



Klein–Gordon equation

The **Klein–Gordon equation** (**Klein–Fock–Gordon equation** or sometimes **Klein–Gordon–Fock equation**) is a relativistic wave equation, related to the Schrödinger equation. It is second-order in space and time and manifestly Lorentz-covariant. It is a differential equation version of the relativistic energy–momentum relation $E^2 = (\mathbf{p}c)^2 + (m_0c^2)^2$.

Statement

The Klein–Gordon equation can be written in different ways. The equation itself usually refers to the position space form, where it can be written in terms of separated space and time components (t, \mathbf{x}) or by combining them into a four-vector $x^\mu = (ct, \mathbf{x})$. By Fourier transforming the field into momentum space, the solution is usually written in terms of a superposition of plane waves whose energy and momentum obey the energy-momentum dispersion relation from special relativity. Here, the Klein–Gordon equation is given for both of the two common metric signature conventions $\eta_{\mu\nu} = \text{diag}(\pm 1, \mp 1, \mp 1, \mp 1)$.

Klein–Gordon equation in normal units with metric signature $\eta_{\mu\nu} = \text{diag}(\pm 1, \mp 1, \mp 1, \mp 1)$

	Position space $x^\mu = (ct, \mathbf{x})$	Fourier transformation $\omega = E/\hbar, \quad \mathbf{k} = \mathbf{p}/\hbar$	Momentum space $p^\mu = (E/c, \mathbf{p})$
Separated time and space	$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2 c^2}{\hbar^2}\right) \psi(t, \mathbf{x}) = 0$	$\psi(t, \mathbf{x}) = \int \frac{d\omega}{2\pi\hbar} \int \frac{d^3k}{(2\pi\hbar)^3} e^{\mp i(\omega t - \mathbf{k}\cdot\mathbf{x})} \psi(\omega, \mathbf{k})$	$E^2 = \mathbf{p}^2 c^2 + m^2 c^4$
Four-vector form	$(\square + \mu^2)\psi = 0, \quad \mu = mc/\hbar$	$\psi(x^\mu) = \int \frac{d^4p}{(2\pi\hbar)^4} e^{-ip_\mu x^\mu/\hbar} \psi(p^\mu)$	$p^\mu p_\mu = \pm m^2 c^2$

Here, $\square = \pm \eta^{\mu\nu} \partial_\mu \partial_\nu$ is the wave operator and ∇^2 is the Laplace operator. The speed of light c and Planck constant \hbar are often seen to clutter the equations, so they are therefore often expressed in natural units where $c = \hbar = 1$.

Klein–Gordon equation in natural units with metric signature $\eta_{\mu\nu} = \text{diag}(\pm 1, \mp 1, \mp 1, \mp 1)$

	Position space $x^\mu = (t, \mathbf{x})$	Fourier transformation $\omega = E, \quad \mathbf{k} = \mathbf{p}$	Momentum space $p^\mu = (E, \mathbf{p})$
Separated time and space	$(\partial_t^2 - \nabla^2 + m^2) \psi(t, \mathbf{x}) = 0$	$\psi(t, \mathbf{x}) = \int \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} e^{\mp i(\omega t - \mathbf{k}\cdot\mathbf{x})} \psi(\omega, \mathbf{k})$	$E^2 = \mathbf{p}^2 + m^2$
Four-vector form	$(\square + m^2)\psi = 0$	$\psi(x^\mu) = \int \frac{d^4p}{(2\pi)^4} e^{-ip_\mu x^\mu} \psi(p^\mu)$	$p^\mu p_\mu = \pm m^2$

Unlike the Schrödinger equation, the Klein–Gordon equation admits two values of ω for each k : one positive and one negative. Only by separating out the positive and negative frequency parts does one obtain an equation describing a relativistic wavefunction. For the time-independent case, the Klein–Gordon equation becomes

$$\left[\nabla^2 - \frac{m^2 c^2}{\hbar^2}\right] \psi(\mathbf{r}) = 0,$$

which is formally the same as the homogeneous screened Poisson equation. In addition, the Klein–Gordon equation can also be represented as:^[1]

$$\hat{p}^\mu \hat{p}_\mu \psi = m^2 c^2 \psi$$

where, the momentum operator is given as: $\hat{p}^\mu = i\hbar \frac{\partial}{\partial x_\mu} = i\hbar \left\{ \frac{\partial}{\partial(ct)}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z} \right\} = \left\{ \frac{E}{c}, \mathbf{p} \right\}$.

