The relaxation spectrum of interacting particle systems

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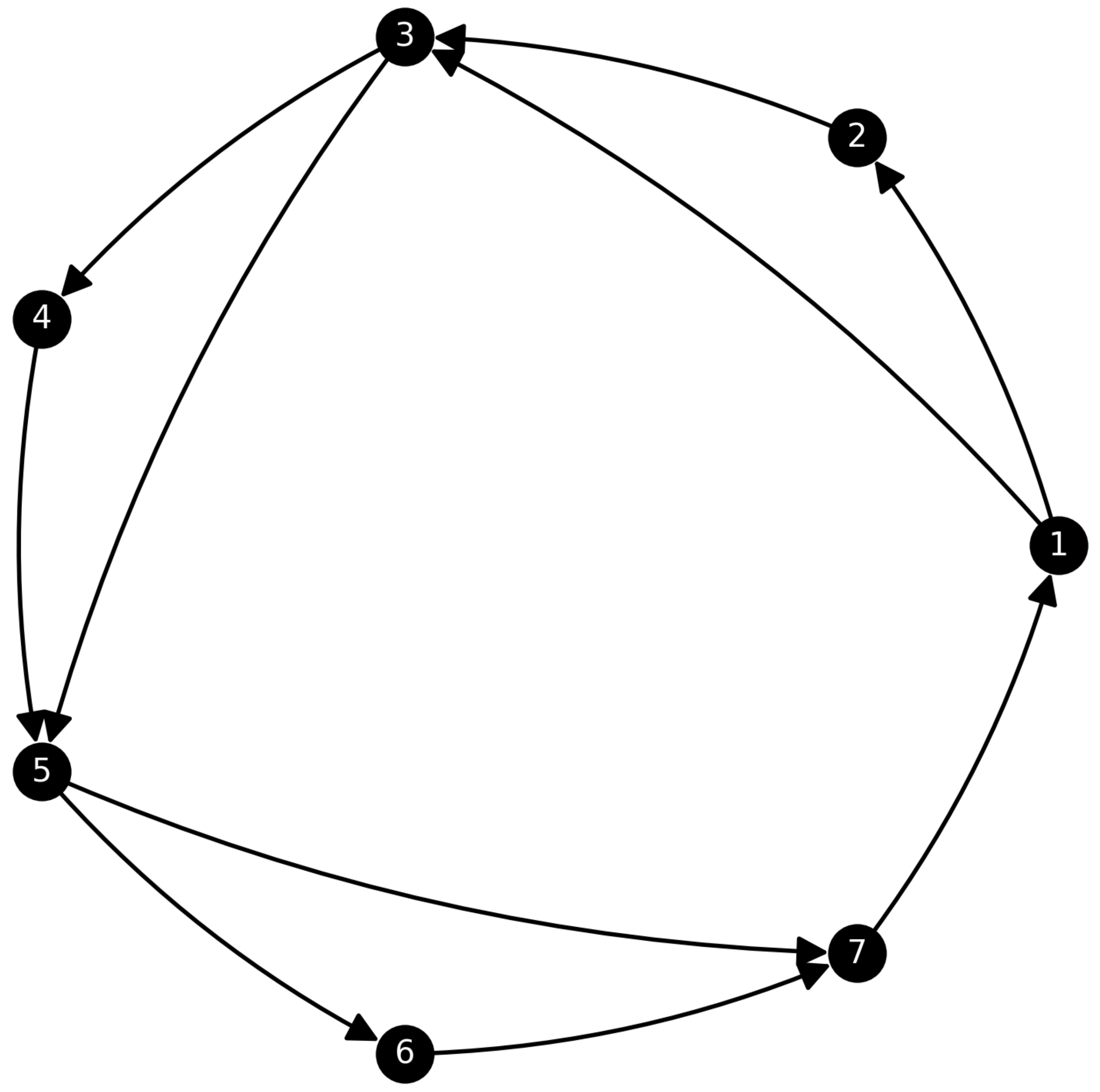
Defining the relaxation spectrum

A master equation formalism

- Consider a master equation defined by $\partial_t |P\rangle = M|P\rangle$,
- Probability vector $|P\rangle = \left(P_1, P_2, \ldots\right)$.



