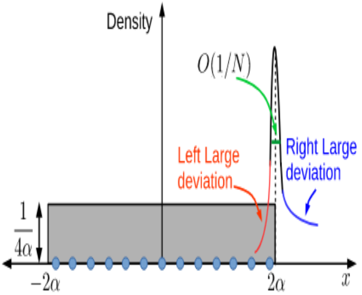

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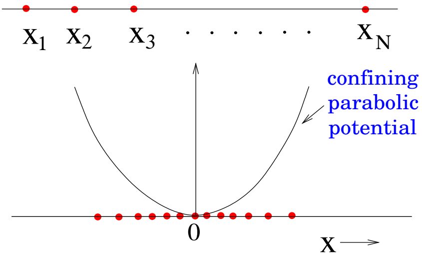
- Extreme, order, gap, full counting statistics $\Longrightarrow$ recently computed

Dhar, Kundu, S.M., Sabhapandit, Schehr, PRL, 119, 060601 (2017); J. Phys. A: Math. Theor. 51, 295001 (2018)

Flack, S.M., Schehr, J. Stat. Mech. 053211 (2022)

Ex 3: harmonically confined Riesz gas in 1-d



Energy function (with $k > -2$):

$$E[\{x_i\}] = \frac{1}{2}\sum_{i=1}^{N} x_i^2 + \frac{J\,\mathrm{sgn}(k)}{2}\sum_{i\neq j}\frac{1}{|x_i - x_j|^k}$$

M. Riesz, 1938

Recent survey: M. Lewin, 2022

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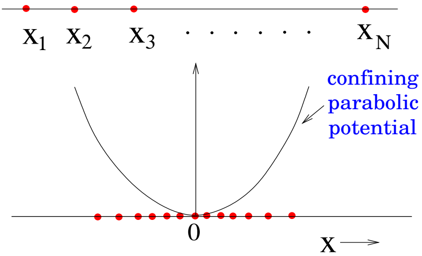

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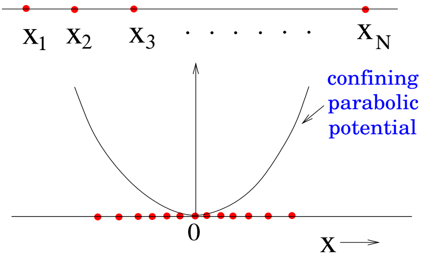
Special cases:

$k = -1$ (Jellium model), $k \to 0^+$ (Log-gas) and $k = 2$ (Calogero model)

Ex 3: harmonically confined Riesz gas in 1-d



Energy function (with $k > -2$):

$$E[\{x_i\}] = \frac{1}{2}\sum_{i=1}^{N} x_i^2 + \frac{J\,\mathrm{sgn}(k)}{2}\sum_{i\neq j}\frac{1}{|x_i - x_j|^k}$$

M. Riesz, 1938

Recent survey: M. Lewin, 2022

Special cases:

$k = -1$ (Jellium model), $k \to 0^+$ (Log-gas) and $k = 2$ (Calogero model)

Average density $\rho(x, N)$ in the large N limit

$\Longrightarrow$ computed recently for all $k > -2$

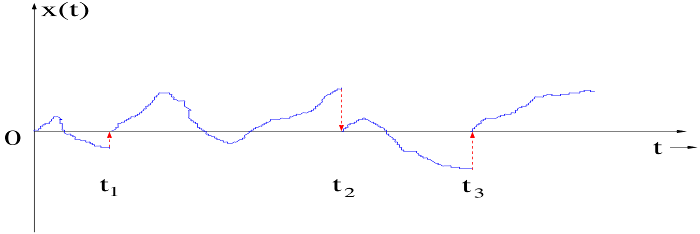
Agarwal, Dhar, Kulkarni, S.M., Mukamel, Schehr, PRL, 123, 100603 (2019)

Kethepalli et. al., J. Stat. Mech., 103209 (2021); J. Stat. Mech. 033203 (2022)

Santra et. al. PRL, 128, 170603 (2022)