Relevance

The equation is to be understood first as a classical continuous scalar field equation that can be quantized. The quantization process introduces then a quantum field whose quanta are spinless particles. Its theoretical relevance is similar to that of the Dirac equation.^[2] The equation solutions include a scalar or pseudoscalar field. In the realm of particle physics electromagnetic interactions can be incorporated, forming the topic of scalar electrodynamics, the practical utility for particles like pions is limited.^{[nb 1][3]} There is a second version of the equation for a complex scalar field that is theoretically important being the equation of the Higgs Boson. In the realm of condensed matter it can be used for many approximations of quasi-particles without spin.^{[4][5][nb 2]}

The equation can be put into the form of a Schrödinger equation. In this form it is expressed as two coupled differential equations, each of first order in time.^[6] The solutions have two components, reflecting the charge degree of freedom in relativity.^{[6][7]} It admits a conserved quantity, but this is not positive definite. The wave function cannot therefore be interpreted as a probability amplitude. The conserved quantity is instead interpreted as electric charge, and the norm squared of the wave function is interpreted as a charge density. The equation describes all spinless particles with positive, negative, and zero charge.

Any solution of the free Dirac equation is, for each of its four components, a solution of the free Klein–Gordon equation. Despite historically it was invented as a single particle equation the Klein–Gordon equation cannot form the basis of a consistent quantum relativistic *one-particle* theory, any relativistic theory implies creation and annihilation of particles beyond a certain energy threshold.^{[8] [nb 3]}

Solution for free particle

Here, the Klein–Gordon equation in natural units, $(\square + m^2)\psi(x) = 0$, with the metric signature $\eta_{\mu\nu} = \mathbf{diag}(+1, -1, -1, -1)$ is solved by Fourier transformation. Inserting the Fourier transformation

$$\psi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot x} \psi(p)$$

and using orthogonality of the complex exponentials gives the dispersion relation

$$p^2 = (p^0)^2 - \mathbf{p}^2 = m^2$$

This restricts the momenta to those that lie on shell, giving positive and negative energy solutions

$$p^0 = \pm E(\mathbf{p}) \quad \text{where} \quad E(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}.$$

For a new set of constants $C(\mathbf{p})$, the solution then becomes

$$\psi(x) = \int \frac{d^4p}{(2\pi)^4} e^{ip \cdot x} C(\mathbf{p}) \delta((p^0)^2 - E(\mathbf{p})^2).$$

It is common to handle the positive and negative energy solutions by separating out the negative energies and work only with positive p^0 :

$$\begin{aligned} \psi(x) &= \int \frac{d^4p}{(2\pi)^4} \delta((p^0)^2 - E(\mathbf{p})^2) \left(A(\mathbf{p}) e^{-ip^0 x^0 + ip^i x^i} + B(\mathbf{p}) e^{+ip^0 x^0 + ip^i x^i} \right) \theta(p^0) \\ &= \int \frac{d^4p}{(2\pi)^4} \delta((p^0)^2 - E(\mathbf{p})^2) \left(A(\mathbf{p}) e^{-ip^0 x^0 + ip^i x^i} + B(-\mathbf{p}) e^{+ip^0 x^0 - ip^i x^i} \right) \theta(p^0) \\ &\rightarrow \int \frac{d^4p}{(2\pi)^4} \delta((p^0)^2 - E(\mathbf{p})^2) \left(A(\mathbf{p}) e^{-ip \cdot x} + B(\mathbf{p}) e^{+ip \cdot x} \right) \theta(p^0) \end{aligned}$$

