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Gauge Symmetry and the SM Higgs Boson

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CONTENTS

I. Preliminaries 2

II. $U(1)_Q$ Gauge Symmetry [1] 3
A. Spin-0 field 3
B. Massive spin-1 field 3
C. Massless spin-1 field 5
D. Covariant derivatives 6
E. Ward identity 6
F. Photon progator 7
G. Decomposition of a vector field 8

III. Yang-Mills Theory [1] 9
A. SU(2) 9
B. Gauge invariance and Wilson lines: Ablelian case 10
C. SU(N) 12

IV. SM Higgs Boson 14
A. The Higgs mechanism in the Standard Model [2] 14
1. Preliminaries: gauge sector 14
2. Preliminaries: fermion sector 15
3. The SM Higgs mechanism 16
4. Gauge boson masses and couplings to the Higgs boson 19
5. Fermion masses, the CKM matrix, and couplings to the Higgs boson 22
6. Higgs self-couplings 27
B. Summary 28

References 30

I. PRELIMINARIES

- Natural Units:

$$c = 2.998 \times 10^8 \text{ meters/second} = 1$$
$$\hbar = \frac{h}{2\pi} = 1.054572 \times 10^{-34} \text{ joules} \cdot \text{seconds} = 1 \quad (1)$$

- Mass dimensions:
  - Length, Time, Energy, Momentum

$$c = \hbar = 1\,, E = \hbar\nu \longrightarrow [\text{Length}] = [\text{Time}]\,; \quad [\text{Energy}] = -[\text{Time}]$$
$$E = mc^2 \longrightarrow [\text{Energy}] = [\text{Mass}] = +1\,; \quad [\text{Length}] = [\text{Time}] = -1$$
$$p = mc \longrightarrow [\text{Momentum}] = [\text{Mass}] = +1\,;$$
$$[\text{Energy}] = [\text{Momentum}] = +1\,; \quad [\text{Length}] = [\text{Time}] = -1 \quad (2)$$

  - Action, Lagrangian, Fields

$$S = \int d^4x\mathcal{L} \,:\, [S] = 0\,, [d^4x] = -4 \longrightarrow [\mathcal{L}] = 4$$
$$\mathcal{L}_{\text{spin}-0\ \text{kinetic}} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) \longrightarrow [\phi] = 1$$
$$\mathcal{L}_{\text{spin}-1/2\ \text{kinetic}} = \bar{\psi}\,(i\partial_\mu\gamma^\mu)\,\psi \longrightarrow [\psi] = 3/2 \quad (3)$$

- Conversion: cross section [1]

$$1 \text{ GeV} = 1.602 \times 10^{-10} \text{ joules}$$
$$\frac{1}{\text{GeV}^2} = \frac{1}{\text{GeV}^2}\hbar^2 c^2 = \frac{(1.054572 \times 10^{-34})^2 (2.998 \times 10^8)^2}{(1.602 \times 10^{-10})^2} \text{ meters}^2 = 3.894 \times 10^{-32=8-12-28} \text{ meters}^2$$
$$= 3.894 \times 10^8 \text{ picobarn} \quad (4)$$

- LHC:

TABLE I. *The SM cross sections from Ref. [3] taking $M_H = 125$ GeV: ggF from Table 191, VBF from Tables 25 and 26, WH from Table 223, ZH from Table 225, ttH from Table 231, tHq from Table 237, and bbH from Table 247.*

$\sqrt{s}$ (TeV)	ggF (pb)	VBF (fb)	WH (fb)	ZH (fb)	ttH (fb)	tHq (fb)	tHW (fb)	bbH (fb)
7	16.85	1241.4	577.30	339.10	88.78	12.26	–	155.20
8	21.42	1601.2	702.50	420.70	133.0	18.69	–	202.10
13	48.57	3781.7	1373.00	883.70	507.2	74.25	15.2	488.00

  - Run 1 (2011-2012): 5/fb @ 7 TeV + 20/fb @ 6 TeV
  - Run 2 (2015-2018): ∼150/fb @ 13 TeV per experiment: $N_H = \sigma \times$ Luminosity $\simeq 60$ pb $\times$ 150/fb $= 9,000,000$
  - Run 3 (2022-2025): ∼300(?)/fb @ 13.6 TeV
  - · · ·

[1] The origin of the term barn comes from the fact that inducing nuclear fission by hitting ^{235}U with neutrons is as easy as hitting the broad side of a barn. The inelastic neutron-^{235}U scattering cross section is around 1 barn$= 10^{-28}$ m^2 at $E \sim 1$ MeV.

II. $U(1)_Q$ GAUGE SYMMETRY [1]

In this section, we follow the notations and conventions in Chapter 8 of Ref. [1].

We aim to find:

- Maxwell tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{5}$$

which is invariant under the gauge tansformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x) \tag{6}$$

for any function $\alpha(x)$. Incidentally, the mass term $\frac{1}{2}m^2 A_\mu^2$ breaks the gauge symmetry.

- Covariant derivative: $D_\mu \phi \equiv (\partial_\mu - ieQA_\mu)\phi$

$$\begin{aligned} \phi &\longrightarrow e^{iQ\alpha(x)}\phi \\ A_\mu &\longrightarrow A_\mu \ + \ \frac{1}{e}\partial_\mu \alpha(x) \\ D_\mu\phi &\longrightarrow e^{iQ\alpha(x)} D_\mu \phi \end{aligned} \tag{7}$$

The charge of the field ϕ is denoted by Q, $1/e$ appears in the gauge transformation of A_μ, and the combination $-ieQ$ is for the covariant detivative:

A. Spin-0 field

Let's start with a spin-0 scalar field $\phi(x)$ with

- Lagrangian:

$$\mathcal{L}(x) = \frac{1}{2}\,\partial_\mu\phi(x)\,\partial_\mu\phi(x) \ - \ \frac{1}{2}\,m^2\,\phi(x)^2 \tag{8}$$

- Equation of motion: [2]

$$\left(\Box + m^2\right)\phi \ = \ 0 \tag{9}$$

- Energy Density: [3]

$$\mathcal{E} = \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}(\partial_t\phi) - \mathcal{L} = (\partial_t\phi)^2 - \frac{1}{2}\left[(\partial_t\phi)^2 - \left(\vec{\nabla}\phi\right)^2 - m^2\phi^2\right] = \frac{1}{2}\left[(\partial_t\phi)^2 + \left(\vec{\nabla}\phi\right)^2 + m^2\phi^2\right] \tag{10}$$

which is positive definite and bounded from below by 0.

B. Massive spin-1 field

From the Lagrangian for a spin-0 field,

---

[2] $\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right] = 0.$

[3] $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1).$

- Lagrangian of four massive scalar fields of A_0, A_1, A_2, and A_3 which somehow represent a massive spin-1 vector field A_μ with the Lorentz-invariant length $A_\mu^2 = A_\mu A^\mu = A_0^2 - A_1^2 - A_2^2 - A_3^2$:

$$\begin{aligned}\mathcal{L}(x) &= +\frac{1}{2}\sum_{i=0,1,2,3}\left[\partial_\nu A_i\,\partial_\nu A_i - m^2\,A_i^2\right] \\ &\to -\frac{1}{2}\left[\partial_\nu A_0\,\partial_\nu A_0 - m^2\,A_0^2\right] + \frac{1}{2}\sum_{i=1,2,3}\left[\partial_\nu A_i\,\partial_\nu A_i - m^2\,A_i^2\right] \\ &= -\frac{1}{2}\partial_\nu A_\mu\,\partial_\nu A_\mu + \frac{1}{2}m^2\,A_\mu^2 \end{aligned} \tag{11}$$

- Equation of motion: $\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right] = 0$.

$$+\left(\Box + m^2\right)A_0 \;=\; 0\,, \quad -\left(\Box + m^2\right)A_{i=1,2,3} \;=\; 0 \;\;\to\;\; \left(\Box + m^2\right)A_\mu \;=\; 0\,; \tag{12}$$

- Energy density:

$$\mathcal{E} = \sum_{i=0,1,2,3}\frac{\partial\mathcal{L}}{\partial(\partial_t A_i)}(\partial_t A_i) - \mathcal{L} = -\frac{1}{2}\left[(\partial_t A_0)^2 + \left(\vec\nabla A_0\right)^2 + m^2 A_0^2\right] + \sum_{i=1,2,3}\frac{1}{2}\left[(\partial_t A_i)^2 + \left(\vec\nabla A_i\right)^2 + m^2 A_i^2\right] \tag{13}$$

which has a negative sign for the A_0 field and, accordingly, will not produce a physical theory...

What's wrong? Definitely, there are three degrees of freedom for spin 1 with $m > 0$, not four! Hmm... then, can we *simply* drop A_0 or remove one degree of freedom from A_μ which has four components?

Actually, there is one more Lorentz-invariant two-derivative kinetic term of

$$A_\mu\partial_\mu\partial_\nu A_\nu = -\left(\partial_\mu A_\mu\right)^2 + \text{t.d.} = -\partial_\nu A_\mu\partial_\mu A_\nu + \text{t.d.} \tag{14}$$

in addition to $\partial_\nu A_\mu\partial_\nu A_\mu = -A_\mu\Box A_\mu + \text{t.d.}$

- The most general Lagrangian:

$$\mathcal{L} = \frac{a}{2}A_\mu\Box A_\mu + \frac{b}{2}A_\mu\partial_\mu\partial_\nu A_\nu + \frac{1}{2}m^2 A_\mu^2 \tag{15}$$

The equations of motion are

$$\begin{aligned} &a\,\Box A_\mu \;+\; b\,\partial_\mu(\partial_\nu A_\nu) \;+\; m^2\,A_\mu = 0\,; \\ &\xrightarrow{\partial_\mu} \left[(a+b)\,\Box + m^2\right]\,(\partial_\mu A_\mu) = 0 \end{aligned} \tag{16}$$

If $a + b = 0$, when $m > 0$, the second equation reduces to $\partial_\mu A_\mu = 0$ which is Lorentz invariant and indeed removes one degree of freedom! Then, taking $a = -b = 1$, we have arrived at

$$\mathcal{L} = \frac{1}{2}A_\mu\Box A_\mu - \frac{1}{2}A_\mu\partial_\mu\partial_\nu A_\nu + \frac{1}{2}m^2 A_\mu^2 = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m^2 A_\mu^2 \tag{17}$$

with the Maxwell tensor [4]

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{18}$$

- Energy Density: With $\vec E = \partial_t\vec A - \vec\nabla A_0$ and $\vec B = \vec\nabla\times\vec A$, one might find

$$\mathcal{E} = \frac{1}{2}\left(\vec E^2 + \vec B^2\right) + \frac{1}{2}m^2\left(A_0^2 + \vec A^2\right) + A_0\partial_t(\partial_\mu A_\mu) - A_0(\Box + m^2)A_0 + \partial_i(A_0 F_{0i}) \tag{19}$$

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[4] Note that $-\frac{1}{4}F_{\mu\nu}^2 = -\frac{1}{4}(\partial_\mu A_\nu\partial_\mu A_\nu + \partial_\nu A_\mu\partial_\nu A_\mu - 2\partial_\mu A_\nu\partial_\nu A_\mu) = -\frac{1}{2}(\partial_\mu A_\nu\partial_\mu A_\nu - \partial_\mu A_\nu\partial_\nu A_\mu) = \frac{1}{2}(A_\nu\Box A_\nu - A_\nu\partial_\mu\partial_\nu A_\mu) + \text{t.d.}$

which give the positive-definite total energy with $\partial_\mu A_\mu = 0$ and $(\Box + m^2)A_0 = 0$.

- Polarizations: One might find the following solution to the equations of motion $(\Box + m^2)A_\mu = 0$ satifying the Lorentz-invariant condition $\partial_\mu A_\mu = 0$:

$$A_\mu(x) = \sum_i \int \frac{d^3\vec{p}}{(2\pi)^3}\, \widetilde{a}_i(\vec{p})\, \epsilon^i_\mu(p)\, e^{ipx} \quad \text{with} \quad p_0 = \sqrt{\vec{p}^2 + m^2} \tag{20}$$

where $\widetilde{a}_i(\vec{p})$ denotes Fourier components and the three basis 4-vectors $\epsilon^i_\mu(p)$ constitute the polarization vectors which satisfy the Lorentz-invariant condition

$$p^\mu\, \epsilon^i_\mu(p) = 0 \tag{21}$$

The polarizations vectors are conventionally normalized as

$$\epsilon^*_\mu \epsilon^\mu = -1 \tag{22}$$

To be explicit, if p^μ points to the z direction

$$p^\mu = (E, 0, 0, p_z) \ \text{ with } E^2 - p_z^2 = m^2\,;$$
$$\epsilon^\mu(\lambda = \pm 1 = R/L) = \frac{1}{\sqrt{2}}(0, -\lambda, -i, 0)\,, \quad \epsilon^\mu(\lambda = 0) = \left(\frac{p_z}{m}, 0, 0, \frac{E}{m}\right) \tag{23}$$

with $\lambda = \pm 1\,(0)$ denote the transverse (longitudinal) polarizations. Note that $\epsilon^\mu \epsilon^*_\mu = -1$ and $\epsilon^\mu p_\mu = 0$.

C. Massless spin-1 field

- Massive spin-1 field in the $m \to 0$ limit:
  - $m^2(\partial_\mu A_\mu) = 0$: we no longer automatically have $\partial_\mu A_\mu = 0$
  - The longitudinal polarization $\epsilon^\mu(\lambda = 0) = \left(\frac{p_z}{m}, 0, 0, \frac{E}{m}\right)$ blows up: $p^\mu \to (E, 0, 0, E)$ and $\epsilon^\mu(\lambda = 0) \to p^\mu$.
  - There should be only two polarizations for a massless spin-1 particle

Instead of trying to analyze what happens to the massive modes in the massless limit, let us just postulate the following Lagrangian and start over with analyzing the degrees of freedom:

$$\mathcal{L} = -\frac{1}{4} F^2_{\mu\nu} \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{24}$$

which is invariant under the tansformation (gauge invariance)

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x) \tag{25}$$

for any function $\alpha(x)$. Note that the mass term $\frac{1}{2} m^2 A^2_\mu$ violates the gauge invariance.