Nonequilibrium Stationary State
induced by
Stochastic Resettting

Stochastic Resetting



- Natural dynamics $\Longrightarrow$ deterministic/stochastic/classical/quantum
- Resetting at random times and then natural dynamics restarts afresh
- Interval between resettings $\Longrightarrow$ $p(\tau)$ independently

  $\Longrightarrow$ renewal process
- If $p(\tau) = r\, e^{-r\,\tau}$ $\Longrightarrow$ Poissonian resetting

M. R. Evans & S.M., PRL, 106, 160601 (2011)

Reviews: Evans, S.M., Schehr, J. Phys. A. : Math. Theor. 53, 193001 (2020); Pal, Kostinski, Reuveni, J. Phys. A. : Math. Theor. 55, 021001 (2022)

Stochastic resetting in many-body systems

Any many-body system evolving under its own stochastic dynamics:

Ex: fluctuating interfaces, Ising model with Glauber dynamics etc.

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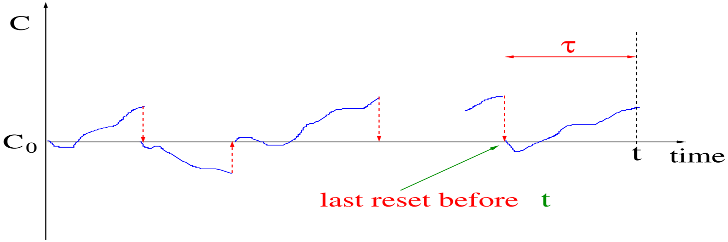
$P_r(C,t) \longrightarrow$ Prob. that the system is in config. C at time t

Question : How does $P_r(C,t)$ evolve with time ?

A renewal equation for $P_r(C,t)$



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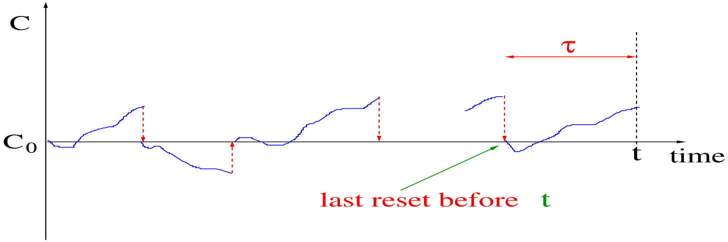

Renewal equation: Setting $\tau \to$ time since last resetting before t

$$P_r(C,t) = e^{-r\,t}\, P_0(C,t) + \int_0^t d\tau\, (r\, e^{-r\,\tau})\, P_0(C,\tau)$$

[S. Gupta, S.M., G. Schehr, PRL, 112, 220601 (2014)]

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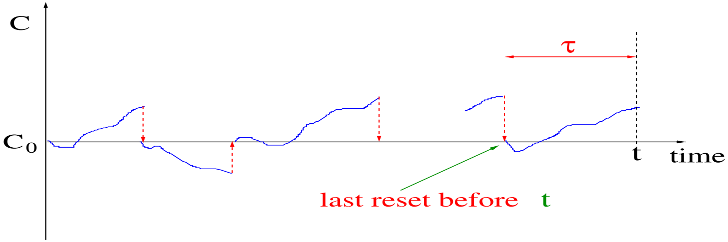

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As $t \to \infty$, the nonequilibrium stationary state:

$$P_r(C) = \int_0^\infty d\tau\, (r\, e^{-r\,\tau})\, P_0(C, \tau)$$

Nonequilibrium Stationary State

At long times, the system reaches a nonequilibrium stationary state

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To determine this stationary state, we need to know the full time-dependent $P_0(C, \tau)$ for the system without resetting at **all times** τ

$\Longrightarrow$ makes it **hard**

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Few cases where analytical progress can be made

Examples: Diffusion-Coagulation process, Fluctuating interfaces, Exclusion processes, N independent Brownian motions, Ising model etc.

Durang, Henkel & Park, J. Phys. A, 47, 045002 (2014), ; Gupta, S.M., Schehr, PRL, 112, 220601 (2014); Basu, Kundu, Pal, PRE, 100, 032136 (2019); Magoni, S.M., Schehr, PRR, 2, 033182 (2020),...