In the last step, $B(\mathbf{p}) \rightarrow B(-\mathbf{p})$ was renamed. Now we can perform the p^0 -integration, picking up the positive frequency part from the delta function only:

$$\begin{aligned} \psi(x) &= \int \frac{d^4p}{(2\pi)^4} \frac{\delta(p^0 - E(\mathbf{p}))}{2E(\mathbf{p})} \left(A(\mathbf{p}) e^{-ip \cdot x} + B(\mathbf{p}) e^{+ip \cdot x} \right) \theta(p^0) \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E(\mathbf{p})} \left(A(\mathbf{p}) e^{-ip \cdot x} + B(\mathbf{p}) e^{+ip \cdot x} \right) \Bigg|_{p^0 = +E(\mathbf{p})}. \end{aligned}$$

This is commonly taken as a general solution to the free Klein–Gordon equation. Note that because the initial Fourier transformation contained Lorentz invariant quantities like $\mathbf{p} \cdot \mathbf{x} = \mathbf{p}_\mu \mathbf{x}^\mu$ only, the last expression is also a Lorentz invariant solution to the Klein–Gordon equation. If one does not require Lorentz invariance, one can absorb the $1/2E(\mathbf{p})$ -factor into the

coefficients ***A***(***p***) and ***B***(***p***).

History

The equation was named after the physicists [Oskar Klein](#)^[9] and [Walter Gordon](#),^[10] who in 1926 proposed that it describes relativistic electrons. [Vladimir Fock](#) also discovered the equation independently in 1926 slightly after Klein's work,^[11] in that Klein's paper was received on 28 April 1926, Fock's paper was received on 30 July 1926 and Gordon's paper on 29 September 1926. Other authors making similar claims in that same year [Johann Kudar](#), [Théophile de Donder](#) and [Frans-H. van den Dungen](#), and [Louis de Broglie](#). Although it turned out that modeling the electron's spin required the [Dirac equation](#), the Klein–Gordon equation correctly describes the spinless relativistic [composite particles](#), like the [pion](#). On 4 July 2012, European Organization for Nuclear Research [CERN](#) announced the discovery of the [Higgs boson](#). Since the [Higgs boson](#) is a spin-zero particle, it is the first observed ostensibly [elementary particle](#) to be described by the Klein–Gordon equation. Further experimentation and analysis is required to discern whether the [Higgs boson](#) observed is that of the [Standard Model](#) or a more exotic, possibly composite, form.

The Klein–Gordon equation was first considered as a quantum wave equation by [Erwin Schrödinger](#) in his search for an equation describing [de Broglie waves](#). The equation is found in his notebooks from late 1925, and he appears to have prepared a manuscript applying it to the hydrogen atom. Yet, because it fails to take into account the electron's spin, the equation predicts the hydrogen atom's fine structure incorrectly, including overestimating the overall magnitude of the splitting pattern by a factor of $\frac{4n}{2n-1}$ for the *n*-th energy level. The Dirac equation relativistic spectrum is, however, easily recovered if the orbital-momentum quantum number *l* is replaced by total angular-momentum quantum number *j*.^[12] In January 1926, Schrödinger submitted for publication instead *his* equation, a non-relativistic approximation that predicts the Bohr energy levels of hydrogen without [fine structure](#).

In 1926, soon after the Schrödinger equation was introduced, [Vladimir Fock](#) wrote an article about its generalization for the case of [magnetic fields](#), where forces were dependent on [velocity](#), and independently derived this equation. Both Klein and Fock used [Kaluza and Klein's method](#). Fock also determined the [gauge theory](#) for the [wave equation](#). The Klein–Gordon equation for a free particle has a simple [plane-wave](#) solution.

Derivation

The non-relativistic equation for the energy of a free particle is

$$\frac{\mathbf{p}^2}{2m} = E.$$

By quantizing this, we get the non-relativistic Schrödinger equation for a free particle:

$$\frac{\hat{\mathbf{p}}^2}{2m} \psi = \hat{E} \psi,$$

where

$$\hat{\mathbf{p}} = -i\hbar\nabla$$

is the [momentum operator](#) (∇ being the [del operator](#)), and

$$\hat{E} = i\hbar \frac{\partial}{\partial t}$$

is the [energy operator](#).

The Schrödinger equation suffers from not being [relativistically invariant](#), meaning that it is inconsistent with [special relativity](#).