- Equations of motion:

$$\begin{aligned}
& \Box A_\mu \ - \ \partial_\mu(\partial_\nu A_\nu) \ = 0\,; \\
\mu = 0 = t \ &: \ (\partial_t^2 - \partial_j^2)A_0 - \partial_t(\partial_t A_0 - \partial_j A_j) = -\partial_j^2 A_0 + \partial_t(\partial_j A_j) = 0 \\
\mu = i = 1, 2, 3 \ &: \ \Box A_i - \partial_i(\partial_t A_0 - \partial_j A_j) = 0
\end{aligned} \tag{26}$$

- Gauge fixing: use the freedom of transforming the fields $A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x)$ to impose constraints on A_μ
  - One can choose α so that $\partial_j A_j = 0$ (Coulomb gauge)

$$A'_j = A_j + \partial_j \alpha \text{ with } \partial_j A_j \neq 0 \longrightarrow \text{ one can make } \ \partial_j A'_j = 0 \text{ by choosing } \alpha \text{ such that } \partial_i^2 \alpha = -\partial_j A_j \tag{27}$$

  Note that, once in the Coulomb gauge, there still is the freedom of gauge transformation $A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x)$ for any α satisfying $\partial_i^2 \alpha = 0$: Coulomb gauge is preserved if $\partial_i^2 \alpha = 0$

– One can set $A_0 = 0$: In Coulomb gauge, $A'_0 = A_0 + \partial_t \alpha$ with $\partial_i^2 \alpha = 0$. [5] Then one can set $A'_0 = 0$ by choosing α such that $\partial_t \alpha = -A_0$.

- In Coulomb gauge and setting $A_0 = 0$, the equations of motion become

$$\Box A_i = 0 \quad \text{for} \quad i = 1,2,3 \tag{28}$$

to which the solutions might be given by

$$A_\mu(x) = \int \frac{d^4p}{(2\pi)^4} \epsilon_\mu(p) e^{ipx} \tag{29}$$

with $\epsilon_0 = 0$ (gauge choice), $p_i \epsilon_i = 0$ (Coulomb gauge: $\vec{\epsilon} \perp \vec{p}$), and $p^2 = 0$ (equation of motion). In the frame $p^\mu = (E, 0, 0, E)$, one might have the two basis 4-vectors

$$\epsilon^\mu(\lambda = \pm 1 = R/L) = \frac{1}{\sqrt{2}}(0, -\lambda, -i, 0)\,, \tag{30}$$

representing the two circularly polarized light or the helicity eigenstates.

D. Covariant derivatives

In order not to affect our counting of degrees of freedom, the interactions in the Lagrangian must respect gauge invariance. Naively,

$$\mathcal{L}_{\text{int}} \sim A_\mu\, \phi \partial_\mu \phi \longrightarrow A_\mu\, \phi \partial_\mu \phi \;+\; (\partial_\mu \alpha)\, \phi \partial_\mu \phi \tag{31}$$

hmm... We must be able to make ϕ transform to compensate for the gauge transformation of A_μ in order to cancel the $\partial_\mu \alpha$ term. In fact, we need at least two real fileds ϕ_1 and ϕ_1 which form a complex field $\phi = \phi_1 + i\phi_2$. Then, under a gauge transformation, it transforms [6]

$$\phi \longrightarrow e^{iQ\alpha(x)} \phi \quad \text{together with} \quad A_\mu \longrightarrow A_\mu \;+\; \frac{1}{e} \partial_\mu \alpha(x) \tag{32}$$

- Mass term $\frac{1}{2} m^2 \phi^* \phi = \frac{1}{2} m^2 |\phi|^2$ is gauge invariant
- Derivative term is not gauge invariant

$$\partial_\mu \phi \longrightarrow e^{iQ\alpha(x)} [\partial_\mu + iQ\, \partial_\mu \alpha(x)] \phi \tag{33}$$

- Hmm...

$$-ieQ A_\mu \phi \longrightarrow -ieQ \left[A_\mu + \frac{1}{e} \partial_\mu \alpha(x) \right] e^{iQ\alpha(x)} \phi = e^{iQ\alpha(x)} \left[-ieQ A_\mu - iQ\, \partial_\mu \alpha(x) \right] \phi \tag{34}$$

- Covariant derivative $D_\mu \phi \equiv (\partial_\mu - ieQA_\mu)\phi$ tansforms like ϕ leading to gauge invariant $|D_\mu \phi|^2$:

$$\begin{aligned} D_\mu \phi &= (\partial_\mu - ieQA_\mu)\phi \longrightarrow e^{iQ\alpha(x)} (\partial_\mu - ieQA_\mu)\phi = e^{iQ\alpha(x)} D_\mu \phi \\ |D_\mu \phi|^2 &= [(\partial_\mu + ieQA_\mu)\, \phi^*]\; [(\partial_\mu - ieQA_\mu)\, \phi] = \partial_\mu \phi^* \partial_\mu \phi + ieQA_\mu\, (\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*) + e^2 Q^2 |\phi|^2 A_\mu A_\mu \end{aligned} \tag{35}$$

E. Ward identity

Let's simply remember that some amplitude $\mathcal{M}_\mu$ which is to be contracted with the polarization 4-vectors to result in Lorentz invariant amplitude of $\epsilon_\mu \mathcal{M}_\mu$, one should have

$$p^\mu \mathcal{M}_\mu = 0 \tag{36}$$

---

[5] Note that the A_0 equation of motion $\partial_j^2 A_0 = 0$ (for $\mu = 0 = t$) is preserved in Coulomb gauge since $\partial_j^2 \alpha = 0$.

[6] The charge of the field ϕ is denoted by Q, $1/e$ appears in the gauge transformation of A_μ, and the combination $-ieQ$ is for the covariant detivative. Note that eQ is the coupling strength between A_μ and the Noether current $J_\mu = i(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*)$ in the free theory ($e = 0$).

which is know as the Ward identity which is guranteed by Lorentz invariance and the fact that unitary representations for massless spin-l particles have two polarizations.

F. Photon progator

Recall that, for the massless spin-1 field, we have [7]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2\,; \quad \Box A_\mu - \partial_\mu(\partial_\nu A_\nu) = 0 \tag{37}$$

which might lead to, in momentum space,

$$(-p^2 g_{\mu\nu} + p_\mu p_\nu)A_\mu = 0 \tag{38}$$

and one may find the photon propagator by inverting $(-p^2 g_{\mu\nu} + p_\mu p_\nu)$:

$$(-p^2 g_{\mu\nu} + p_\mu p_\nu)\Pi_{\nu\alpha} = g_{\mu\alpha} \tag{39}$$

The problem is that $\det(-p^2 g_{\mu\nu} + p_\mu p_\nu) = 0$ and make it non-invertiable which is a manifestation of gauge invariance.

What should we do? gauge fixing? ... $\partial_\mu A_\mu = 0$. OK, then how? A Lagrange multiplier?! :

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2\xi}(\partial_\mu A_\mu)^2\,; \quad \Box A_\mu - \left(1-\frac{1}{\xi}\right)\partial_\mu(\partial_\nu A_\nu) = 0 \tag{40}$$

and we would like to invert

$$\left[-p^2 g_{\mu\nu} + \left(1-\frac{1}{\xi}\right)p_\mu p_\nu\right] \tag{41}$$

One may find

$$\Pi_{\mu\nu} = -\frac{g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}}{p^2} \tag{42}$$

by checking that

$$\begin{aligned}
&\left[p^2 g_{\mu\alpha} - \left(1-\frac{1}{\xi}\right)p_\mu p_\alpha\right]\left[p^2 g_{\alpha\nu} - (1-\xi)p_\alpha p_\nu\right]\frac{1}{p^4} \\
&= g_{\mu\nu} + \left[-(1-1/\xi) - (1-\xi) + (1-1/\xi)(1-\xi)\right]\frac{p_\mu p_\nu}{p^2} = g_{\mu\nu}
\end{aligned} \tag{43}$$

So, the time-ordered Feynman propagator for a photon might be givenn by

$$i\Pi_{\mu\nu}(p) = \frac{-i}{p^2 + i\varepsilon}\left[g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right] \tag{44}$$

in covariant or R_ξ-gauge.

- Feynman-'t Hooft gauge $\xi = 1$: $i\Pi_{\mu\nu}(p) = \frac{-ig_{\mu\nu}}{p^2+i\varepsilon}$: for most calculations
- Lorentz gauge $\xi = 0$ [8]: $i\Pi_{\mu\nu}(p) = -i\frac{g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}}{p^2+i\varepsilon}$: $\partial_\mu A_\mu = 0$ or $p_\mu \Pi_{\mu\nu} = 0$ enforced
- Unitary gauge $\xi \to \infty$: the propagator blows up ... useless for QED but ...

Note that physical results should be independent of ξ.

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[7] See Eq. (15) and below.
[8] We could not set $\xi = 0$ and then invert the kinetic term, but we can invert and then set $\xi = O$.

G. Decomposition of a vector field

Any vector field can be written as [9]

$$A_\mu(x) = A_\mu^T(x) + \partial_\mu \pi(x) \;\; \text{with} \;\; \partial_\mu A_\mu^T = 0. \tag{45}$$

The beauty of this decomposition is that it lets us see whether the non-transverse polarizations are physical or not simply by looking at the Lagrangian: find conditions to remove unphysical terms.

Performing the decomposition in the most general Lorentz-invariant Lagrangin for a vector field A_μ[10]

$$\mathcal{L} = \frac{a}{2} A_\mu \Box A_\mu + \frac{b}{2} A_\mu \partial_\mu \partial_\nu A_\nu + \frac{1}{2} m^2 A_\mu^2 \tag{46}$$

we have [11]

$$\mathcal{L} = \frac{a}{2} A_\mu^T \Box A_\mu^T + \frac{m^2}{2} (A_\mu^T)^2 - \frac{a+b}{2} \pi \Box^2 \pi - \frac{m^2}{2} \pi \Box \pi \tag{47}$$

For π field, in momentum space, we have $-(a+b)p^4 + m^2 p^2$ and, by imverting it, we have the π's propagator which reads

$$\Pi_\pi = \frac{-1}{(a+b)p^4 - m^2 p^2} = \frac{1}{m^2} \left[\frac{1}{p^2} - \frac{(a+b)}{(a+b)p^2 - m^2} \right] \tag{48}$$

Thus, π really represents two fields: one of which has negative norm for generic a and b and therefore represents a ghost with a wrong-sign kinetic term. For $(a+b) \neq 0$, there are ghosts and the theory cannot be unitary. More generally, a kinetic term with more than two derivatives always indicates that a theory is not unitary.

We can remove the dangerous 4-derivative kinetic terms by choosing $a = -b = 1$ and we have finally arrived at

$$\mathcal{L} = \frac{1}{2} A_\mu \Box A_\mu - \frac{1}{2} A_\mu \partial_\mu \partial_\nu A_\nu + \frac{1}{2} m^2 A_\mu^2 = -\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} m^2 A_\mu^2 \tag{49}$$

which is invariant under the gauge transformation $A_\mu \to A_\mu + \partial_\mu \alpha(x)$ when $m^2 = 0$. In this case, we see that the longitudinal modes get a kinetic term from the mass term, as expected.

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[9] This decomposition is invariant under shifts $A_\mu^T \to A_\mu^T + \partial_\mu \alpha$ and $\pi \to \pi - \alpha$ and we can pick α so that the field is in Lorenz gauge where $\partial_\mu A_\mu^T = 0$.

[10] Eq. (15).

[11] For the mass term, $A_\mu^2 = (A_\mu^T + \partial_\mu \pi)(A_\mu^T + \partial_\mu \pi) = (A_\mu^T)^2 + (\partial_\mu \pi)(\partial_\mu \pi) + A_\mu^T(\partial_\mu \pi) + (\partial_\mu \pi) A_\mu^T = (A_\mu^T)^2 - \pi \Box \pi + \text{t.d.}$.

III. YANG-MILLS THEORY [1]

In this section, we follow early sections of Chapter 25 of Ref. [1].

- QED is to embed a massles spin-1 particle, whose irreducible representation of the Poincare group has two degrees of freedom, in a vector field $A_\mu(x)$, which has four degrees of freedom. The two extra degrees of freedom are removed in quantum field theory through gauge invariance under the transformation

$$A_\mu(x) \longrightarrow A_\mu(x) \; + \; \frac{1}{e}\partial_\mu \alpha(x) \tag{50}$$

resulting in a gauge-invariant kinetic Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{51}$$

To have the photon interact with matter, we replace ∂_μ in the matter kinetic term with the covartiant derivative D_μ:

$$D_\mu = \partial_\mu - ieQA_\mu \tag{52}$$

which gives, for example,

$$\mathcal{L} = \overline{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi \tag{53}$$

which is invariant under the gauge transformations given by Eq. (50) and

$$\psi \longrightarrow e^{iQ\alpha(x)}\psi \tag{54}$$

- Yang-Mills theories are the unique generalizations of QED in which Lagrangians are constrained by non-Abelian gauge invariance having renormalizable self-interactions among massless spin-l particles

A. SU(2)

- A global SU(2) symmetry: Consider two complex fields ϕ_1 and ϕ_2. Then we might have the following kinetic Lagrangian for them:

$$\mathcal{L}_{\rm kin} = \left(\partial_\mu\phi_1^*\right)\left(\partial_\mu\phi_1\right) + \left(\partial_\mu\phi_2^*\right)\left(\partial_\mu\phi_2\right) = \left(\partial_\mu\Phi\right)^\dagger\left(\partial_\mu\Phi\right) \tag{55}$$

with

$$\Phi \equiv \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{56}$$

Note that $\mathcal{L}_{\rm kin}$ is invariant under a global SU(2) symmetry

$$\Phi \longrightarrow e^{i(\alpha^1 t^1 + \alpha^2 t^2 + \alpha^3 t^3)}\,\Phi = e^{i\alpha^a t^a}\,\Phi \tag{57}$$

where $\alpha^{1,2,3}$ are real numbers and $t^a = \sigma^a/2$ with σ^a begin the Pauli matrices, [12]

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{58}$$

The normazlization of the t matrices is chosen so that

$$\left[t^a\,,t^b\right] \;=\; i\epsilon^{abc}t^c \tag{59}$$

where ϵ^{abc} is the Levi-Civita tensor.