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Example: N **noninteracting** particles in a switching optical trap

M. Biroli, H. Larralde, S. M., G. Schehr, PRL, 130, 207101 (2023)

Correlated Resetting Gas

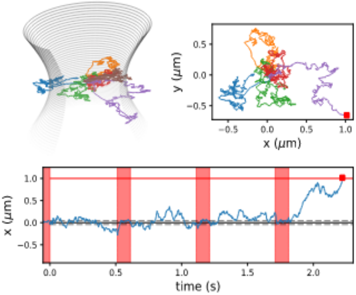
Recent Experiments on Stochastic Resetting

Recent experiments on stochastic resetting using optical traps set-up:

Tal-Friedman, Pal, Sekhon, Reuveni, Roichman, J. Phys. Chem. Lett. 11, 7350 (2020)

Besga, Bovon, Petrosyan, S.M., Ciliberto, Phys. Rev. Res. 2, 032029 (2020) $\longrightarrow$ 1-dimension

Faisant, Besga, Petrosyan, Ciliberto, S.M. J. Stat. Mech. 113203 (2021) $\longrightarrow$ 2-dimension

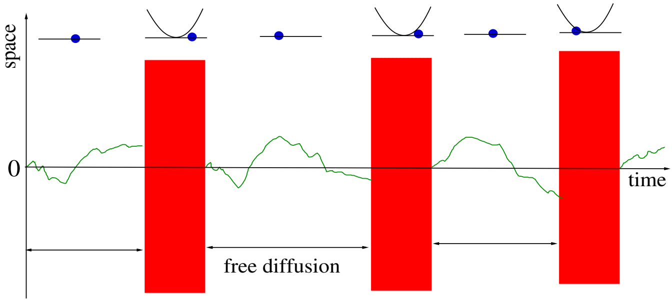


Experimental protocol for resetting

1. Free diffusion for a certain period (deterministic or random)

2. Switch on an optical harmonic trap and the let the particle relax to its equilibrium distribution using Engineered Swift Equilibration (ESE) technique $\Longrightarrow$ mimics instantaneous resetting

Steps 1 and 2 alternate ...

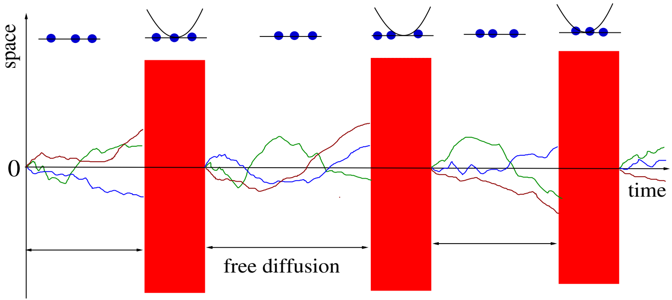


Exp. protocol for resetting

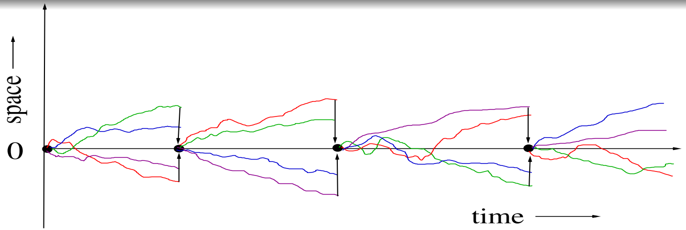
1. Free diffusion of N noninteracting particles during an exponentially distributed period

2. Switch on an optical harmonic trap and the let the particles relax to their equilibrium distribution $\Longrightarrow$ mimics instantaneous resetting

Steps 1 and 2 alternate ...