It is natural to try to use the identity from special relativity describing the energy:

$$\sqrt{\mathbf{p}^2 c^2 + m^2 c^4} = E.$$

Then, just inserting the quantum-mechanical operators for momentum and energy yields the equation

$$\sqrt{(-i\hbar\nabla)^2 c^2 + m^2 c^4} \psi = i\hbar \frac{\partial}{\partial t} \psi.$$

The square root of a differential operator can be defined with the help of [Fourier transformations](#), but due to the asymmetry of space and time derivatives, Dirac found it impossible to include external electromagnetic fields in a relativistically invariant way. So he looked for another equation that can be modified in order to describe the action of electromagnetic forces. In addition, this equation, as it stands, is [nonlocal](#) (see also [Introduction to nonlocal equations \(http://www.ma.utexas.edu/media/wiki/index.php/Introduction_to_nonlocal_equations\)](http://www.ma.utexas.edu/media/wiki/index.php/Introduction_to_nonlocal_equations)).

Klein and Gordon instead began with the square of the above identity, i.e.

$$\mathbf{p}^2 c^2 + m^2 c^4 = E^2,$$

which, when quantized, gives

$$((-i\hbar\nabla)^2 c^2 + m^2 c^4) \psi = \left(i\hbar \frac{\partial}{\partial t}\right)^2 \psi,$$

which simplifies to

$$-\hbar^2 c^2 \nabla^2 \psi + m^2 c^4 \psi = -\hbar^2 \frac{\partial^2}{\partial t^2} \psi.$$

Rearranging terms yields

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \psi - \nabla^2 \psi + \frac{m^2 c^2}{\hbar^2} \psi = 0.$$

Since all reference to imaginary numbers has been eliminated from this equation, it can be applied to fields that are [real-valued](#), as well as those that have [complex values](#).

Rewriting the first two terms using the inverse of the [Minkowski metric](#) $\text{diag}(-c^2, 1, 1, 1)$, and writing the Einstein summation convention explicitly we get

$$-\eta^{\mu\nu} \partial_\mu \partial_\nu \psi \equiv \sum_{\mu=0}^3 \sum_{\nu=0}^3 -\eta^{\mu\nu} \partial_\mu \partial_\nu \psi = \frac{1}{c^2} \partial_0^2 \psi - \sum_{\nu=1}^3 \partial_\nu \partial_\nu \psi = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \psi - \nabla^2 \psi.$$

Thus the Klein–Gordon equation can be written in a covariant notation. This often means an abbreviation in the form of

$$(\square + \mu^2) \psi = 0,$$

where

$$\mu = \frac{mc}{\hbar},$$

and

$$\square = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2.$$

This operator is called the [wave operator](#).

Today this form is interpreted as the relativistic [field equation](#) for [spin-0](#) particles.^[6] Furthermore, any *component* of any solution to the free [Dirac equation](#) (for a [spin-1/2](#) particle) is automatically a solution to the free Klein–Gordon equation. This generalizes to particles of any spin due to the [Bargmann–Wigner equations](#). Furthermore, in [quantum field theory](#), every component of every quantum field must satisfy the free Klein–Gordon equation,^[13] making the equation a generic expression of quantum fields.

Klein–Gordon equation in a potential

The Klein–Gordon equation can be generalized to describe a field in some potential $V(\psi)$ as^[14]

$$\square \psi + \frac{\partial V}{\partial \psi} = 0.$$

Then the Klein–Gordon equation is the case $V(\psi) = M^2 \bar{\psi} \psi$.

Another common choice of potential which arises in interacting theories is the ϕ^4 potential for a real scalar field ϕ ,

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \lambda\phi^4.$$

Higgs sector

The pure Higgs boson sector of the Standard model is modelled by a Klein–Gordon field with a potential, denoted \mathbf{H} for this section. The Standard model is a gauge theory and so while the field transforms trivially under the Lorentz group, it transforms as a \mathbf{C}^2 -valued vector under the action of the $\mathbf{SU}(2)$ part of the gauge group. Therefore while it is a vector field $\mathbf{H} : \mathbb{R}^{1,3} \rightarrow \mathbf{C}^2$, it is still referred to as a scalar field, as scalar describes its transformation (formally, representation) under the Lorentz group. This is also discussed below in the scalar chromodynamics section.