[12] Note that $(\sigma^i)^\dagger = \sigma^i$.

- We can promote the global SU(2) symmetry to a local symmetry by elevating the real numbers α^a to real functions of space-time $\alpha^a(x)$. Then one can make $\mathcal{L}_{\rm kin}$ gauge invariant by replacing $\partial_\mu \Phi$ with the covariant derivative

$$D_\mu \Phi = \left(\partial_\mu - igA^a_\mu t^a\right)\,\Phi\,. \tag{60}$$

Then the unique gauge-invariant kinetic term for the spin-1 fields is given by

$$\mathcal{L}_{\rm YM} = -\frac{1}{4}\sum_a \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu\right)^2 \equiv -\frac{1}{4}\sum_a (F^a_{\mu\nu})^2 \tag{61}$$

One should check that $\mathcal{L}_{\rm kin}$ with $\partial_\mu \to D_\mu$ and $\mathcal{L}_{\rm YM}$ are invariant under the SU(2) gauge transformations

$$\begin{aligned} \Phi &\longrightarrow e^{i\alpha^a(x)t^a}\,\Phi\,, \\ A^a_\mu(x) &\longrightarrow A^a_\mu(x) + \frac{1}{g}\partial_\mu\alpha^a(x) - \epsilon^{abc}\alpha^b(x)A^c_\mu(x) \end{aligned} \tag{62}$$

HW#1: Derive the infinitesimal SU(2) gauge transformation rule of A^a_μ given by the above equation and show that, under which, $\mathcal{L}_{\rm YM}$ is invariant.

Hint: one may consider infinitesimal transformation conveniently. From $D_\mu\Phi \to (D_\mu\Phi)' = e^{i\alpha^a t^a} D_\mu\Phi$, one have

$$\left(\partial_\mu - igA'^a_\mu\tau^a\right)\left(1 + i\alpha^b t^b\right)\Phi = \left(1 + i\alpha^b t^b\right)\left(\partial_\mu - igA^a_\mu t^a\right)\Phi$$

which is solved up to the first order of α to give

$$A'^a_\mu(x) = A^a_\mu(x) + \frac{1}{g}\partial_\mu\alpha^a(x) - \epsilon^{abc}\alpha^b(x)A^c_\mu(x)$$

Then, under the infinitesimal transformation, you might be able to show

$$F^a_{\mu\nu} \longrightarrow F^a_{\mu\nu} - \epsilon^{abc}\alpha^b F^c_{\mu\nu}$$

which might make $\mathcal{L}_{\rm YM}$ invariant under $A^a_\mu \to A'^a_\mu$. One might need the identity $\epsilon^{abc}\epsilon^{ade} = \delta^{bd}\delta^{ce} - \delta^{be}\delta^{cd}$... or not.

B. Gauge invariance and Wilson lines: Ablelian case

Consider a complex field $\phi(x)$ with $Q = 1$. Then, how can we tell if $\phi(x) = \phi(y)$? The difference, under the local gauge transformation, transforoms as

$$\phi(y) - \phi(x) \longrightarrow e^{i\alpha(y)}\phi(y) - e^{i\alpha(x)}\phi(x) \tag{63}$$

which makes, for example, $|\phi(y) - \phi(x)|$ depend on our choice of local phases. How one can have well-defined comparisons of field values at difference points or a well-defined derivative? The answer is to introduce a new bi-local field $W(x, y)$ called a Wilson line which transforms as

$$W(x, y) \longrightarrow e^{i\alpha(x)}W(x, y)e^{-i\alpha(y)} \tag{64}$$

- The absolute value of the quantity $W(x, y)\phi(y) - \phi(x)$ is independent of our choice of local phases:

$$\begin{aligned} W(x, y)\phi(y) - \phi(x) &\longrightarrow e^{i\alpha(x)}W(x, y)e^{-i\alpha(y)}\,e^{i\alpha(y)}\phi(y) - e^{i\alpha(x)}\phi(x) \\ &= e^{i\alpha(x)}\left[W(x, y)\phi(y) - \phi(x)\right] \end{aligned} \tag{65}$$

- The covariant derivative by use of the quantiy $W(x, y)\phi(y) - \phi(x)$ and the gauge field as a **connection** allowing us to compare field values at different points: [13]

$$D_\mu\phi \equiv \lim_{\delta x^\mu \to 0} \frac{W(x, x+\delta x)\phi(x+\delta x) - \phi(x)}{\delta x^\mu}$$
$$W(x, x+\delta x) \equiv 1 - ie(\delta x)^\mu A_\mu(x) + \mathcal{O}(\delta x^2) \tag{66}$$

Then, from $W(x,y) \longrightarrow e^{i\alpha(x)}W(x,y)e^{-i\alpha(y)}$ and the definition of $D_\mu\phi$, we can obtain the gauge transformation rule for the vector field and the explict form for the covariant derivative !!!:

$$A_\mu(x) \longrightarrow A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x)\,; \quad D_\mu\phi = \partial_\mu\phi - ieA_\mu\phi \tag{67}$$

Proofs:

$$\begin{aligned} 1 - ieA_\mu\delta x^\mu \longrightarrow 1 - ieA'_\mu\delta x^\mu &= e^{i\alpha(x)}(1 - ieA_\mu\delta x^\mu)e^{-i\alpha(x+\delta x)} \\ &= (1 - ieA_\mu\delta x^\mu)(1 - i\partial_\mu\alpha\delta x^\mu) \\ &\simeq 1 - ie\delta x^\mu\,(A_\mu + \partial_\mu\alpha/e) \end{aligned} \tag{68}$$

$$\begin{aligned} D_\mu\phi &= \lim_{\delta x^\mu \to 0} \frac{(1 - ieA_\nu\delta x^\nu)\phi(x+\delta x) - \phi(x)}{\delta x^\mu} \\ &\simeq \lim_{\delta x^\mu \to 0} \frac{\phi(x+\delta x) - \phi(x)}{\delta x^\mu} - ieA_\mu\phi(x) = (\partial_\mu - ieA_\mu)\phi(x) \end{aligned} \tag{69}$$

- A closed-form expression for the Wilson line $W(x,y)$ using the path integral of the vector field from y to x

$$W_P(x,y) = \exp\left(ie\int_y^x A_\mu(z)dz^\mu\right) \tag{70}$$

One can see that $W_P(x,x) = 1$ and, using $A_\mu \to A_\mu + \partial_\mu\alpha/e$, it indeed satisfy the supposed transformation property

$$W_P(x,y) \longrightarrow W_P(x,y) = \exp\left(ie\int_y^x \left[A_\mu(z) + \partial_\mu\alpha(z)/e\right]dz^\mu\right)A = e^{i\alpha(x)}W_P(x,y)e^{-i\alpha(y)} \tag{71}$$

- A Wilson loop: if we set $x = y$, we get

$$W_P^{\rm loop} = \exp\left(ie\oint_P A_\mu(x)dx^\mu\right) = \exp\left(i\frac{e}{2}\int_\Sigma F_{\mu\nu}d\sigma^{\mu\nu}\right) \tag{72}$$

where, in the second step, we apply Stokes' therem over the surface Σ with surface element $\sigma^{\mu\nu}$ which the closed-path P bounds. One can see that the Wilson loop is gauge invariant by observing that it dependes only on $F_{\mu\nu}$.

- The Maxwell tensor $F_{\mu\nu}$ from a commutator of covariant derivatives [14]

$$[D_\mu\,, D_\nu]\,\phi(x) = ([\partial_\mu\,, \partial_\nu] - ie[\partial_\mu\,, A_\nu] + ie[\partial_\nu\,, A_\mu])\,\phi(x) = -ieF_{\mu\nu}\phi(x) \tag{73}$$

The commutator $[D_\mu\,, D_\nu]$ is not an operator and the field strength for QED can be defined as

$$F_{\mu\nu} \equiv \frac{i}{e}[D_\mu\,, D_\nu] \tag{74}$$

whcih is the result of comparing field values around an infinitesimal closed loop in the $\mu - \nu$ plane.

---

[13] Note that $W(x,x) = 1$.
[14] $[\partial_\mu\,, A_\nu]\phi = \partial_\mu(A_\nu\phi) - A_\nu(\partial_\mu\phi) = (\partial_\mu A_\nu)\phi$.

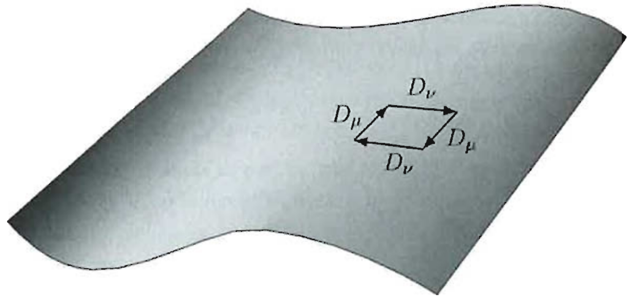

FIG. 1. $F_{\mu\nu}$ from a commutator of covariant derivatives: $F_{\mu\nu} \equiv \frac{i}{e}[D_\mu\,, D_\nu]$

C. SU(N)

Consider the kinetic Lagrangian with N Dirac fermions

$$\mathcal{L} = \sum_{j=1}^{N} \overline{\psi}_j (i\partial\!\!\!/ - m)\psi_j = \overline{\Psi}(i\partial\!\!\!/ - m)\Psi \tag{75}$$

with $\Psi = (\psi_1, \psi_2, \cdots, \psi_N)^T$. This kinetic term is invariant under a global SU(N) symmetry

$$\Psi \longrightarrow \left(e^{i\alpha^a T^a}\right)\Psi = U\Psi \tag{76}$$

where α^a and, accordingly, U do not depend on x. T^a's are the SU(N) generators in the fundamental representation satisfying

$$[T^a\,, T^b] \;=\; if^{abc}\,T^c \tag{77}$$

Note that

$$(T^a)^\dagger = T^a \;\;\text{and}\;\; U^\dagger = U^{-1} \tag{78}$$

One can promote the global SU(N) symmetry to the local one by taking space-time dependent $\alpha^a(x)$. Then we have

$$U(x) \;=\; e^{i\alpha^a(x)T^a} \tag{79}$$

The non-Abelian vector fields, the convariant derivative, the gauge transformation rule of the vector fields, and the gauge-invariant field strength could be obtained by:

- Wilson line and a Lie-algebra-valued field $\mathbf{A}_\mu = A^a_\mu T^a$

$$W_P(x,y) = P\left\{\exp\left(ig\int_y^x A^a_\mu(z)T^a dz^\mu\right)\right\} \equiv P\left\{\exp\left(ig\int_y^x \mathbf{A}_\mu(z)dz^\mu\right)\right\} \tag{80}$$

  with $P\{\cdots\}$ denoting a path-ordering operator

- Covariant derivative: The infinitesimal expansion of the Wilson line is

$$W_P(x, x+\delta x) = \mathbb{1} - ig\mathbf{A}_\mu \delta x^\mu \tag{81}$$

  and we have

$$D_\mu\Psi = \lim_{\delta x^\mu \to 0} \frac{(\mathbb{1} - ig\mathbf{A}_\nu\delta x^\nu)\Psi(x+\delta x) - \Psi(x)}{\delta x^\mu} = (\partial_\mu - ig\mathbf{A}_\mu)\Psi(x) \tag{82}$$

- How A^a_μ transform?

$$D_\mu\Psi \longrightarrow (D_\mu\Psi)' = (\partial_\mu - ig\mathbf{A}'_\mu)U\Psi = U(\partial_\mu - ig\mathbf{A}_\mu)\Psi$$
$$\partial_\mu U - ig\mathbf{A}'_\mu U = -igU\mathbf{A}_\mu$$
$$\mathbf{A}_\mu \longrightarrow \mathbf{A}'_\mu = U\mathbf{A}_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger \tag{83}$$

Using $[T^a\,, T^b] = if^{abc}T^c$, one might have the infinitesimal version

$$A^a_\mu \longrightarrow A^a_\mu + \frac{1}{g}\partial_\mu\alpha^a(x) - f^{abc}\alpha^b(x)A^c_\mu \tag{84}$$

- Field strength

$$\mathbf{F}_{\mu\nu} = F^a_{\mu\nu}T^a = \frac{i}{g}[D_\mu\,, D_\nu] = \frac{i}{g}\Big\{ -ig\,(\partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu) - g^2[\mathbf{A}_\mu\,, \mathbf{A}_\nu]\Big\} = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu - ig[\mathbf{A}_\mu\,, \mathbf{A}_\nu]$$
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{bca}A^b_\mu A^c_\nu = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \tag{85}$$

Last but not least, note some identities for SU(N) which are used in almost every QCD ($N = 3$) calculation:

$$\mathrm{tr}\left(T^aT^b\right) = T_F\delta^{ab}$$
$$\sum_a (T^aT^a)_{ij} = C_F\delta_{ij}$$
$$f^{acd}f^{bcd} = C_A\delta^{ab} \tag{86}$$

with $T_F = 1/2$, $C_A = N$, and $C_F = \frac{N^2-1}{2N}$

HW#2: Show Eq. (84) and the second line of Eq. (85).

IV. SM HIGGS BOSON

The SM gauge structure is SU(3)$_c$×SU(2)$_L$×U(1)$_Y$. The corresponding gauge transformations can be written as follows (for the SU(2)$_L$ and SU(3)$_c$ gauge field transformations, we give only the infinitesimal form):

$$\begin{aligned}
\mathrm{U}(1)_Y: &\quad \psi \to \exp[i\lambda_Y(x)Y]\psi, &\quad B_\mu \to B_\mu + \frac{1}{g'}\partial_\mu \lambda_Y(x) \\
\mathrm{SU}(2)_L: &\quad \psi \to \exp[i\lambda_L^a(x)T^a]\psi, &\quad W_\mu^a \to W_\mu^a + \frac{1}{g}\partial_\mu \lambda_L^a(x) + \epsilon^{abc}W_\mu^b\lambda_L^c(x) \\
\mathrm{SU}(3)_c: &\quad \psi \to \exp[i\lambda_c^a(x)t^a]\psi, &\quad G_\mu^a \to G_\mu^a + \frac{1}{g_s}\partial_\mu \lambda_c^a(x) + f^{abc}G_\mu^b\lambda_c^c(x)
\end{aligned} \tag{87}$$

TABLE II. The SM fermions and Higgs. Note that $Q = T_3 + Y$.