A simple model of Correlated resetting gas



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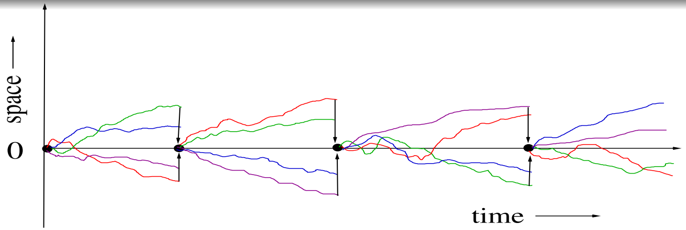

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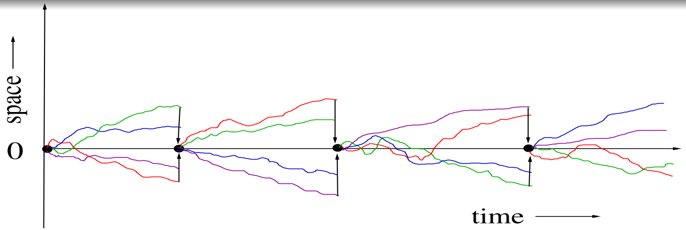

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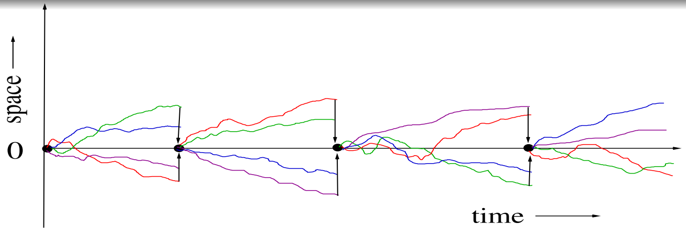

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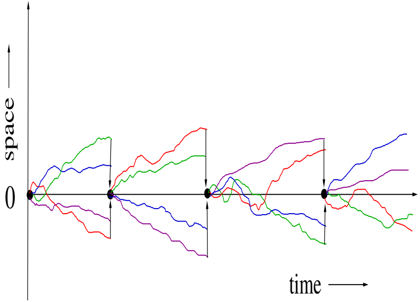
The joint distribution does not **factorize** $\Longrightarrow$ **correlated** resetting gas

M. Biroli, H. Larralde, S. M., G. Schehr, PRL, 130, 207101 (2023)

Solvable Correlated Gas



Joint distribution:

$$P_r^{\mathrm{st}}(\{x_i\}) = r \int_0^\infty d\tau \, e^{-r\tau} \prod_{i=1}^{N} p_0(x_i,\tau)$$

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In this model, interactions between particles are not built-in, but the correlations are generated by the dynamics (**simultaneous resetting**), that persist all the way to the stationary state

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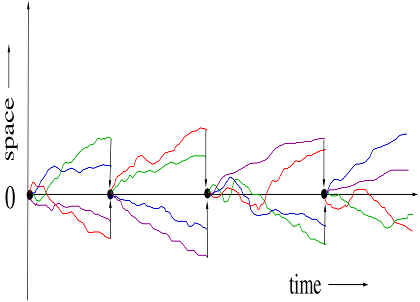

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The gas is **strongly** correlated in the NESS

$$\langle x_i^2 x_j^2\rangle - \langle x_i^2\rangle\langle x_j^2\rangle = 4\frac{D^2}{r^2} \Longrightarrow$$ **attractive all-to-all** interaction

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Despite **strong correlations**, several physical observables can be computed **exactly** in the NESS $\Longrightarrow$ (Solvable)

- Compute any observable for the ideal gas $\Rightarrow$ I.I.D variables with distribution $p_0(x,\tau)$ parametrized by $\tau \Longrightarrow$ easy
- Average over the **random** parameter τ using $p(\tau) = r\, e^{-r\,\tau}$

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Examples:

- Average density
- Distribution of the k-th maximum: Order statistics
- Spacing distribution
- Full Counting Statistics

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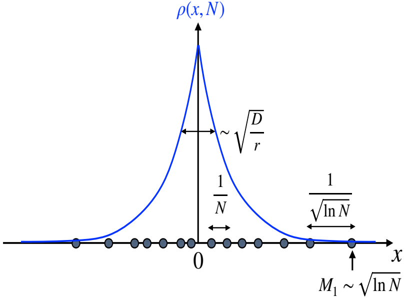
Average Density



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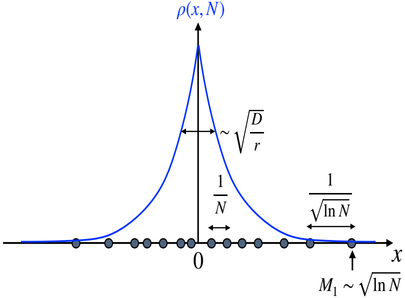