The Higgs field is modelled by a potential

$$V(\mathbf{H}) = -m^2\mathbf{H}^\dagger\mathbf{H} + \lambda(\mathbf{H}^\dagger\mathbf{H})^2,$$

which can be viewed as a generalization of the ϕ^4 potential, but has an important difference: it has a circle of minima. This observation is an important one in the theory of spontaneous symmetry breaking in the Standard model.

Conserved U(1) current

The Klein–Gordon equation (and action) for a complex field ψ admits a $\mathbf{U}(1)$ symmetry. That is, under the transformations

$$\begin{aligned}\psi(\mathbf{x}) &\mapsto e^{i\theta}\psi(\mathbf{x}), \\ \bar{\psi}(\mathbf{x}) &\mapsto e^{-i\theta}\bar{\psi}(\mathbf{x}),\end{aligned}$$

the Klein–Gordon equation is invariant, as is the action (see below). By Noether's theorem for fields, corresponding to this symmetry there is a current \mathbf{J}^μ defined as

$$\mathbf{J}^\mu(\mathbf{x}) = \frac{e}{2m} (\bar{\psi}(\mathbf{x})\partial^\mu\psi(\mathbf{x}) - \psi(\mathbf{x})\partial^\mu\bar{\psi}(\mathbf{x})).$$

which satisfies the conservation equation $\partial_\mu\mathbf{J}^\mu(\mathbf{x}) = \mathbf{0}$. The form of the conserved current can be derived systematically by applying Noether's theorem to the $\mathbf{U}(1)$ symmetry. We will not do so here, but simply verify that this current is conserved.

From the Klein–Gordon equation for a complex field $\psi(\mathbf{x})$ of mass \mathbf{M} , written in covariant notation and *mostly plus* signature,

$$(\square + m^2)\psi(\mathbf{x}) = \mathbf{0}$$

and its complex conjugate

$$(\square + m^2)\bar{\psi}(\mathbf{x}) = \mathbf{0}.$$

Multiplying by the left respectively by $\bar{\psi}(\mathbf{x})$ and $\psi(\mathbf{x})$ (and omitting for brevity the explicit \mathbf{x} dependence),

$$\begin{aligned}\bar{\psi}(\square + m^2)\psi &= \mathbf{0}, \\ \psi(\square + m^2)\bar{\psi} &= \mathbf{0}.\end{aligned}$$

Subtracting the former from the latter, we obtain

$$\bar{\psi}\square\psi - \psi\square\bar{\psi} = \mathbf{0},$$

or in index notation,

$$\bar{\psi}\partial_\mu\partial^\mu\psi - \psi\partial_\mu\partial^\mu\bar{\psi} = \mathbf{0}.$$

Applying this to the derivative of the current $\mathbf{J}^\mu(\mathbf{x}) \equiv \psi^*(\mathbf{x})\partial^\mu\psi(\mathbf{x}) - \psi(\mathbf{x})\partial^\mu\psi^*(\mathbf{x})$, one finds

$$\partial_\mu\mathbf{J}^\mu(\mathbf{x}) = \mathbf{0}.$$

This **U(1)** symmetry is a global symmetry, but it can also be gauged to create a local or gauge symmetry: see below scalar QED. The name of gauge symmetry is somewhat misleading: it is really a redundancy, while the global symmetry is a genuine symmetry.

Lagrangian formulation

The Klein–Gordon equation can also be derived by a variational method, arising as the Euler–Lagrange equation of the action

$$S = \int (-\hbar^2 \eta^{\mu\nu} \partial_\mu \bar{\psi} \partial_\nu \psi - M^2 c^2 \bar{\psi} \psi) d^4 x,$$

In natural units, with signature *mostly minus*, the actions take the simple form

Klein–Gordon action for a real scalar field

$$S = \int d^4 x \left(\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 \right)$$

for a real scalar field of mass m , and

Klein–Gordon action for a complex scalar field

$$S = \int d^4 x (\partial^\mu \psi \partial_\mu \bar{\psi} - M^2 \psi \bar{\psi})$$

for a complex scalar field of mass M .