	$Q_L \equiv \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	u_R	d_R	$L_L \equiv \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	e_R	$\Phi \equiv \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$
Hypercharge Y	1/6	2/3	−1/3	−1/2	−1	1/2
Color	triplet	triplet	triplet	singlet	singlet	singlet

The gauge interactions of fermions or scalars are encoded in the covariant derivative,

$$\mathcal{D}_\mu = \partial_\mu - ig'B_\mu Y - igW_\mu^a T^a - ig_s G_\mu^a t^a, \tag{88}$$

where g' is the coupling strength of the hypercharge interaction, Y is the hypercharge operator, and T^a and t^a are the SU(2) and SU(3) generators, respectively. When acting upon a doublet representation of SU(2), T^a is just $\sigma^a/2$ where σ^a are the Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{89}$$

A. The Higgs mechanism in the Standard Model [2]

This subsection IV.A is just a copy of Section 2 of Ref. [2]

1. *Preliminaries: gauge sector*

Let's start with a review of the gauge and fermion parts of the SM Lagrangian. The SM gauge structure is SU(3)$_c$×SU(2)$_L$×U(1)$_Y$, comprising respectively the strong interactions (subscript c for color), weak isospin (subscript L for the left-handed fermions it couples to), and hypercharge (subscript Y for the hypercharge operator). The gauge boson dynamics are encoded in the Lagrangian in terms of the field strength tensors:[15]

$$\mathcal{L}_{gauge} = -\frac{1}{4}G_{\mu\nu}^a G^{a\mu\nu} - \frac{1}{4}W_{\mu\nu}^a W^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}, \tag{90}$$

where repeated indices are always taken as summed. Here the field strength tensors are given as follows. For the U(1)$_Y$ interaction, the field strength tensor takes the same form as in electromagnetism,

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \tag{91}$$

For SU(3)$_c$, and non-abelian theories in general, the field strength tensor takes a more complicated form,

$$G_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a + g_s f^{abc} G_\mu^b G_\nu^c, \tag{92}$$

[15] I use the metric $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, so that $p^2 \equiv p_\mu p^\mu = m^2$ for an on-shell particle.

	$Q_L \equiv \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	u_R	d_R	$L_L \equiv \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	e_R
Hypercharge	1/6	2/3	−1/3	−1/2	−1
Color	triplet	triplet	triplet	singlet	singlet

TABLE III. The chiral fermion content of a single generation of the Standard Model.

where g_s is the strong interaction coupling strength, a, b, c run from 1 to 8, and f^{abc} are the (antisymmetric) structure constants of SU(3), defined in terms of the group generators t^a according to

$$[t^a, t^b] = i f^{abc} t^c. \tag{93}$$

For SU(2), a, b, c run from 1 to 3 and $f^{abc} = \epsilon^{abc}$, the totally antisymmetric three-index tensor defined so that $\epsilon^{123} = 1$. Therefore, the field strength tensor for $\mathrm{SU}(2)_L$ can be written as

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\epsilon^{abc} W^b_\mu W^c_\nu, \tag{94}$$

where g is the weak interaction coupling strength.

The gauge interactions of fermions or scalars are encoded in the covariant derivative,

$$\mathcal{D}_\mu = \partial_\mu - ig' B_\mu Y - igW^a_\mu T^a - ig_s G^a_\mu t^a, \tag{95}$$

where g' is the coupling strength of the hypercharge interaction, Y is the hypercharge operator, and T^a and t^a are the SU(2) and SU(3) generators, respectively. When acting upon a doublet representation of SU(2), T^a is just $\sigma^a/2$ where σ^a are the Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{96}$$

The corresponding gauge transformations can be written as follows (for the $\mathrm{SU}(2)_L$ and $\mathrm{SU}(3)_c$ gauge field transformations, we give only the infinitesimal form):

$$\begin{aligned}
\mathrm{U}(1)_Y: &\quad \psi \to \exp[i\lambda_Y(x)Y]\psi, &\quad B_\mu &\to B_\mu + \frac{1}{g'}\partial_\mu \lambda_Y(x) \\
\mathrm{SU}(2)_L: &\quad \psi \to \exp[i\lambda^a_L(x)T^a]\psi, &\quad W^a_\mu &\to W^a_\mu + \frac{1}{g}\partial_\mu \lambda^a_L(x) + \epsilon^{abc} W^b_\mu \lambda^c_L(x) \\
\mathrm{SU}(3)_c: &\quad \psi \to \exp[i\lambda^a_c(x)t^a]\psi, &\quad G^a_\mu &\to G^a_\mu + \frac{1}{g_s}\partial_\mu \lambda^a_c(x) + f^{abc} G^b_\mu \lambda^c_c(x).
\end{aligned} \tag{97}$$

A mass term for a gauge boson would take the form

$$\mathcal{L} \supset \frac{1}{2} m_B^2 B_\mu B^\mu. \tag{98}$$

This is not gauge invariant and thus cannot be inserted by hand into the Lagrangian. Therefore, (unbroken) gauge invariance implies that gauge bosons are all massless.

2. *Preliminaries: fermion sector*

The SM contains three copies (generations) of a collection of chiral fermion fields with different gauge transformation properties under $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$. The content of a single generation is given in Table III, along with their hypercharge assignments[16] (the value of the quantum number Y) and their $\mathrm{SU}(3)_c$ (color) transformation properties. The fields Q_L and L_L transform as doublets under $\mathrm{SU}(2)_L$, while the remaining fields transform as singlets.

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[16] A careful observer will notice that the electric charge of each field is given by $Q = T^3 + Y$. We will derive this relationship in Sec. IV A 3.

The left- and right-handed chiral fermion states are obtained from an unpolarized Dirac spinor using the projection operators

$$P_R = \frac{1}{2}(1+\gamma^5), \qquad P_L = \frac{1}{2}(1-\gamma^5), \tag{99}$$

in such a way that

$$P_R\psi \equiv \psi_R, \qquad P_L\psi \equiv \psi_L. \tag{100}$$

Using the anticommutation relations $\{\gamma^\mu, \gamma^5\} = 0$ and the fact that γ^5 is Hermitian, we also have

$$\bar{\psi}P_R = \psi^\dagger\gamma^0 P_R = \psi^\dagger P_L\gamma^0 = (P_L\psi)^\dagger\gamma^0 = \bar{\psi}_L, \tag{101}$$

and similarly $\bar{\psi}P_L = \bar{\psi}_R$. Finally, the projection operators obey $P_R + P_L = 1$ and $P_R^2 = P_R$, $P_L^2 = P_L$.

We can use this to rewrite the Dirac Lagrangian in terms of chiral fermion fields as follows. We start with the Lagrangian for a generic fermion ψ with mass m,

$$\mathcal{L} = \bar{\psi}i\partial_\mu\gamma^\mu\psi - m\bar{\psi}\psi. \tag{102}$$

The first term can be split into two terms involving left- and right-handed chiral fermion fields by inserting a factor of $1 = (P_L^2 + P_R^2)$ before the ψ and using the anticommutation relation to pull one factor of the projection operator through the γ^μ in each term:

$$\bar{\psi}i\partial_\mu\gamma^\mu\psi = \bar{\psi}P_R i\partial_\mu\gamma^\mu P_L\psi + \bar{\psi}P_L i\partial_\mu\gamma^\mu P_R\psi = \bar{\psi}_L i\partial_\mu\gamma^\mu\psi_L + \bar{\psi}_R i\partial_\mu\gamma^\mu\psi_R. \tag{103}$$

The kinetic term separates neatly into one term involving only ψ_L and one involving only ψ_R. We can then incorporate the gauge transformation properties by promoting the derivative ∂_μ to a covariant derivative $\mathcal{D}_\mu$ and these two terms will be gauge invariant for any of the fermion fields given in Table III.

Now let's consider the mass term. Using the same tricks, we have,

$$-m\bar{\psi}\psi = -m\bar{\psi}P_L^2\psi - m\bar{\psi}P_R^2\psi = -m\bar{\psi}_R\psi_L - m\bar{\psi}_L\psi_R. \tag{104}$$

(Note that the second term is just the Hermitian conjugate of the first term.) The mass terms each involve fermions of both chiralities. Because the left-handed and right-handed fermions of the SM carry different $\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y$ gauge charges, such mass terms are not gauge invariant and thus cannot be inserted by hand into the Lagrangian. Therefore, given the gauge charges of the SM fermions, (unbroken) gauge invariance implies that all the SM fermions are massless. [17]

3. *The SM Higgs mechanism*

We have established that the theoretical explanation of the experimentally-observed nonzero masses of the W and Z bosons and the SM fermions requires a new ingredient. Such an explanation is achieved by introducing a single $\mathrm{SU}(2)_L$-doublet scalar field, which causes spontaneous breaking of the $\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y$ gauge symmetry via the Higgs mechanism.

We add to the SM a field Φ, an $\mathrm{SU}(2)_L$-doublet of complex scalar fields that can be written as

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}, \tag{105}$$

where $\phi_1, \phi_2, \phi_3, \phi_4$ are properly normalized real scalar fields. We assign Φ a hypercharge $Y = 1/2$ and make it a color singlet. The new terms in the Lagrangian involving Φ are given by [18]

$$\mathcal{L}_\Phi = (\mathcal{D}_\mu\Phi)^\dagger(\mathcal{D}^\mu\Phi) - V(\Phi) + \mathcal{L}_{\mathrm{Yukawa}}, \tag{106}$$

where the first term contains the kinetic and gauge-interaction terms via the covariant derivative, the second term is

[17] Some models beyond the SM contain left- and right-handed chiral fermions that carry the same $\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y$ gauge charges, and can thus form a massive Dirac fermion without any reference to electroweak symmetry breaking. Such fermions are called vectorlike fermions, because of their pure vector (as opposed to axial-vector) couplings to the Z boson.

[18] $(\mathcal{D}_\mu\Phi)^\dagger(\mathcal{D}^\mu\Phi) \to$ **Sec. IV.A.4**: $\mathcal{L}_{\mathrm{Yukawa}} \to$ **Sec. IV.A.5**: $V(\Phi) \to$ **Sec. IV.A.6**.

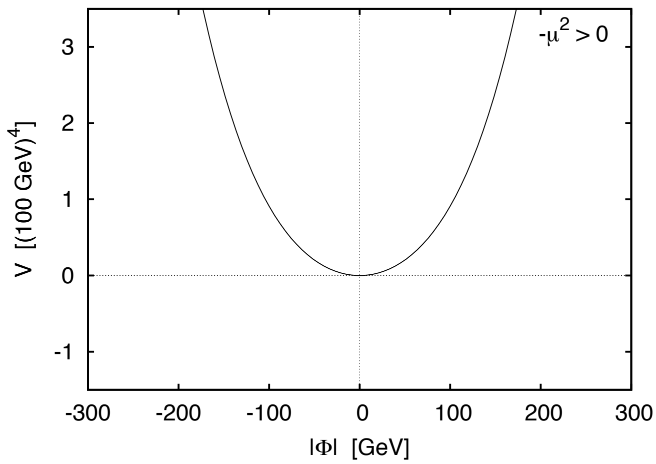


FIG. 2. Plots of $V(\Phi) = -\mu^2\Phi^\dagger\Phi + \lambda(\Phi^\dagger\Phi)^2$ as a function of $|\Phi| \equiv \sqrt{\Phi^\dagger\Phi}$ for the cases $-\mu^2 > 0$ (left) and $-\mu^2 < 0$ (right). For the SM parameters I used $|-\mu^2| \simeq (88.4 \text{ GeV})^2$ and $\lambda \simeq 0.129$, obtained from the measured values $m_h \simeq 125$ GeV and $v \simeq 246$ GeV. In the case that $-\mu^2 < 0$ (right), the minimum of the potential is at $|\Phi| = v/\sqrt{2} = (246/\sqrt{2})$ GeV.

a potential energy function involving Φ, and the third term contains Yukawa couplings of the scalar field to pairs of fermions. We will treat each term in turn, starting with the potential energy function.

The most general gauge invariant potential energy function, or scalar potential, involving Φ is given by

$$V(\Phi) = -\mu^2\Phi^\dagger\Phi + \lambda(\Phi^\dagger\Phi)^2. \tag{107}$$

Consider the possible signs of the coefficients of the two terms in V:

- If λ is negative, then V is unbounded from below and there is no stable vacuum state.
- When $-\mu^2$ and λ are both positive, the potential energy function has a minimum at $|\Phi| \equiv \sqrt{\Phi^\dagger\Phi} = 0$ (left panel of Fig. 2). In this case the electroweak symmetry is unbroken in the vacuum, because a gauge transformation acting on the vacuum state $\Phi = 0$ does not change the vacuum state.
- When $-\mu^2$ is negative and λ is positive, the potential energy function has a minimum away from $|\Phi| = 0$ (right panel of Fig. 2). In this case the vacuum, or minimum energy state, is not invariant under $SU(2)_L \times U(1)_Y$ transformations: the gauge symmetry is spontaneously broken in the vacuum.

Let's take a closer look at the symmetry-breaking case. The Higgs field Φ is a complex scalar field with two isospin components; we can thus write it in terms of four real scalar degrees of freedom as in Eq. (105), where the $1/\sqrt{2}$ normalization ensures that the kinetic energy terms for the real scalars will have the correct normalization, $\mathcal{L} \supset \frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i$. Then

$$\Phi^\dagger\Phi = \frac{1}{2}\left(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2\right), \tag{108}$$

which can be thought of as the square of the length of a four-component vector. Minimizing the potential in Eq. (107) fixes the length of this vector to satisfy

$$\Phi^\dagger\Phi = \frac{\mu^2}{2\lambda}, \tag{109}$$

which is a positive quantity when $-\mu^2$ is negative. This picks out a spherical surface in four dimensions upon which the potential is minimized.[19]

In this language, $SU(2)_L \times U(1)_Y$ gauge transformations correspond to rotations in this four-dimensional space.[20] Under such rotations V is invariant—the value of the potential depends only on the distance from the origin— but a particular

[19] For the topologically inclined, the vacuum manifold is S^3.