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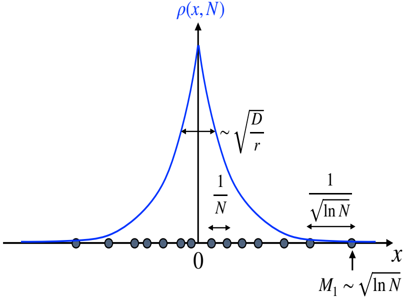
$$= r \int_0^\infty d\tau\, e^{-r\tau}\, p_0(x,\tau) = \frac{\alpha_0}{2}\, \exp[-\alpha_0\, |x|]$$

where $\alpha_0 = \sqrt{r/D}$

$\Longrightarrow$ same as the single particle position distribution

Order Statistics



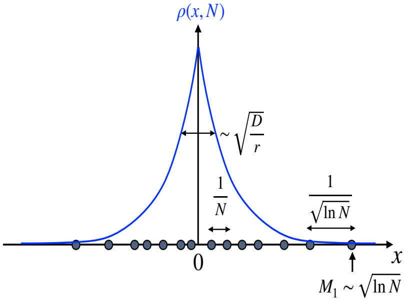
$M_k \implies k$-th maximum

Set $k = \alpha N$

$\alpha \sim O(1) \implies$ **bulk**

$\alpha \sim O(1/N) \implies$ **edge**

Order Statistics



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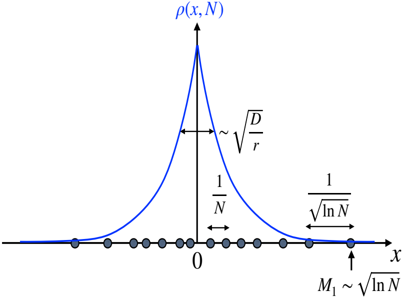
Order Statistics



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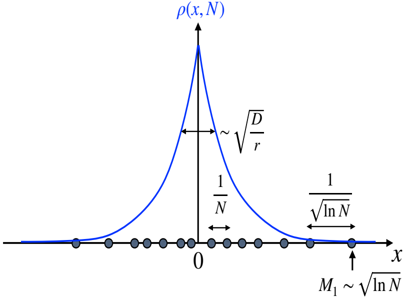

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The scaling function $\mathbf{f(z) = 2\,z\,e^{-z^2}\,\theta(z)} \Longrightarrow$ **universal** (indep. of α)

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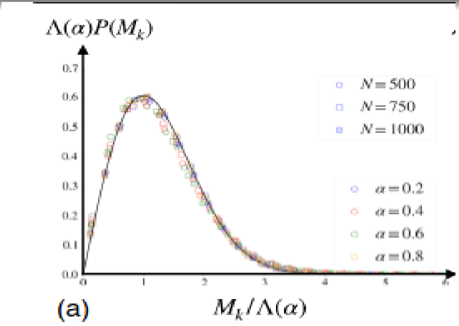

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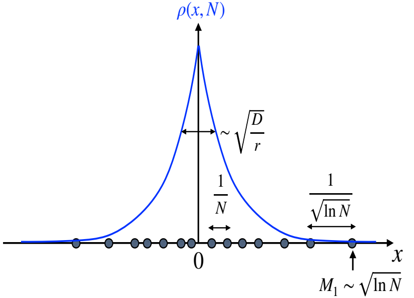
Gap/Spacing Statistics



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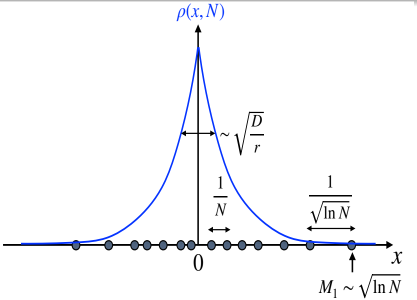

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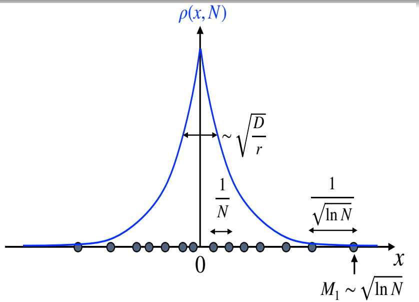