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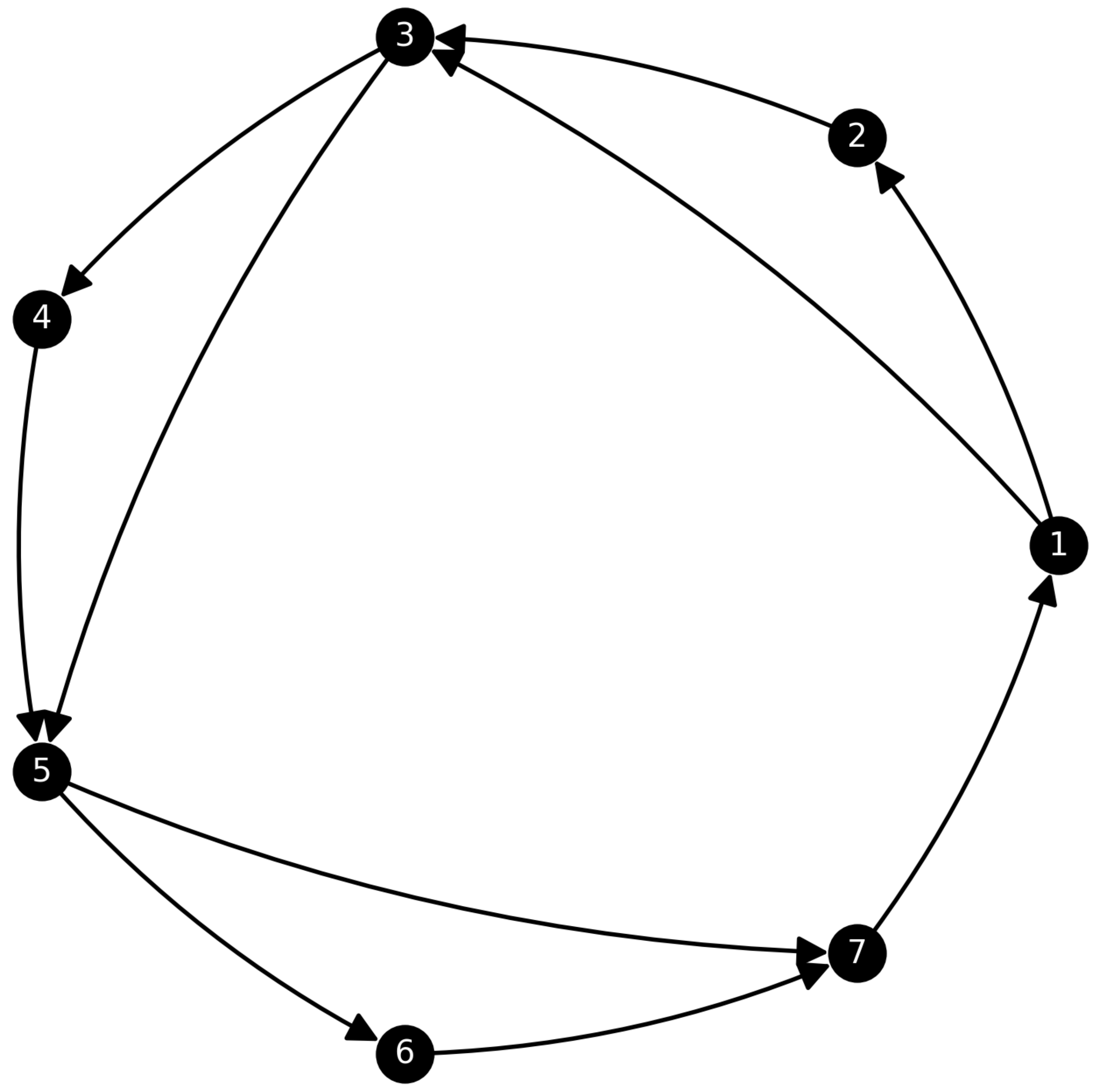