[20] Note that there are four independent $SU(2)_L \times U(1)_Y$ gauge transformations in Eq. (97) but only three independent rotation directions for a vector in a four-dimensional space. In fact, there is always one combination of the $SU(2)_L$ and $U(1)_Y$ transformations that leaves the vacuum state invariant. This particular combination of gauge transformations will remain unbroken by the Higgs field and corresponds to the gauge transformation of electromagnetism.

vacuum state (a particular vector of length $\sqrt{\mu^2/2\lambda}$) transforms nontrivially: it is rotated into a new vector of the same length but pointing in a different direction.

We also acquire a physical picture for excitations around such a vacuum state. Excitations in any of the three rotational directions cost zero energy, because the potential is flat in those directions. These correspond to massless modes or Goldstone modes. An excitation in the radial direction, on the other hand, feels an approximate harmonic oscillator potential about the minimum and gives rise to a massive particle.

Let's see how this works explicitly. The potential is given in terms of the four real scalars by

$$V = -\frac{\mu^2}{2}\left(\phi_1^2+\phi_2^2+\phi_3^2+\phi_4^2\right)+\frac{\lambda}{4}\left(\phi_1^2+\phi_2^2+\phi_3^2+\phi_4^2\right)^2. \tag{110}$$

We are free to choose the basis of states $\phi_1, \cdots, \phi_4$ to be oriented however we like relative to the local vacuum value; let's choose the vacuum expectation values ("vevs") of the four fields to be

$$\langle\phi_3\rangle \equiv v = \sqrt{\frac{\mu^2}{\lambda}}, \qquad \langle\phi_1\rangle = \langle\phi_2\rangle = \langle\phi_4\rangle = 0. \tag{111}$$

We can also define a new real scalar field h with zero vacuum value, $\langle h\rangle = 0$, according to

$$\phi_3 = h + v. \tag{112}$$

Then our field becomes

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + i\phi_2\\ v+h+i\phi_4\end{pmatrix}, \tag{113}$$

and the potential becomes

$$V = -\frac{\mu^2}{2}\left(\phi_1^2+\phi_2^2+(h+v)^2+\phi_4^2\right)+\frac{\lambda}{4}\left(\phi_1^2+\phi_2^2+(h+v)^2+\phi_4^2\right)^2. \tag{114}$$

In particular, we have expressed the potential entirely in terms of constants and fields with zero vacuum value. This lets us treat the fields in terms of small excitations as usual in quantum field theory. Multiplying out the terms in V and using $\mu^2 = \lambda v^2$ to eliminate μ^2, we find [21]

$$V = \text{constant} + 0\cdot\phi_1^2 + 0\cdot\phi_2^2 + \lambda v^2 h^2 + 0\cdot\phi_4^2 + \text{cubic} + \text{quartic}. \tag{115}$$

These quadratic terms are the mass terms for the real scalars. We see that ϕ_1, ϕ_2, and ϕ_4 are massless in accordance with our intuitive picture above, while h has a mass $m_h = \sqrt{2\lambda v^2}$.[22]

To learn more about the nature of the massless modes, we can rewrite Φ in another convenient form,

$$\Phi = \frac{1}{\sqrt{2}}\exp\left(\frac{i\xi^a\sigma^a}{v}\right)\begin{pmatrix}0\\ v+h\end{pmatrix}. \tag{116}$$

Here h and ξ^a are fields, σ^a are the Pauli matrices as in Eq. (96), and a is summed over $1,2,3$. This expression is equivalent to Eq. (113) up to linear order in the fields, i.e., for infinitesimal fluctuations about the vacuum. [23]

HW#3: Show that $\xi^1 = \phi_2$, $\xi^2 = \phi_1$, and $\xi^3 = -\phi_4$ to linear order in the fields to make the above expression equivalent to Eq. (113.

Now consider the gauge transformations of Φ:

$$\begin{aligned}\mathrm{U}(1)_Y: &\quad \Phi \to \exp\left(i\lambda_Y(x)\cdot\frac{1}{2}\right)\Phi,\\ \mathrm{SU}(2)_L: &\quad \Phi \to \exp\left(i\lambda_L^a(x)\frac{\sigma^a}{2}\right)\Phi.\end{aligned} \tag{117}$$

---

[21] $0 = -\mu^2/2 + \lambda/4\,[2v^2]$ and $\lambda v^2 = -\mu^2/2 + \lambda/4\,[2v^2 + (2v)^2]$

[22] Recall that for a real scalar ϕ with mass m, $V \supset \frac{1}{2}m^2\phi^2$.

[23] To linear order, $\xi^1 = \phi_2$, $\xi^2 = \phi_1$, and $\xi^3 = -\phi_4$.

If we choose $\lambda_L^a(x) = -2\xi^a/v$ at each point in spacetime, we arrive at a gauge in which

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v+h \end{pmatrix}, \tag{118}$$

i.e., we have gauged away the fields ξ^a, or equivalently ϕ_1, ϕ_2, ϕ_4.[24] These fields have been entirely removed from the Lagrangian by means of a gauge transformation![25] This means that it must be possible to interpret the theory in a way in which these fields are absent (but with the gauge fixed): they are not physical degrees of freedom. This gauge choice is known as unitary or unitarity gauge. The massive field h remains present and always shows up in the combination $(v+h)$.

4. Gauge boson masses and couplings to the Higgs boson

We now examine the gauge-kinetic term,

$$\mathcal{L} \supset (\mathcal{D}_\mu \Phi)^\dagger (\mathcal{D}^\mu \Phi). \tag{119}$$

When acting on Φ, the covariant derivative reads

$$\mathcal{D}_\mu = \partial_\mu - i\frac{g'}{2} B_\mu - i\frac{g}{2} W_\mu^a \sigma^a. \tag{120}$$

Applying this to Φ in the unitarity gauge we find

$$\mathcal{D}_\mu \Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} -\frac{i}{2} g (W_\mu^1 - i W_\mu^2)(v+h) \\ \partial_\mu h + \frac{i}{2}(g W_\mu^3 - g' B_\mu)(v+h) \end{pmatrix}. \tag{121}$$

Dotting this into its Hermitian conjugate gives,

$$(\mathcal{D}_\mu \Phi)^\dagger (\mathcal{D}^\mu \Phi) = \frac{1}{2}(\partial_\mu h)(\partial^\mu h) + \frac{1}{8} g^2 (v+h)^2 (W_\mu^1 - i W_\mu^2)(W^{1\mu} + i W^{2\mu}) + \frac{1}{8}(v+h)^2 \left(-g' B_\mu + g W_\mu^3\right)^2. \tag{122}$$

HW#4: Derive the above equation.

Let us consider the three terms in turn. The first is the properly normalized kinetic term for the real scalar field h (the Higgs boson). For the second term, we note that the combinations $W^1 \pm i W^2$ correspond to the charged W bosons.[26]

$$\begin{aligned} \frac{W_\mu^1 - i W_\mu^2}{\sqrt{2}} \left(\bar{u}\ \bar{d}\right) \sigma^+ \gamma^\mu P_L \begin{pmatrix} u \\ d \end{pmatrix} &= \frac{W_\mu^1 - i W_\mu^2}{\sqrt{2}} \bar{u} \gamma^\mu P_L d \quad \Rightarrow \quad \frac{W_\mu^1 - i W_\mu^2}{\sqrt{2}} = W_\mu^+, \\ \frac{W_\mu^1 + i W_\mu^2}{\sqrt{2}} \left(\bar{u}\ \bar{d}\right) \sigma^- \gamma^\mu P_L \begin{pmatrix} u \\ d \end{pmatrix} &= \frac{W_\mu^1 + i W_\mu^2}{\sqrt{2}} \bar{d} \gamma^\mu P_L u \quad \Rightarrow \quad \frac{W_\mu^1 + i W_\mu^2}{\sqrt{2}} = W_\mu^-. \end{aligned} \tag{126}$$

$$W_\mu^+ = \frac{W_\mu^1 - i W_\mu^2}{\sqrt{2}}, \qquad W_\mu^- = \frac{W_\mu^1 + i W_\mu^2}{\sqrt{2}}. \tag{127}$$

---

[24] Note that we could have gauged away ξ^3 by doing an appropriate combination of $\mathrm{SU}(2)_L$ and $\mathrm{U}(1)_Y$ gauge transformations.

[25] This removal of the Goldstone modes by means of a gauge transformation is sometimes described as the Goldstones being "eaten" by the corresponding gauge bosons.

[26] Which combination corresponds to W^+ and which to W^-? This can be checked by noting that

$$\overline{Q}_L W_\mu^a \sigma^a \gamma^\mu Q_L \supset W_\mu^1 \sigma^1 + W_\mu^2 \sigma^2 = \frac{1}{2}(W_\mu^1 - i W_\mu^2)(\sigma^1 + i\sigma^2) + \frac{1}{2}(W_\mu^1 + i W_\mu^2)(\sigma^1 - i\sigma^2) = \sqrt{2}\frac{W_\mu^1 - i W_\mu^2}{\sqrt{2}} \sigma^+ + \sqrt{2}\frac{W_\mu^1 + i W_\mu^2}{\sqrt{2}} \sigma^-, \tag{123}$$

where

$$(\sigma^1 + i\sigma^2) = 2\sigma^+ = 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad (\sigma^1 - i\sigma^2) = 2\sigma^- = 2\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{124}$$

When the covariant derivative acts on the left-handed fermion doublets we get terms of the following form, from which we can identify W^+ and W^- using charge conservation. Recall:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{125}$$

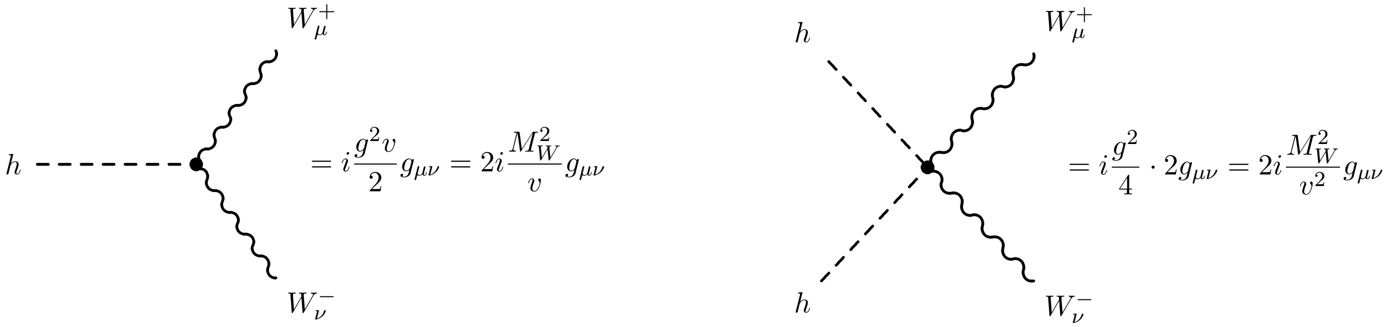


FIG. 3. Feynman rules for the hWW and $hhWW$ vertices, as derived from the Lagrangian in Eq. (128). The extra factor of 2 in the first expression for the $hhWW$ coupling is a symmetry factor accounting for the two identical Higgs bosons. See also Eq. (130).

The second term in Eq. (122) becomes

$$\begin{aligned}\mathcal{L} \supset & \frac{1}{8} g^2 (v+h)^2 (W^1_\mu - iW^2_\mu)(W^{1\mu} + iW^{2\mu}) \\ &= \frac{1}{4} g^2 (v+h)^2 W^+_\mu W^{-\mu} \\ &= \frac{g^2 v^2}{4} W^+_\mu W^{-\mu} + \frac{g^2 v}{2} h W^+_\mu W^{-\mu} + \frac{g^2}{4} hh W^+_\mu W^{-\mu}. \end{aligned} \tag{128}$$

The first term here is a mass term for the W boson, with

$$M_W^2 = \frac{g^2 v^2}{4}. \tag{129}$$

The Higgs vacuum expectation value (vev) has given the W boson a mass! Because M_W and g have been directly measured, we can determine $v \simeq 246$ GeV.[27] The second and third terms in Eq. (128) give interactions of one or two Higgs bosons with W^+W^-. The corresponding Feynman rules (see Fig. 3) are

$$\begin{aligned} hW^+_\mu W^-_\nu : \quad & i\frac{g^2 v}{2} g_{\mu\nu} = igM_W g_{\mu\nu} = 2i\frac{M_W^2}{v} g_{\mu\nu}, \\ hhW^+_\mu W^-_\nu : \quad & i\frac{g^2}{4} \times 2!\, g_{\mu\nu} = 2i\frac{M_W^2}{v^2} g_{\mu\nu}, \end{aligned} \tag{130}$$

where the 2! in the second expression is a combinatorical factor from the two identical Higgs bosons in the Lagrangian term. Note that the W mass, the hWW coupling, and the $hhWW$ coupling all come from the same term in the Lagrangian and are generated by expanding out the factor $(v+h)^2$. Thus the hWW and $hhWW$ couplings are uniquely predicted in the SM once the W mass and v are known.

We now consider the third term of Eq. (122). We first write the linear combination of W^3_μ and B_μ that appears in this term as a properly normalized real field:

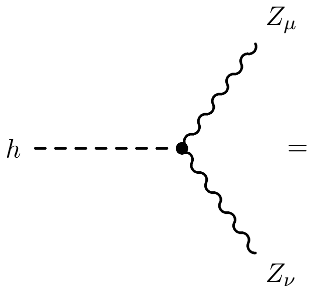
$$\begin{aligned} (gW^3_\mu - g' B_\mu) &= \sqrt{g^2 + g'^2} \left(\frac{g}{\sqrt{g^2+g'^2}} W^3_\mu - \frac{g'}{\sqrt{g^2+g'^2}} B_\mu \right) \\ &\equiv \sqrt{g^2+g'^2} \left(c_W W^3_\mu - s_W B_\mu \right) \\ &\equiv \sqrt{g^2+g'^2}\, Z_\mu, \end{aligned} \tag{131}$$

where we have defined $s_W = \sin\theta_W$, $c_W = \cos\theta_W$, where θ_W is the weak mixing angle or Weinberg angle. We have also defined the field combination Z_μ, which will receive a mass from the Higgs vev and be identified as the Z boson.