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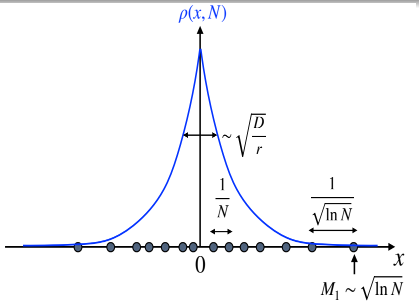

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- Bulk: $\mathrm{Prob.}[d_k = g] \approx \frac{1}{\lambda_N(\alpha)}\, h\left(\frac{g}{\lambda_N(\alpha)}\right)$ where $\lambda_N(\alpha) = \frac{1}{b\sqrt{r}\,N}$ with $b = \exp\left(-[\mathrm{erfc}^{-1}(2\,\alpha)]^2\right)/\sqrt{4\pi D}$

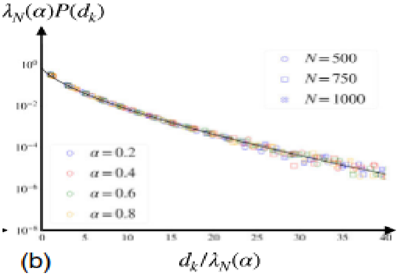
- Edge: $\mathrm{Prob.}[d_k = g] \approx \frac{1}{l_N(k)}\, h\left(\frac{g}{l_N(k)}\right)$ where $l_N(k) = \sqrt{\frac{D}{r\,k^2\,\ln N}}$

The scaling function $h(z) = 2\int_0^\infty du\, e^{-u^2 - z/u} \quad (z \geq 0)$

$\implies$ **universal** (indep. of α)

M. Biroli, H. Larralde, S. M., G. Schehr, PRL, 130, 207101 (2023)

Gap/Spacing Statistics



The gap scaling function:

$h(z) = 2 \int_0^\infty du\, e^{-u^2 - z/u}$

$h(z) \to \sqrt{\pi}$ as $z \to 0$

$h(z) \sim \exp[-3(z/2)^{2/3}]$ as $z \to \infty$

- Bulk: $\mathrm{Prob.}[d_k = g] \approx \frac{1}{\lambda_N(\alpha)}\, h\left(\frac{g}{\lambda_N(\alpha)}\right)$ where $\lambda_N(\alpha) = \frac{1}{b\sqrt{r}\,N}$ with $b = \exp\left(-[\mathrm{erfc}^{-1}(2\,\alpha)]^2\right)/\sqrt{4\pi D}$
- Edge: $\mathrm{Prob.}[d_k = g] \approx \frac{1}{l_N(k)}\, h\left(\frac{g}{l_N(k)}\right)$ where $l_N(k) = \sqrt{\frac{D}{r\,k^2\,\ln N}}$

The scaling function $h(z) = 2 \int_0^\infty du\, e^{-u^2 - z/u}$ $(z \geq 0)$

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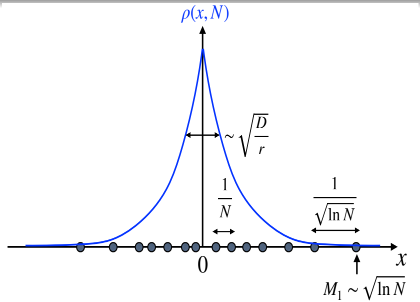
Full Counting Statistics

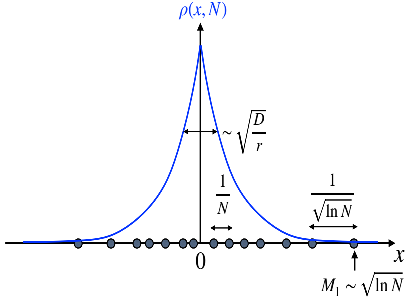


$N_L \Longrightarrow$ number of particles in $[-L, L]$

Clearly, $0 \leq N_L \leq N$

$P(N_L, N) = ?$

Full Counting Statistics



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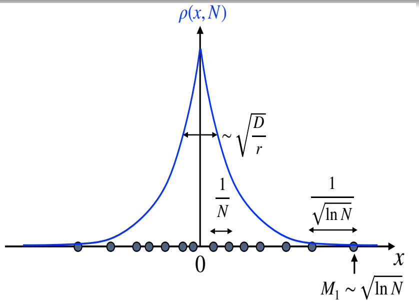