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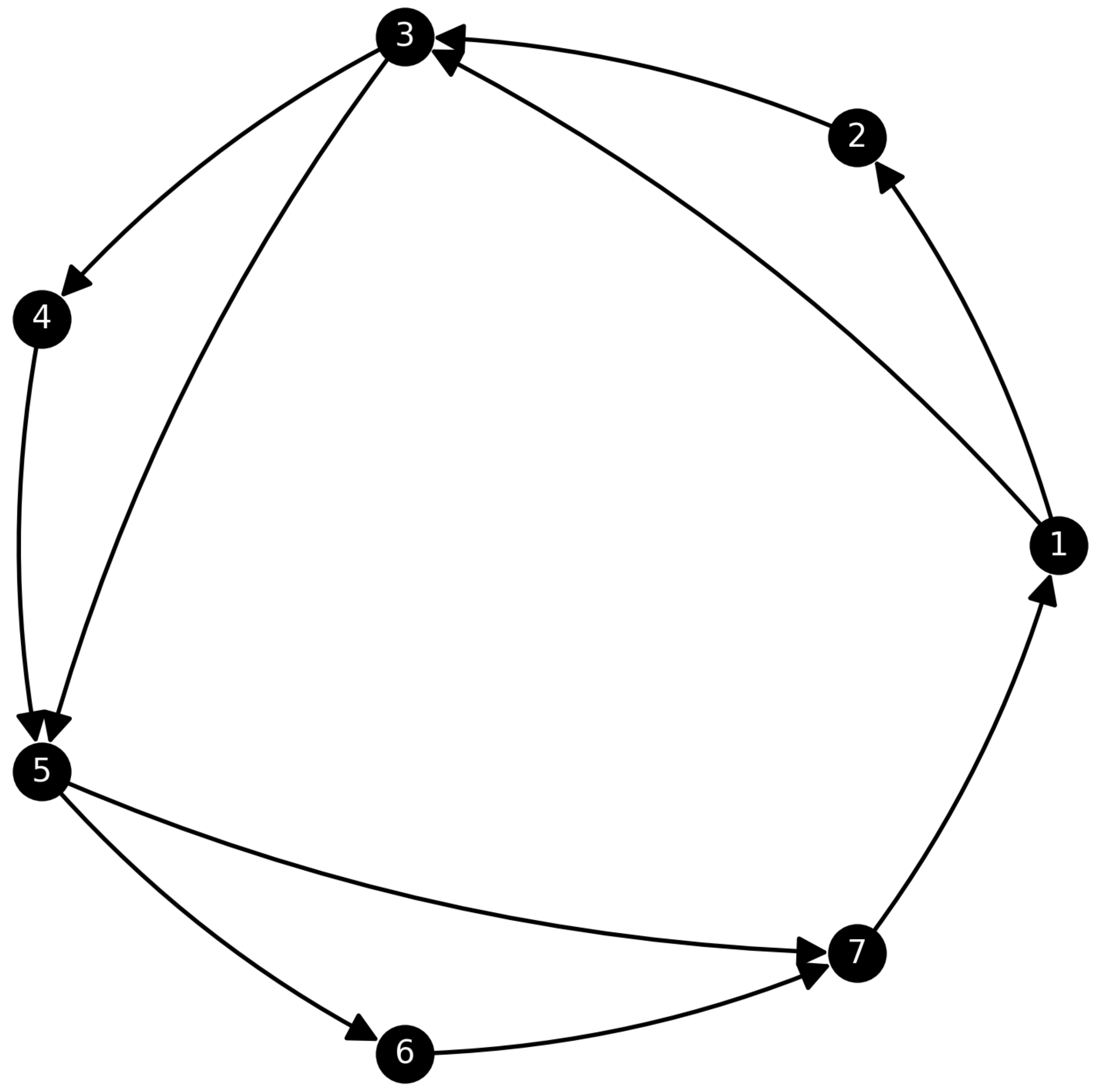