Applying the formula for the stress–energy tensor to the Lagrangian density (the quantity inside the integral), we can derive the stress–energy tensor of the scalar field. It is

$$T^{\mu\nu} = \hbar^2 (\eta^{\mu\alpha} \eta^{\nu\beta} + \eta^{\mu\beta} \eta^{\nu\alpha} - \eta^{\mu\nu} \eta^{\alpha\beta}) \partial_\alpha \bar{\psi} \partial_\beta \psi - \eta^{\mu\nu} M^2 c^2 \bar{\psi} \psi.$$

and in natural units,

$$T^{\mu\nu} = 2\partial^\mu \bar{\psi} \partial^\nu \psi - \eta^{\mu\nu} (\partial^\rho \bar{\psi} \partial_\rho \psi - M^2 \bar{\psi} \psi)$$

By integration of the time–time component T^{00} over all space, one may show that both the positive- and negative-frequency plane-wave solutions can be physically associated with particles with *positive* energy. This is not the case for the Dirac equation and its energy–momentum tensor.^[6]

The stress energy tensor is the set of conserved currents corresponding to the invariance of the Klein–Gordon equation under space-time translations $x^\mu \mapsto x^\mu + c^\mu$. Therefore each component is conserved, that is, $\partial_\mu T^{\mu\nu} = 0$ (this holds only **on-shell**, that is, when the Klein–Gordon equations are satisfied). It follows that the integral of $T^{0\nu}$ over space is a conserved quantity for each ν . These have the physical interpretation of total energy for $\nu = 0$ and total momentum for $\nu = i$ with $i \in \{1, 2, 3\}$.

Non-relativistic limit

Classical field

Taking the non-relativistic limit ($v \ll c$) of a classical Klein–Gordon field $\psi(\mathbf{x}, t)$ begins with the ansatz factoring the oscillatory rest mass energy term,

$$\psi(\mathbf{x}, t) = \phi(\mathbf{x}, t) e^{-\frac{i}{\hbar} mc^2 t} \quad \text{where} \quad \phi(\mathbf{x}, t) = u_E(\mathbf{x}) e^{-\frac{i}{\hbar} E' t}.$$

Defining the kinetic energy $E' = E - mc^2 = \sqrt{m^2 c^4 + c^2 p^2} - mc^2 \approx \frac{p^2}{2m}$, $E' \ll mc^2$ in the non-relativistic limit $v = p/m \ll c$, and hence

$$i\hbar \frac{\partial \phi}{\partial t} = E' \phi \ll mc^2 \phi \quad \text{and} \quad (i\hbar)^2 \frac{\partial^2 \phi}{\partial t^2} = (E')^2 \phi \ll (mc^2)^2 \phi.$$

Applying this yields the non-relativistic limit of the second time derivative of ψ ,

$$\begin{aligned}\frac{\partial\psi}{\partial t} &= \left(-i\frac{mc^2}{\hbar}\phi + \frac{\partial\phi}{\partial t}\right) e^{-\frac{i}{\hbar}mc^2t} \approx -i\frac{mc^2}{\hbar}\phi e^{-\frac{i}{\hbar}mc^2t} \\ \frac{\partial^2\psi}{\partial t^2} &= -\left(i\frac{2mc^2}{\hbar}\frac{\partial\phi}{\partial t} + \left(\frac{mc^2}{\hbar}\right)^2\phi - \frac{\partial^2\phi}{\partial t^2}\right) e^{-\frac{i}{\hbar}mc^2t} \approx -\left(i\frac{2mc^2}{\hbar}\frac{\partial\phi}{\partial t} + \left(\frac{mc^2}{\hbar}\right)^2\phi\right) e^{-\frac{i}{\hbar}mc^2t}\end{aligned}$$

Substituting into the free Klein–Gordon equation, $c^{-2}\partial_t^2\psi = \nabla^2\psi - m^2\psi$, yields

$$-\frac{1}{c^2}\left(i\frac{2mc^2}{\hbar}\frac{\partial\phi}{\partial t} + \left(\frac{mc^2}{\hbar}\right)^2\phi\right) e^{-\frac{i}{\hbar}mc^2t} \approx \left(\nabla^2 - \left(\frac{mc}{\hbar}\right)^2\right)\phi e^{-\frac{i}{\hbar}mc^2t}$$

which (by dividing out the exponential and subtracting the mass term) simplifies to

$$i\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\phi.$$

This is a *classical Schrödinger field*.