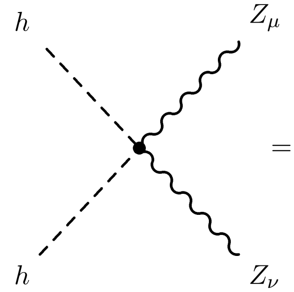
We note that the orthogonal state,

$$\left(s_W W^3_\mu + c_W B_\mu \right) \equiv A_\mu, \tag{132}$$

[27] This value of v actually comes from the Fermi constant, $G_F = 1/\sqrt{2} v^2$.



$$= i\frac{(g^2+g'^2)v}{4}\cdot 2g_{\mu\nu} = 2i\frac{M_Z^2}{v}g_{\mu\nu}$$



$$= i\frac{(g^2+g'^2)}{8}\cdot 2\cdot 2g_{\mu\nu} = 2i\frac{M_Z^2}{v^2}g_{\mu\nu}$$

FIG. 4. Feynman rules for the hZZ and $hhZZ$ vertices, as derived from the Lagrangian in Eq. (133). The extra factor of 2 in the first expression for the hZZ coupling is a symmetry factor accounting for the two identical Z bosons. The $hhZZ$ coupling contains two extra factors of 2 which are the symmetry factors accounting respectively for the two identical Higgs bosons and two identical Z bosons. See also Eq. (135).

does not couple to the Higgs field and thus does not acquire a mass through the Higgs mechanism. This state will be identified as the photon.[28]

The third term in Eq. (122) becomes

$$\begin{aligned}
\mathcal{L} &\supset \frac{1}{8}(v+h)^2\left(-g'B_\mu + gW_\mu^3\right)^2 \\
&= \frac{1}{8}(g^2+g'^2)(v+h)^2 Z_\mu Z^\mu \\
&= \frac{(g^2+g'^2)v^2}{8}Z_\mu Z^\mu + \frac{(g^2+g'^2)v}{4}hZ_\mu Z^\mu + \frac{(g^2+g'^2)}{8}hhZ_\mu Z^\mu. \qquad (133)
\end{aligned}$$

The first term here is a mass term for the Z boson,[29]

$$M_Z^2 = \frac{(g^2+g'^2)v^2}{4} = \frac{g^2+g'^2}{g^2}\frac{g^2v^2}{4} = \frac{M_W^2}{c_W^2} \qquad (134)$$

The second and third terms in Eq. (133) give interactions of one or two Higgs bosons with ZZ. The corresponding Feynman rules (see Fig. 4) are

$$\begin{aligned}
hZ_\mu Z_\nu : &\quad i\frac{(g^2+g'^2)v}{4}\times 2!\, g_{\mu\nu} = i\sqrt{g^2+g'^2}M_Z g_{\mu\nu} = 2i\frac{M_Z^2}{v}g_{\mu\nu}, \\
hhZ_\mu Z_\nu : &\quad i\frac{(g^2+g'^2)}{8}\times 2!\times 2!\, g_{\mu\nu} = 2i\frac{M_Z^2}{v^2}g_{\mu\nu}, \qquad (135)
\end{aligned}$$

where each coupling contains a 2! from the two identical Z bosons, and the second expression contains an extra 2! from the two identical Higgs bosons in the Lagrangian term. As before, the Z mass, the hZZ coupling, and the $hhZZ$ coupling all come from the same term in the Lagrangian and are generated by expanding out the factor $(v+h)^2$. Thus the hZZ and $hhZZ$ couplings are uniquely predicted in the SM once the Z mass and v are known.

We can now rewrite the covariant derivative in terms of our new basis of electroweak gauge bosons, W^+, W^-, Z, and A. Starting from Eq. (95), we make the following substitutions:

$$\begin{aligned}
B_\mu &= c_W A_\mu - s_W Z_\mu, \\
W_\mu^3 &= s_W A_\mu + c_W Z_\mu, \\
W^1T^1 + W^2T^2 &= W^1\frac{\sigma^1}{2} + W^2\frac{\sigma_2}{2} = \frac{1}{\sqrt{2}}(W^+T^+ + W^-T^-), \qquad (136)
\end{aligned}$$

---

[28] The choice of basis of the Higgs field, i.e., in which component we put the vev, does not affect this conclusion. There will always remain one massless gauge boson, corresponding to the combination of $SU(2)_L$ and $U(1)_Y$ gauge transformations that leaves our chosen vacuum state invariant. This combination will not couple to $(v+h)^2$, will not acquire a mass, and will thus be identified with the known massless electroweak gauge boson, the photon. Since electric charge is defined in terms of the couplings of the photon, the SM Higgs vev and physical Higgs boson will always be what we call electrically neutral.

[29] Remember that the mass term for a real vector field takes the form $\mathcal{L} \supset \frac{1}{2}M_Z^2 Z_\mu Z^\mu$.

where $T^\pm$ are the raising and lowering operators of $SU(2)_L$, with $T^\pm = \sigma^\pm$ in the doublet representation. This yields,

$$\mathcal{D}_\mu = \partial_\mu - ig_s G^a_\mu t^a - i\frac{g}{\sqrt{2}}\left(W^+_\mu T^+ + W^-_\mu T^-\right) - iZ_\mu\left(gc_W T^3 - g's_W Y\right) - iA_\mu\left(gs_W T^3 + g'c_W Y\right). \tag{137}$$

We first examine the photon coupling. Using the definitions $s_W = g'/\sqrt{g^2+g'^2}$, $c_W = g/\sqrt{g^2+g'^2}$, we can simplify the coefficient

$$\left(gs_W T^3 + g'c_W Y\right) = \frac{gg'}{\sqrt{g^2+g'^2}}\left(T^3 + Y\right) \equiv eQ, \tag{138}$$

where e is the electromagnetic coupling and Q is the electric charge operator. By convention, we identify

$$e = \frac{gg'}{\sqrt{g^2+g'^2}} = gs_W = g'c_W, \qquad\qquad Q = T^3 + Y. \tag{139}$$

The photon coupling then takes the familiar form $\mathcal{D}_\mu \supset -ieA_\mu Q$.

Now let's examine the Z boson coupling. We can use $Y = Q - T^3$ to write

$$\left(gc_W T^3 - g's_W Y\right) = \frac{g^2+g'^2}{\sqrt{g^2+g'^2}}T^3 - \frac{g'^2}{\sqrt{g^2+g'^2}}Q = \sqrt{g^2+g'^2}\left(T^3 - s_W^2 Q\right). \tag{140}$$

Putting it all together, we obtain the covariant derivative in the gauge boson mass basis,

$$\mathcal{D}_\mu = \partial_\mu - ig_s G^a_\mu t^a - i\frac{g}{\sqrt{2}}\left(W^+_\mu T^+ + W^-_\mu T^-\right) - i\frac{e}{s_W c_W}Z_\mu\left(T^3 - s_W^2 Q\right) - ieA_\mu Q, \tag{141}$$

where we note that $g = e/s_W$ and $e/s_W c_W = g/c_W = \sqrt{g^2+g'^2}$. From this expression we can derive the familiar electroweak fermion-antifermion-gauge boson Feynman rules using the fermion gauge-kinetic terms,

$$\mathcal{L} \supset \bar{\psi}_L i\mathcal{D}_\mu\gamma^\mu\psi_L + \bar{\psi}_R i\mathcal{D}_\mu\gamma^\mu\psi_R. \tag{142}$$

5. *Fermion masses, the CKM matrix, and couplings to the Higgs boson*

Now let's look at the couplings of the Higgs doublet Φ to fermions. We'll start with the leptons and neglect neutrino masses[30] for simplicity.

- Lepton masses

  The construction of the Lagrangian terms that describe the Higgs couplings to fermions is pretty straightforward. Lorentz invariance (conservation of spin) requires that fermion spinors appear in pairs, $\bar{\psi}\psi$. Because the fermion field has mass dimension 3/2, $\bar{\psi}\psi$ has mass dimension 3; combining this with a single Higgs doublet (with mass dimension 1) already yields mass dimension 4. Thus we can construct renormalizable Higgs-fermion couplings involving only one each of $\bar{\psi}$, ψ, and Φ. Furthermore, Φ is an $SU(2)_L$ doublet; for our Lagrangian term to be gauge invariant, we must couple it to one $SU(2)_L$ doublet fermion field (e.g., $L_L = (\nu_L, e_L)^T$, see Table III) and one $SU(2)_L$ singlet (e.g., e_R).

  Following this logic, the most general gauge-invariant renormalizable Lagrangian terms involving the Higgs doublet and leptons are, for a single generation,[31] Recall: $Y(e_R) = -1$, $Y(L_L) = -1/2$, and $Y(\Phi) = 1/2$

$$\mathcal{L}_{\text{Yukawa}} \supset -\left[y_e\bar{e}_R\Phi^\dagger L_L + y_e^*\bar{L}_L\Phi e_R\right], \tag{143}$$

  where the second term is the Hermitian conjugate of the first and y_e is a dimensionless constant. The coupling y_e is complex in general, but its phase can be absorbed into a physically-undetectable rephasing of the right-handed electron field e_R; therefore we'll treat it as real in what follows.

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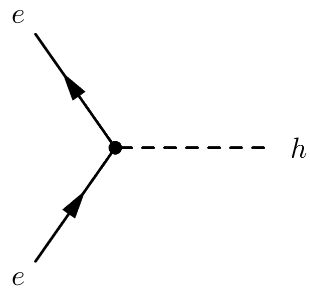
[30] I'll make some comments on neutrino masses later in this subsection.

[31] You can add up the hypercharges of the fields in these Lagrangian terms, remembering that a Hermitian-conjugated field carries minus the hypercharge of the original field, and see that the net hypercharge of each term is zero, i.e., that these terms are also gauge invariant under $U(1)_Y$. The same is true for the up- and down-type quark Yukawa terms that we will write down below. Aren't we lucky that the hypercharges of the left-handed fermions, right-handed fermions, and Higgs doublet work out just right to allow for the generation of fermion masses via electroweak symmetry breaking! Why this works out so nicely is a mystery in the SM, possibly to be explained by grand unification of the gauge interactions.



$$= -i\frac{y_e}{\sqrt{2}} = -i\frac{m_e}{v}$$

FIG. 5. Feynman rule for the $h\bar{e}e$ vertex, as derived from the Lagrangian in Eq. (146). See also Eq. (148).

In unitarity gauge,

$$\Phi = \begin{pmatrix} 0 \\ (v+h)/\sqrt{2} \end{pmatrix}, \tag{144}$$

and

$$\Phi^\dagger L_L = \left(0, \frac{v+h}{\sqrt{2}}\right)\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L = \frac{v+h}{\sqrt{2}} e_L, \tag{145}$$

so [using Eq. (104) in the second step]

$$\begin{aligned} \mathcal{L}_{\text{Yukawa}} &\supset -y_e \frac{1}{\sqrt{2}}\left[(v+h)\bar{e}_R e_L + (v+h)\bar{e}_L e_R\right] \\ &= -\frac{y_e}{\sqrt{2}}(v+h)\bar{e}e \\ &= -\left(\frac{y_e v}{\sqrt{2}}\right)\bar{e}e - \frac{y_e}{\sqrt{2}}h\bar{e}e. \end{aligned} \tag{146}$$

The first term in the last line is a mass term for the electron,

$$m_e = \frac{y_e v}{\sqrt{2}}. \tag{147}$$

The Higgs vacuum expectation value has given the electron a mass! Using the known value of v as determined from the W boson mass, we can deduce the value of y_e and hence the he^+e^- Feynman rule (see Fig. 5), which is

$$h\bar{e}e: \quad \frac{-iy_e}{\sqrt{2}} = \frac{-im_e}{v}. \tag{148}$$

Thus the $h\bar{e}e$ coupling is uniquely predicted in the SM once the electron mass and v are known.

The Higgs-electron coupling is really very small:

$$\frac{y_e}{\sqrt{2}} = \frac{m_e}{v} = \frac{511\ \text{keV}}{246\ \text{GeV}} \simeq 2.1 \times 10^{-6}. \tag{149}$$

We can write down a similar Higgs coupling and mass term for the muon and for the tau lepton. The tau Yukawa coupling is more "respectable," though still kind of small:

$$\frac{y_\tau}{\sqrt{2}} = \frac{m_\tau}{v} = \frac{1.78\ \text{GeV}}{246\ \text{GeV}} \simeq 7.2 \times 10^{-3}. \tag{150}$$

The SM does not provide any explanation for these numbers or their sizes; they are just parameters to be measured. One can hope that a more complete theory of flavor would provide an explanation for the pattern of fermion masses.

Note that we have not generated any masses or Higgs couplings to neutrinos, because we did not introduce three right-handed neutrinos ν_R to participate in the Higgs couplings. More on this after we deal with the quark masses.

- Quark masses and mixing

  We start by following our noses and writing a term just like for the charged leptons: Recall: $Y(d_R) = -1/3$, $Y(Q_L) = 1/6$, and $Y(\Phi) = 1/2$

$$\mathcal{L}_{\text{Yukawa}} \supset -\left[y_d \bar{d}_R \Phi^\dagger Q_L + y_d^* \bar{Q}_L \Phi d_R\right], \tag{151}$$

where again the second term is just the Hermitian conjugate of the first, and we will again assume that the dimensionless constant y_d is real for now. As for the leptons, we multiply out the $\text{SU}(2)_L$ doublets in unitarity gauge,

$$\Phi^\dagger Q_L = \left(0, \frac{v+h}{\sqrt{2}}\right)\begin{pmatrix} u_L \\ d_L \end{pmatrix} = \frac{v+h}{\sqrt{2}} d_L, \tag{152}$$

so that

$$\mathcal{L}_{\text{Yukawa}} \supset -\left(\frac{y_d v}{\sqrt{2}}\right)\bar{d}d - \frac{y_d}{\sqrt{2}} h\bar{d}d. \tag{153}$$

The first term is a mass for the down quark, $m_d = y_d v/\sqrt{2}$, and the second is an $h\bar{d}d$ coupling.