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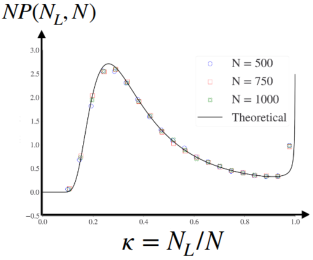
where the scaling function:

$$H(\kappa) = \gamma\sqrt{\pi}\,[u(\kappa)]^{-3}\, \exp\left[-\gamma\, u^{-2}(\kappa) + u^2(\kappa)\right]$$

with $\gamma = r\, L^2/(4D)$ and $u(\kappa) = \mathrm{erf}^{-1}(\kappa)$

M. Biroli, H. Larralde, S. M., G. Schehr, PRL, 130, 207101 (2023)

Full Counting Statistics



The scaling function $H(\kappa)$

$$H(\kappa) \to \frac{8\gamma}{\pi\,\kappa^3}\,\exp\left[-\frac{4\gamma}{\pi\,\kappa^2}\right] \quad \text{as } \kappa \to 0$$

$$H(\kappa) \to \frac{\gamma\sqrt{\pi}}{(1-\kappa)\,[\ln(1-\kappa)]^{3/2}} \quad \text{as } \kappa \to 1$$

Full Counting Statistics: $P(N_L, N) \approx \frac{1}{N}\, H\left(\frac{N_L}{N} = \kappa\right) \quad (0 \le \kappa \le 1)$

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M. Biroli, H. Larralde, S. M., G. Schehr, PRL, 130, 207101 (2023)

Generalisations

The structure of the joint distribution for N independent particles driven by simultaneous resetting is very general:

$$P_r^{\rm st}\left(\{x_i\}\right) = r\int_0^{\infty} d\tau\, e^{-r\,\tau}\prod_{i=1}^{N} p_0(x_i,\tau)$$

where $p_0(x,\tau)$ can represent **any** single particle motion, not necessarily difusion

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where $p_0(x,\tau)$ can represent **any** single particle motion, not necessarily difusion

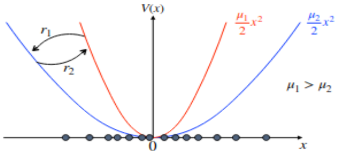
Ex: ballistic motion, Lévy flights etc.

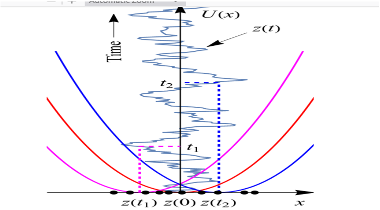
$\Longrightarrow$ a whole class of **solvable** correlated gases in their nonequilibrium stationary state

$\Longrightarrow$ a new application of stochastic resetting

M. Biroli, H. Larralde, S. M., G. Schehr, Phys. Rev. E **109**, 014101 (2024)

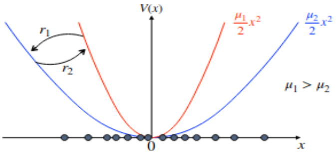
Exact stationary states for two other protocols

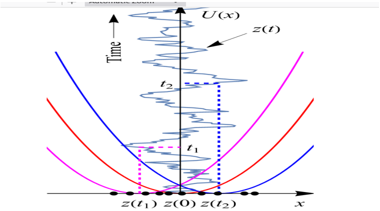




N noninteracting particles in a harmonic trap

Exact stationary states for two other protocols





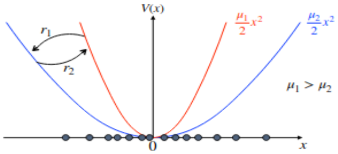
N noninteracting particles in a harmonic trap