Quantum field

The analogous limit of a quantum Klein–Gordon field is complicated by the non-commutativity of the field operator. In the limit $v \ll c$, the creation and annihilation operators decouple and behave as independent quantum Schrödinger fields.

Scalar electrodynamics

There is a way to make the complex Klein–Gordon field ψ interact with electromagnetism in a gauge-invariant way. We can replace the (partial) derivative with the gauge-covariant derivative. Under a local **U(1)** gauge transformation, the fields transform as

$$\begin{aligned}\psi &\mapsto \psi' = e^{i\theta(\mathbf{x})}\psi, \\ \bar{\psi} &\mapsto \bar{\psi}' = e^{-i\theta(\mathbf{x})}\bar{\psi},\end{aligned}$$

where $\theta(\mathbf{x}) = \theta(t, \mathbf{x})$ is a function of spacetime, thus making it a local transformation, as opposed to a constant over all of spacetime, which would be a global **U(1)** transformation. A subtle point is that global transformations can arise as local ones, when the function $\theta(\mathbf{x})$ is taken to be a constant function.

A well-formulated theory should be invariant under such transformations. Precisely, this means that the equations of motion and action (see below) are invariant. To achieve this, ordinary derivatives ∂_μ must be replaced by gauge-covariant derivatives D_μ , defined as

$$\begin{aligned}D_\mu\psi &= (\partial_\mu - ieA_\mu)\psi \\ D_\mu\bar{\psi} &= (\partial_\mu + ieA_\mu)\bar{\psi}\end{aligned}$$

where the 4-potential or gauge field A_μ transforms under a gauge transformation θ as

$$A_\mu \mapsto A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\theta.$$

With these definitions, the covariant derivative transforms as

$$D_\mu\psi \mapsto e^{i\theta}D_\mu\psi$$

In natural units, the Klein–Gordon equation therefore becomes

$$D_\mu D^\mu\psi - M^2\psi = 0.$$

Since an *ungauged* **U(1)** symmetry is only present in complex Klein–Gordon theory, this coupling and promotion to a *gauged* **U(1)** symmetry is compatible only with complex Klein–Gordon theory and not real Klein–Gordon theory.

In natural units and mostly minus signature we have

Scalar QED action

$$S = \int d^4x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + D^\mu \psi D_\mu \bar{\psi} - M^2 \psi \bar{\psi} \right)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is known as the Maxwell tensor, field strength or curvature depending on viewpoint.

This theory is often known as scalar quantum electrodynamics or scalar QED, although all aspects we've discussed here are classical.

Scalar chromodynamics

It is possible to extend this to a non-abelian gauge theory with a gauge group G , where we couple the scalar Klein–Gordon action to a Yang–Mills Lagrangian. Here, the field is actually vector-valued, but is still described as a scalar field: the scalar describes its transformation under space-time transformations, but not its transformation under the action of the gauge group.

For concreteness we fix G to be $SU(N)$, the special unitary group for some $N \geq 2$. Under a gauge transformation $U(x)$, which can be described as a function $U : \mathbb{R}^{1,3} \rightarrow SU(N)$, the scalar field ψ transforms as a \mathbb{C}^N vector

$$\begin{aligned} \psi(x) &\mapsto U(x)\psi(x) \\ \psi^\dagger(x) &\mapsto \psi^\dagger(x)U^\dagger(x). \end{aligned}$$

The covariant derivative is

$$\begin{aligned} D_\mu \psi &= \partial_\mu \psi - igA_\mu \psi \\ D_\mu \psi^\dagger &= \partial_\mu \psi^\dagger + ig\psi^\dagger A_\mu^\dagger \end{aligned}$$

where the gauge field or connection transforms as

$$A_\mu \mapsto UA_\mu U^{-1} - \frac{i}{g} \partial_\mu U U^{-1}.$$

This field can be seen as a matrix valued field which acts on the vector space \mathbb{C}^N .

Finally defining the chromomagnetic field strength or curvature,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g(A_\mu A_\nu - A_\nu A_\mu),$$

we can define the action.