So far so good, but what about the up-type quark masses? To generate these, we take advantage of a useful property of SU(2): the anti-doublet or "conjugate" doublet transforms in the same way as the doublet. [32] The conjugate Higgs doublet is given by

$$\tilde{\Phi} \equiv i\sigma^2 \Phi^* = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \phi^- \\ \phi^{0*} \end{pmatrix} = \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix}, \tag{154}$$

and has hypercharge $Y = -1/2$. Using $\tilde{\Phi}$ we can write another gauge-invariant Lagrangian term, Recall: $Y(u_R) = 2/3$, $Y(Q_L) = 1/6$, and $Y(\Phi) = 1/2$

$$\mathcal{L}_{\text{Yukawa}} \supset -\left[y_u \bar{u}_R \tilde{\Phi}^\dagger Q_L + y_u^* \bar{Q}_L \tilde{\Phi} u_R\right], \tag{155}$$

where again the second term is the Hermitian conjugate of the first, and we will assume that the dimensionless constant y_u is real for now. Writing out the product of the $\text{SU}(2)_L$ doublets in unitarity gauge,

$$\tilde{\Phi}^\dagger Q_L = \left(\frac{v+h}{\sqrt{2}}, 0\right)\begin{pmatrix} u_L \\ d_L \end{pmatrix} = \frac{v+h}{\sqrt{2}} u_L, \tag{156}$$

so that

$$\mathcal{L}_{\text{Yukawa}} \supset -\left(\frac{y_u v}{\sqrt{2}}\right)\bar{u}u - \frac{y_u}{\sqrt{2}} h\bar{u}u. \tag{157}$$

This is exactly what we need to describe the up-quark mass $m_u = y_u v/\sqrt{2}$ and its coupling to the Higgs.

This is fine if we want to describe a single generation of quarks. But in the SM there are three generations of quarks! We should really rewrite our left- and right-handed quark fields with a generation index j,

$$Q_{Lj}, \quad u_{Rj}, \quad d_{Rj}, \qquad j = 1, 2, 3. \tag{158}$$

In general, we can write a gauge-invariant coupling of Q_{L1} to a Higgs doublet and each of u_{Rj} and d_{Rj}, with $j = 1, 2, 3$, and the same for Q_{L2} and Q_{L3}. The most general form of the quark Yukawa Lagrangian is

$$\mathcal{L}^q_{\text{Yukawa}} = -\sum_{i=1}^{3}\sum_{j=1}^{3}\left[y^u_{ij}\bar{u}_{Ri}\tilde{\Phi}^\dagger Q_{Lj} + y^d_{ij}\bar{d}_{Ri}\Phi^\dagger Q_{Lj}\right] + \text{h.c.}, \tag{159}$$

---

[32] Contrast this to the case of SU(3), in which the anti-triplet does not transform in the same way as the triplet.

where h.c. stands for Hermitian conjugate. The dimensionless couplings y^u_{ij} and y^d_{ij} are now the (i,j) entries of 3×3 complex matrices, containing a total of 18 complex coupling parameters! Replacing Φ with its vacuum value $(0, v/\sqrt{2})^T$, we obtain the quark mass terms:

$$\mathcal{L}^q_{\text{Yukawa}} \supset -\left(\bar{u}_1, \bar{u}_2, \bar{u}_3\right)_R \mathcal{M}^u \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_L - \left(\bar{d}_1, \bar{d}_2, \bar{d}_3\right)_R \mathcal{M}^d \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}_L + \text{h.c.}, \tag{160}$$

where

$$\mathcal{M}^u_{ij} = \frac{v}{\sqrt{2}} y^u_{ij}, \qquad \mathcal{M}^d_{ij} = \frac{v}{\sqrt{2}} y^d_{ij} \tag{161}$$

are the <u>quark mass matrices</u> in generation space, each containing 9 complex entries.

We want to find the quark mass eigenstates. To do that, we just need to diagonalize the two complex matrices $\mathcal{M}^u$ and $\mathcal{M}^d$. Any such matrix can be transformed into a real diagonal matrix by multiplying it on the left and right by appropriate unitary transformation matrices. We define four unitary matrices U_L, U_R, D_L, and D_L according to

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_{L,R} = U_{L,R} \begin{pmatrix} u \\ c \\ t \end{pmatrix}_{L,R}, \qquad \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}_{L,R} = D_{L,R} \begin{pmatrix} d \\ s \\ b \end{pmatrix}_{L,R}, \tag{162}$$

where u, c, t, d, s, b are the quark mass eigenstates, such that[33]

$$U_R^{-1} \mathcal{M}^u U_L = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}, \qquad D_R^{-1} \mathcal{M}^d D_L = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}. \tag{163}$$

Note that diagonalizing the mass matrices $\mathcal{M}^u$ and $\mathcal{M}^d$ simultaneously diagonalizes the Yukawa matrices $y^u_{ij} = \frac{\sqrt{2}}{v}\mathcal{M}^u_{ij}$ and $y^d_{ij} = \frac{\sqrt{2}}{v}\mathcal{M}^d_{ij}$: this means that the Higgs couplings to $\bar{q}q$ are real and diagonal in the quark mass basis. In particular, the Feynman rules are just

$$h\bar{q}q: \quad \frac{-iy_q}{\sqrt{2}} = \frac{-im_q}{v}, \tag{164}$$

where y_q is the appropriate eigenvalue of the Yukawa matrix y^u_{ij} or y^d_{ij}.

Notice that we've "broken up" the left-handed quark doublets by rotating the up-type quarks by U_L and the down-type quarks by the different matrix D_L. This shows up in the charged-current weak interactions, which change $u_{Lj} \leftrightarrow d_{Lj}$ within the <u>same</u> (linear combination of) doublets. Because the mass eigenstates of the down-type quarks are no longer matched up to the mass eigenstates of the up-type quarks, there are generation-changing weak interactions, which are described by the Cabibbo-Kobayashi-Maskawa (CKM) matrix.

In the charged-current interaction part of the Lagrangian we have the quark bilinears

$$\bar{u}_{L1}\gamma^\mu d_{L1}, \qquad \bar{u}_{L2}\gamma^\mu d_{L2}, \qquad \bar{u}_{L3}\gamma^\mu d_{L3}. \tag{165}$$

Their sum can be written in matrix form as

$$J_L^{+\mu} = \left(\bar{u}_1, \bar{u}_2, \bar{u}_3\right)_L \gamma^\mu \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}_L = \left(\bar{u}, \bar{c}, \bar{t}\right)_L U_L^\dagger \gamma^\mu D_L \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L = \left(\bar{u}, \bar{c}, \bar{t}\right)_L \gamma^\mu V \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L. \tag{166}$$

The combination $U_L^\dagger D_L \equiv V$ is the CKM matrix. Its elements are denoted by quark symbol subscripts; e.g., V_{ud} is the $(1,1)$ element of V. This indexing convention also helps one remember the form of Eq. (166).

---

[33] For a unitary matrix, $U^{-1} = U^\dagger$.

The CKM matrix is unitary:

$$V^\dagger V = \left(U_L^\dagger D_L\right)^\dagger \left(U_L^\dagger D_L\right) = D_L^\dagger U_L U_L^\dagger D_L = 1. \tag{167}$$

Note also that U_R and D_R have no physical consequences in the SM: u_{Ri} and d_{Ri} are not tied together in any way, so their relative basis rotations do not matter.

In the neutral current interactions, the photon couplings Q and the Z boson couplings $(T^3 - s_W^2 Q)$ are the same for each of the three generations. The fermion bilinears involved in the neutral current can then be written out in generation space as, e.g.,

$$(\bar{u}_1, \bar{u}_2, \bar{u}_3)_L \gamma^\mu \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_L = (\bar{u}, \bar{c}, \bar{t})_L U_L^\dagger \gamma^\mu U_L \begin{pmatrix} u \\ c \\ t \end{pmatrix}_L = (\bar{u}, \bar{c}, \bar{t})_L \gamma^\mu \begin{pmatrix} u \\ c \\ t \end{pmatrix}_L . \tag{168}$$

So the neutral currents are automatically flavor diagonal, so long as the photon and Z boson couplings to all three generations are universal. This is a manifestation of the GIM mechanism (after Glashow, Iliopoulos and Maiani). It is also why "flavor changing neutral currents" (FCNCs) provide such tight constraints on physics beyond the SM: they are absent at tree level in the SM, and the SM FCNCs induced at one-loop by W boson exchange are typically quite small effects.

As a last comment, it is often convenient to work in the weak basis in which the up-type quarks are mass eigenstates. The weak isospin doublets can then be written as

$$\begin{pmatrix} u \\ d' \end{pmatrix}_L, \quad \begin{pmatrix} c \\ s' \end{pmatrix}_L, \quad \begin{pmatrix} t \\ b' \end{pmatrix}_L, \tag{169}$$

where in generation space, $(V = U_L^\dagger D_L = D_L$ in this basis)

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}_L = V \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L . \tag{170}$$

- An aside on neutrino masses

If the neutrinos are Dirac particles (we do not know whether this is true; the other alternative is that they are Majorana particles, which are their own antiparticles), then we can introduce three right-handed neutrino fields ν_{Ri} $(i = 1, 2, 3)$ and write Dirac neutrino masses in the same way as the up-type quark masses: Recall: $Y(\nu_R) = 0$, $Y(L_L) = -1/2$, and $Y(\Phi) = 1/2$

$$\mathcal{L}_{\text{Yukawa}} \supset -y_\nu \bar{\nu}_R \tilde{\Phi}^\dagger L_L + \text{h.c.}, \tag{171}$$

or, including the charged lepton mass terms and the full three-generation structure [compare Eq. (159)],

$$\mathcal{L}^\ell_{\text{Yukawa}} = -\sum_{i=1}^{3}\sum_{j=1}^{3} \left[y^\nu_{ij} \bar{\nu}_{R_i} \tilde{\Phi}^\dagger L_{Lj} + y^\ell_{ij} \bar{e}_{Ri} \Phi^\dagger L_{Lj} \right] + \text{h.c.} \tag{172}$$

Exactly as for the quarks, we get Dirac masses for the charged lepton mass eigenstates e, μ, τ and the neutrino mass eigenstates ν_1, ν_2, ν_3. The weak isospin doublets can be written in the basis in which the charged leptons are mass eigenstates as

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L, \tag{173}$$

where the "flavor eigenstates" of the neutrinos, ν_e, ν_μ, and ν_τ, are related to their mass eigenstates by the lepton analogue of the CKM matrix, called the Maki-Nakagawa-Sakata-Pontecorvo (MNSP, or PMNS depending on your political affiliation) matrix U:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}_L = U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}_L . \tag{174}$$

The elements of the MNSP matrix are denoted by indices as, e.g., U_{e1} for the $(1,1)$ element. This helps one remember the form of Eq. (174).

Note that the Yukawa couplings needed to generate the neutrino masses are extremely—some would say unreasonably—small: for a neutrino mass $m_\nu \sim 0.1$ eV, the corresponding neutrino Yukawa coupling would be

$$\frac{y_\nu}{\sqrt{2}} = \frac{m_\nu}{v} \simeq 4 \times 10^{-13}. \tag{175}$$

The other possibility for neutrinos is a "Majorana mass." In terms of the SM fields, this is a term of the form $m\nu_L\nu_L$ (no bar!). Neutrinos are the only known fermion for which we can construct a Majorana mass because they are electrically neutral, so that the Majorana mass term does not violate electric charge conservation. Such a mass term is not gauge invariant under $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$, but we can generate it after electroweak symmetry breaking by writing a more complicated term involving the Higgs field:[34]

$$\mathcal{L}_{\mathrm{Majorana}} = -\frac{(\tilde{\Phi}^\dagger L_L)^2}{\Lambda}. \tag{177}$$

Counting up the dimensionality of the fields in the numerator of $\mathcal{L}_{\mathrm{Majorana}}$ quickly reveals that the field operator has dimension 5. This is thus a nonrenormalizable interaction, with coefficient $1/\Lambda$ where Λ indicates the cutoff scale beyond which a more complete theory must reveal itself.

Such a term yields a neutrino mass $m_\nu = v^2/2\Lambda$. To get a neutrino mass of $m_\nu \sim 0.1$ eV requires $\Lambda \sim 3 \times 10^{14}$ GeV. The more complete theory that yields the Majorana mass term usually involves a very heavy Majorana right-handed neutrino ν_R with mass of order the scale Λ. This is known as the "Type-I Seesaw."

- CKM matrix parameter counting

  You may have heard that the CKM matrix (and also the MNSP matrix) can be specified by three angles and a phase. Here's where that counting comes from.

  - We start with a 3×3 complex matrix V: in general it contains 9 complex numbers, i.e., 18 independent real parameters.
  - V is unitary, yielding 9 constraints of the form $V^\dagger_{ab} V_{bc} = \delta_{ac}$. This leaves 9 independent real parameters.
  - We are free to absorb a phase out of V into each left-handed field, by redefining $q_L \to e^{i\alpha_{q_L}} q_L$, with $q = u, d$ of each of the three generations. This removes an arbitrary phase from each row or column of V. But a common phase redefinition of all the q_L has no effect on V, so this rephasing actually removes only $6 - 1 = 5$ unphysical phases. This leaves $9 - 5 = 4$ physical free parameters in V.

  To see that these four free parameters comprise three angles and a phase, note that a 3×3 real unitary matrix—i.e., an orthogonal matrix—has three independent parameters (the familiar three Euler angles). Thus $4 - 3 = 1$ of our CKM parameters must be a complex phase. This phase is what gives rise to CP violation in the Standard Model weak interactions.[35]

6. *Higgs self-couplings*

Finally let's return to the Higgs potential,

$$\mathcal{L}_V = -V(\Phi) = \mu^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2, \tag{178}$$

---

[34] The Majorana mass term is more properly written as

$$\mathcal{L}_{\mathrm{Majorana}} = -\frac{y^{\mathrm{Maj}}_{ij}}{\Lambda} \bar{L}^c_{Li} \tilde{\Phi}^* \tilde{\Phi}^\dagger L_{Lj}, \tag{176}$$

where the conjugate spinor $\bar{L}^c_L \equiv -L^T_L C$, where $C = -i\gamma^2\gamma^0$ is known as the charge conjugation matrix. $\bar{L}^c_L$ transforms in the same way as a right-handed spinor $\bar{\psi}_R$ under the Lorentz group. (I also included a generation-dependent prefactor y^{Maj}_{ij} to allow for different Majorana masses for the three generations.) Majorana particles also show up in supersymmetry—in the Minimal Supersymmetric Standard Model, the gluinos and neutralinos are Majorana fermions.