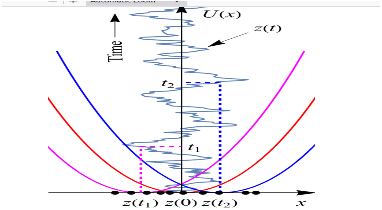
(1) Protocol 1: Stiffness of the harmonic trap changes from $\mu_1 \to \mu_2$ with rate r_1 and $\mu_2 \to \mu_1$ with rate r_2

$\Longrightarrow$ drives the system into a correlated NESS

Biroli, Kulkarni, S.M., Schehr, PRE, 109, L032106 (2024)

Exact stationary states for two other protocols





N noninteracting particles in a harmonic trap

(1) Protocol 1: Stiffness of the harmonic trap changes from $\mu_1 \to \mu_2$ with rate r_1 and $\mu_2 \to \mu_1$ with rate r_2

$\Longrightarrow$ drives the system into a correlated NESS

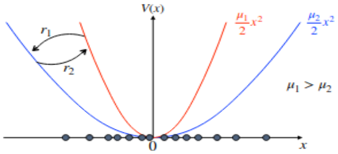
Biroli, Kulkarni, S.M., Schehr, PRE, 109, L032106 (2024)

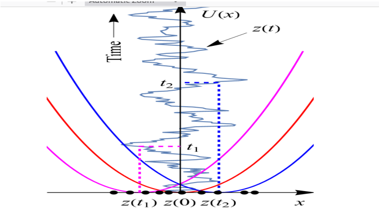
(2) Protocol 2: The center of the harmonic trap performs a stochastic motion

$\Longrightarrow$ drives the system into a correlated NESS

Sabhapandit, S.M., arXiv: 2404.02480

Exact stationary states for two other protocols





In both protocols, the NESS has the CIID (conditionally independent and identically distributed) structure

$$P_{\rm st}(x_1, x_2, \ldots, x_N) = \int_{-\infty}^{\infty} du\, h(u) \prod_{i=1}^{N} p(x_i|u)$$

This CIID structure makes the problem **solvable** for various observables such as average density, spacing distribution, extreme statistics, full counting statistics etc.

Biroli, Kulkarni, S.M., Schehr, PRE, 109, L032106 (2024); Sabhapandit, S.M., arXiv: 2404.02480

Summary and Conclusions

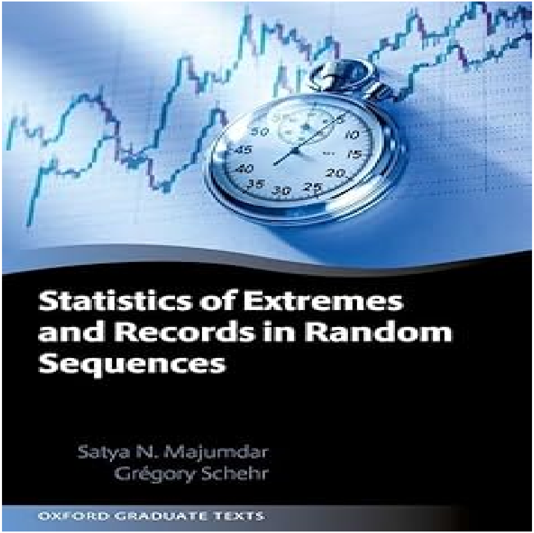
- A simple solvable model of a correlated gas of N diffusing particles in their nonequilibrium stationary state driven by simultaneous stochastic resetting

- Several physical observables are exactly computable and have rich and interesting behaviors, despite being a **strongly correlated** system

- Easily generalisable to a whole new class of solvable correlated gases in their nonequilibrium stationary state $\longrightarrow$ ballistic particles, Lévy flights

- Generalisation to N independent particles with two other protocols

Biroli, Larralde, S.M., Schehr, PRL, **130**, 207101 (2023); Phys. Rev. E **109**, 014101 (2024); Biroli, Kulkarni, S.M., Schehr, PRE, **109**, L032106 (2024); Sabhapandit, S.M., arXiv: 2404.02480


Statistics of Extremes and Records in Random Sequences
Satya N. Majumdar
Grégory Schehr
OXFORD GRADUATE TEXTS