Scalar QCD action

$$S = \int d^4x \left(-\frac{1}{4} \text{Tr}(F^{\mu\nu} F_{\mu\nu}) + D^\mu \psi^\dagger D_\mu \psi - M^2 \psi^\dagger \psi \right)$$

Klein–Gordon on curved spacetime

In general relativity, we include the effect of gravity by replacing partial derivatives with covariant derivatives, and the Klein–Gordon equation becomes (in the mostly pluses signature)^[15]

$$\begin{aligned} 0 &= -g^{\mu\nu} \nabla_\mu \nabla_\nu \psi + \frac{m^2 c^2}{\hbar^2} \psi = -g^{\mu\nu} \nabla_\mu (\partial_\nu \psi) + \frac{m^2 c^2}{\hbar^2} \psi \\ &= -g^{\mu\nu} \partial_\mu \partial_\nu \psi + g^{\mu\nu} \Gamma^\sigma_{\mu\nu} \partial_\sigma \psi + \frac{m^2 c^2}{\hbar^2} \psi, \end{aligned}$$

or equivalently,

$$\frac{-1}{\sqrt{-g}} \partial_\mu \left(g^{\mu\nu} \sqrt{-g} \partial_\nu \psi \right) + \frac{m^2 c^2}{\hbar^2} \psi = 0,$$

where $g^{\alpha\beta}$ is the inverse of the metric tensor that is the gravitational potential field, g is the determinant of the metric tensor, ∇_μ is the covariant derivative, and $\Gamma^\sigma_{\mu\nu}$ is the Christoffel symbol that is the gravitational force field.

With natural units this becomes

Klein–Gordon equation on curved spacetime for a real scalar field

$$\nabla^a \nabla_a \Phi - m^2 \Phi = 0$$

This also admits an action formulation on a spacetime (Lorentzian) manifold M . Using abstract index notation and in *mostly plus* signature this is

Klein–Gordon action on curved spacetime for a real scalar field

$$S = \int_M d^4x \sqrt{-g} \left(-\frac{1}{2} g^{ab} \nabla_a \Phi \nabla_b \Phi - \frac{1}{2} m^2 \Phi^2 \right)$$

or

Klein–Gordon action on curved spacetime for a complex scalar field

$$S = \int_M d^4x \sqrt{-g} \left(-g^{ab} \nabla_a \Psi \nabla_b \bar{\Psi} - M^2 \Psi \bar{\Psi} \right)$$

See also

- [Quantum field theory](#)
- [Quartic interaction](#)
- [Relativistic wave equations](#)
- [Dirac equation](#) (spin 1/2)
- [Proca action](#) (spin 1)
- [Rarita–Schwinger equation](#) (spin 3/2)
- [Scalar field theory](#)
- [Sine–Gordon equation](#)

Remarks

1. common spinless particles like the [pions](#) are unstable and also experience the strong interaction (with unknown interaction term in the Hamiltonian)
2. The [Sine-Gordon equation](#) is an important example of an [Integrable system](#)
3. To reconcile quantum mechanics with special relativity a multiple particle theory and therefore [quantum field theory](#) is needed, in which the Klein–Gordon equation reemerges as the equation obeyed by the components of all free quantum fields.

Steven Weinberg makes a point about this. He leaves out the treatment of relativistic wave mechanics altogether in his otherwise complete introduction to modern applications of quantum mechanics, explaining: "It seems to me that the way this is usually presented in books on quantum mechanics is profoundly misleading." (From the preface in *Lectures on Quantum Mechanics*, referring to treatments of the Dirac equation in its original flavor.)

Others, like Walter Greiner does in his series on theoretical physics, give a full account of the historical development and view of [relativistic quantum mechanics](#) before they get to the modern interpretation, with the rationale that it is highly desirable or even necessary from a pedagogical point of view to take the long route. In quantum field theory, the solutions of the free (noninteracting) versions of the original equations still play a role. They are needed to build the Hilbert space ([Fock space](#)) and to express quantum fields by using complete sets (spanning sets of Hilbert space) of wave functions.

Notes

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External links

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- Introduction to nonlocal equations (http://www.ma.utexas.edu/mediawiki/index.php/Introduction_to_nonlocal_equations).

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