[35] Note also that if we'd had only two generations, the CKM matrix would be fixed in terms of a single mixing angle and no phase. The introduction of the CP-violating phase was part of the original motivation for Kobayashi and Maskawa to introduce the third generation.

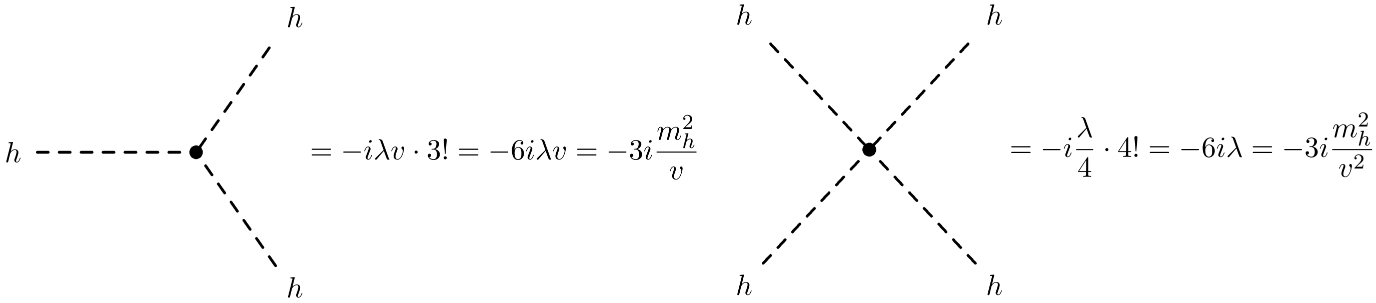


FIG. 6. Feynman rules for the hhh and $hhhh$ vertices, as derived from the Lagrangian in Eq. (180). The hhh coupling contains a symmetry factor of $3! = 6$ from the three identical Higgs bosons, and the $hhhh$ coupling contains a symmetry factor of $4! = 24$ from the four identical Higgs bosons. See also Eqs. (181) and (182).

and work out the self-interactions of the Higgs. In unitarity gauge,

$$\Phi^\dagger \Phi = \frac{1}{2}(h+v)^2, \tag{179}$$

and minimizing the potential gave us the relation $\mu^2 = \lambda v^2$, which we will use to eliminate μ^2.

Plugging in and multiplying out, we obtain [36]

$$\mathcal{L}_V = -\lambda v^2 h^2 - \lambda v h^3 - \frac{\lambda}{4} h^4 + \text{const.} \tag{180}$$

The first term is the mass term for the Higgs, $-\lambda v^2 = -m_h^2/2$. The second term is an interaction vertex involving three Higgs bosons, with Feynman rule (see Fig. 6)

$$hhh: \quad -i\lambda v \times 3! = -6i\lambda v = -3i\frac{m_h^2}{v}, \tag{181}$$

where the 3! is a combinatorical factor from the three identical Higgs bosons in the Lagrangian term. The third term is an interaction vertex involving four Higgs bosons, with Feynman rule (see Fig. 6)

$$hhhh: \quad -i\frac{\lambda}{4} \times 4! = -6i\lambda = -3i\frac{m_h^2}{v^2}, \tag{182}$$

where again the 4! is a combinatorical factor from the four identical Higgs bosons in the Lagrangian term.

B. Summary [37]

The self-interactions of the SM Higgs boson and its interactions with the massive vector bosons are derived from the Higgs Lagrangian:

$$\mathcal{L}_{\text{Higgs}} = (D^\mu \Phi)^\dagger (D_\mu \Phi) \; - \; V_{\text{SM}}(\Phi)\,, \tag{183}$$

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[36]

$$(\lambda/2)v^2(h+v)^2 - (\lambda/4)(h^2+2vh+v^2)^2 = \text{const.} + h[(\lambda/2)v^2(2v) - (\lambda/4)(4v^3)] + h^2[(\lambda/2)v^2 - (\lambda/4)(2v^2+4v^2)]$$
$$+ h^3[-(\lambda/4)(4v)] + h^4[-(\lambda/4)]$$

[37] This is a summary from Ref. [4]

where Φ denotes a complex $\mathrm{SU}(2)_L$ doublet Higgs field with hypercharge $Y = 1/2$ and its covariant derivative is defined as

$$D_\mu \Phi = \left(\partial_\mu - ig\frac{\sigma^a}{2} W^a_\mu - ig'\frac{1}{2}B_\mu \right)\Phi$$
$$= \begin{pmatrix} \partial_\mu - \frac{i}{2}(gW^3_\mu + g'B_\mu) & -\frac{ig}{2}(W^1_\mu - iW^2_\mu) \\ -\frac{ig}{2}(W^1_\mu + iW^2_\mu) & \partial_\mu + \frac{i}{2}(gW^3_\mu - g'B_\mu) \end{pmatrix}\Phi\,, \tag{184}$$

in terms of the $\mathrm{SU}(2)_L$ and $\mathrm{U}(1)_Y$ gauge couplings g and g', respectively, the three $\mathrm{SU}(2)_L$ gauge bosons $W^{1,2,3}_\mu$, and the single $\mathrm{U}(1)_Y$ gauge boson B_μ with the usual three 2×2 Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{185}$$

And the renormalizable SM Higgs potential $V_{\rm SM}(\Phi)$ is given by [38]

$$V_{\rm SM}(\Phi) = \mu^2(\Phi^\dagger\Phi) \;+\; \lambda(\Phi^\dagger\Phi)^2\,, \tag{186}$$

with $\mu^2 < 0$ leading to the spontaneous breakdown of the electroweak gauge symmetry.

Taking $\Phi = (0, v+H)^T/\sqrt{2}$ with the vacuum expectation value (vev) $v = \sqrt{-\mu^2/\lambda}$ and the real scalar field H after rotating away three Goldstone modes and using $W^\pm_\mu = (W^1_\mu \mp iW^2_\mu)/\sqrt{2}$ and $Z_\mu = (gW^3_\mu - g'B_\mu)/\sqrt{g^2+g'^2}$, we can render the kinetic term of the Higgs Lagrangian in Eq. (183) into the form expanded as [39]

$$\begin{aligned}(D^\mu\Phi)^\dagger\,(D_\mu\Phi) &= \frac{1}{2}(\partial_\mu H)(\partial^\mu H) + M_W^2 W^+_\mu W^{\mu -} + \frac{1}{2}M_Z^2 Z_\mu Z^\mu \\ &+ gM_W\left(W^+_\mu W^{\mu-} + \frac{1}{2c_W^2}Z_\mu Z^\mu\right)H + \frac{1}{v^2}\left(M_W^2 W^+_\mu W^{\mu-} + \frac{M_Z^2}{2}Z_\mu Z^\mu\right)H^2\,,\end{aligned} \tag{187}$$

in the unitary gauge. We use the abbreviation $s_W \equiv \sin\theta_W$ for the sine of the weak mixing angle θ_W and $c_W \equiv \cos\theta_W$, $t_W \equiv \sin\theta_W/\cos\theta_W$, etc. The masses of the massive gauge bosons W and Z are given by $M_W = gv/2$ and $M_Z = M_W/c_W$ with $v = \left(\sqrt{2}G_F\right)^{-1/2} \approx 246\,\mathrm{GeV}$ fixed by the Fermi constant G_F. Incidentally, the $\mathrm{SU}(2)_L$ and $\mathrm{U}(1)_Y$ gauge couplings are $g = e/s_W$ and $g' = g\,t_W = e/c_W$, respectively, where the magnitude of the electron electric charge $e = 2\sqrt{\pi\alpha}$ with α being the fine structure constant. On the other hand, the SM Higgs potential takes the form of

$$V_{\rm SM}(H) = -\frac{1}{8}v^2M_H^2 + \frac{1}{2}M_H^2\,H^2 + \frac{1}{3!}\left(\frac{3M_H^2}{v^2}\right)v\,H^3 + \frac{1}{4!}\left(\frac{3M_H^2}{v^2}\right)H^4\,, \tag{188}$$

which is completely fixed in terms of v and the Higgs mass M_H with the replacements of $\mu^2 = -\lambda v^2$ and $\lambda = M_H^2/2v^2$.

The Higgs interactions with the SM fermions are derived by considering the following Yukawa interactions [40]

$$\begin{aligned}-\mathcal{L}_Y &= \overline{U_R}\,\mathbf{h}_u\,Q^T\,(i\sigma^2)\,\Phi - \overline{D_R}\,\mathbf{h}_d\,Q^T\,(i\sigma^2)\,\widetilde{\Phi} - \overline{E_R}\,\mathbf{h}_e\,L^T\,(i\sigma^2)\,\widetilde{\Phi} \;+\; \text{h.c.}\,, \\ &= \overline{U_R}\,\mathbf{h}_u\,\widetilde{\Phi}^\dagger\,Q + \overline{D_R}\,\mathbf{h}_d\,\Phi^\dagger\,Q + \overline{E_R}\,\mathbf{h}_e\,\Phi^\dagger\,L \;+\; \text{h.c.}\,,\end{aligned} \tag{189}$$

where $\widetilde{\Phi} = i\sigma^2\Phi^* = (\phi^{0*}, -\phi^-)^T$ and $Q^T = (U_L\,, D_L)$ and $L^T = (\nu_L\,, E_L)$ with U and D standing for the three up- and down-type quarks, respectively, and ν and E for the three neutrinos and charged leptons, respectively, in the weak eigenstate basis. And the 3×3 Yukawa matrices are denoted by $\mathbf{h}_{u,d,e}$. Taking $\Phi = (0, v+H)^T/\sqrt{2}$ again, we have

$$-\mathcal{L}_{H\bar{f}f} = \sum_{f=u,d,c,s,t,b,e,\mu,\tau} \frac{m_f}{v}\,H\,\overline{f}f\,, \tag{190}$$

with the masses $m_f = h_f\,v/\sqrt{2}$ in the fermion mass eigenstate basis diagonalizing the Higgs-fermion interactions.

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[38] Note the sign flip of the μ^2 term

[39] For the HVV couplings, one might have $gM_W = g(gv/2) = 2M_W^2/v$ and $gM_W/c_W^2 = 2M_W^2/v/c_W^2 = 2M_Z^2/v$ using $M_W = gv/2$ and $M_W = M_Z c_W$.

[40] Note that $(i\sigma^2)\Phi = \widetilde{\Phi}^*$ and $(i\sigma^2)\widetilde{\Phi} = -\Phi^*$ which lead to $Q^T\,(i\sigma^2)\,\Phi = \widetilde{\Phi}^\dagger\,Q$, $Q^T\,(i\sigma^2)\,\widetilde{\Phi} = -\Phi^\dagger\,Q$, and $L^T\,(i\sigma^2)\,\widetilde{\Phi} = -\Phi^\dagger\,L$.

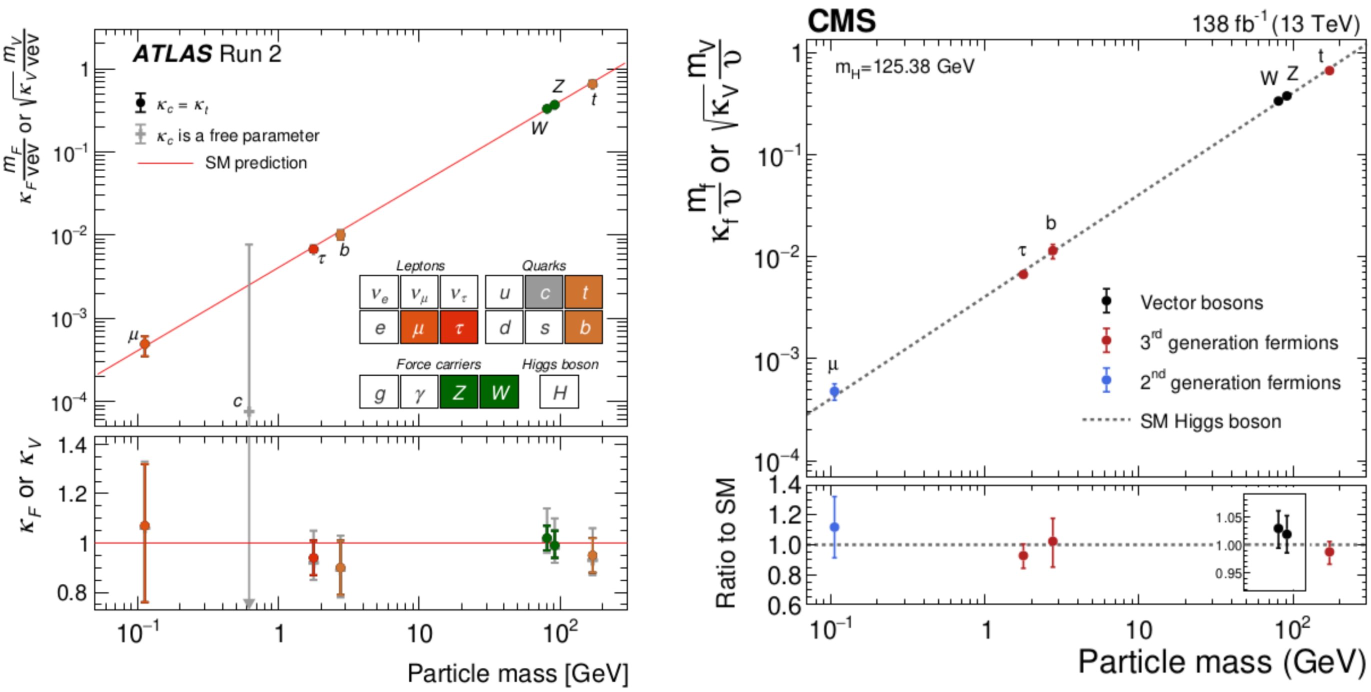

FIG. 7. The measured coupling modifiers of the Higgs boson to fermions and heavy gauge bosons, as functions of fermion or gauge boson mass, see Refs.[5] and [6]. Note that, with the coupling modifiers, the HHVV and Yukawa couplings are given by (two times of) $\kappa_V M_V^2/v^2$ and $\kappa_f m_f/v$, respectively, see Eqs. (187) and (190).

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