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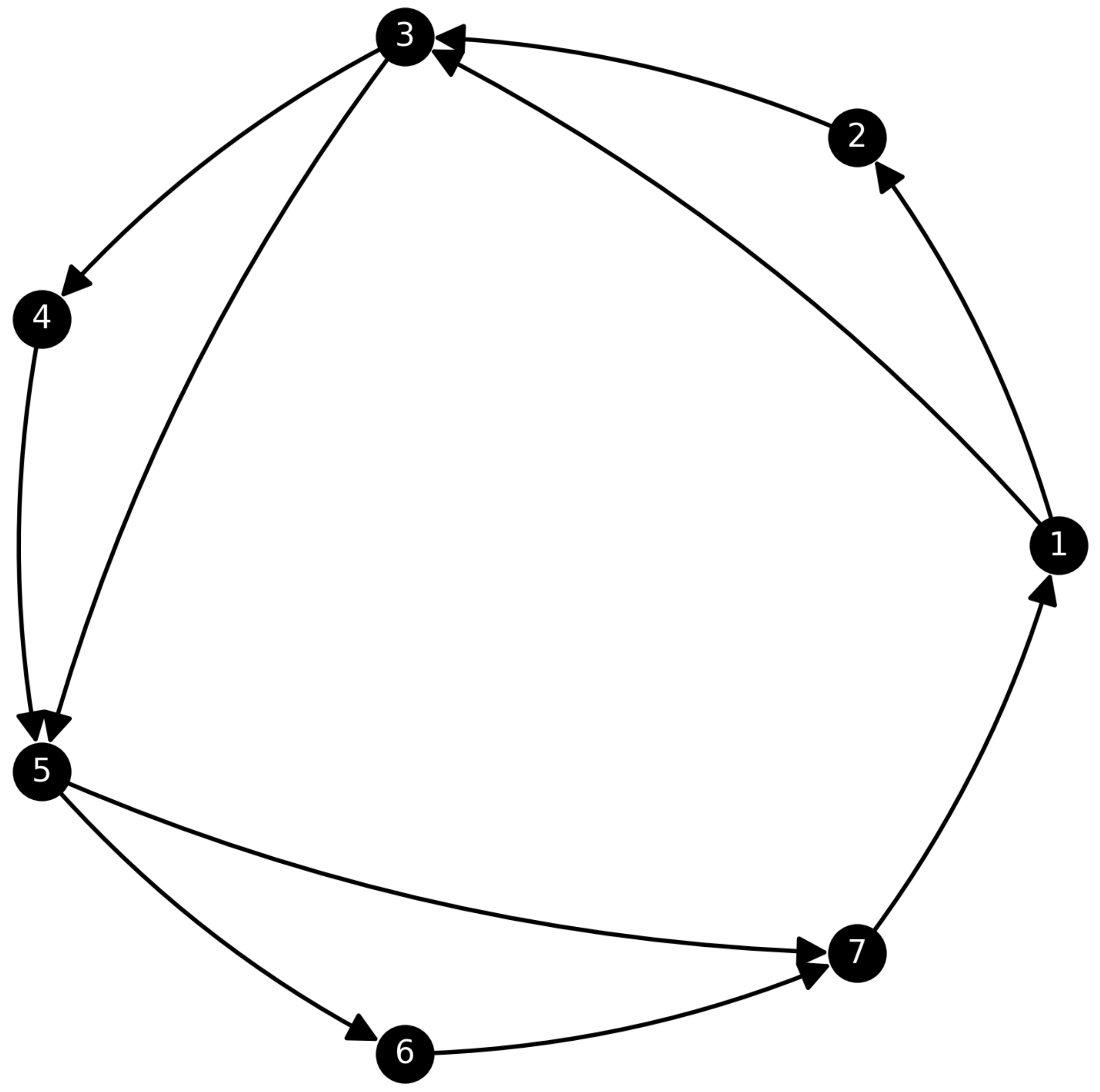

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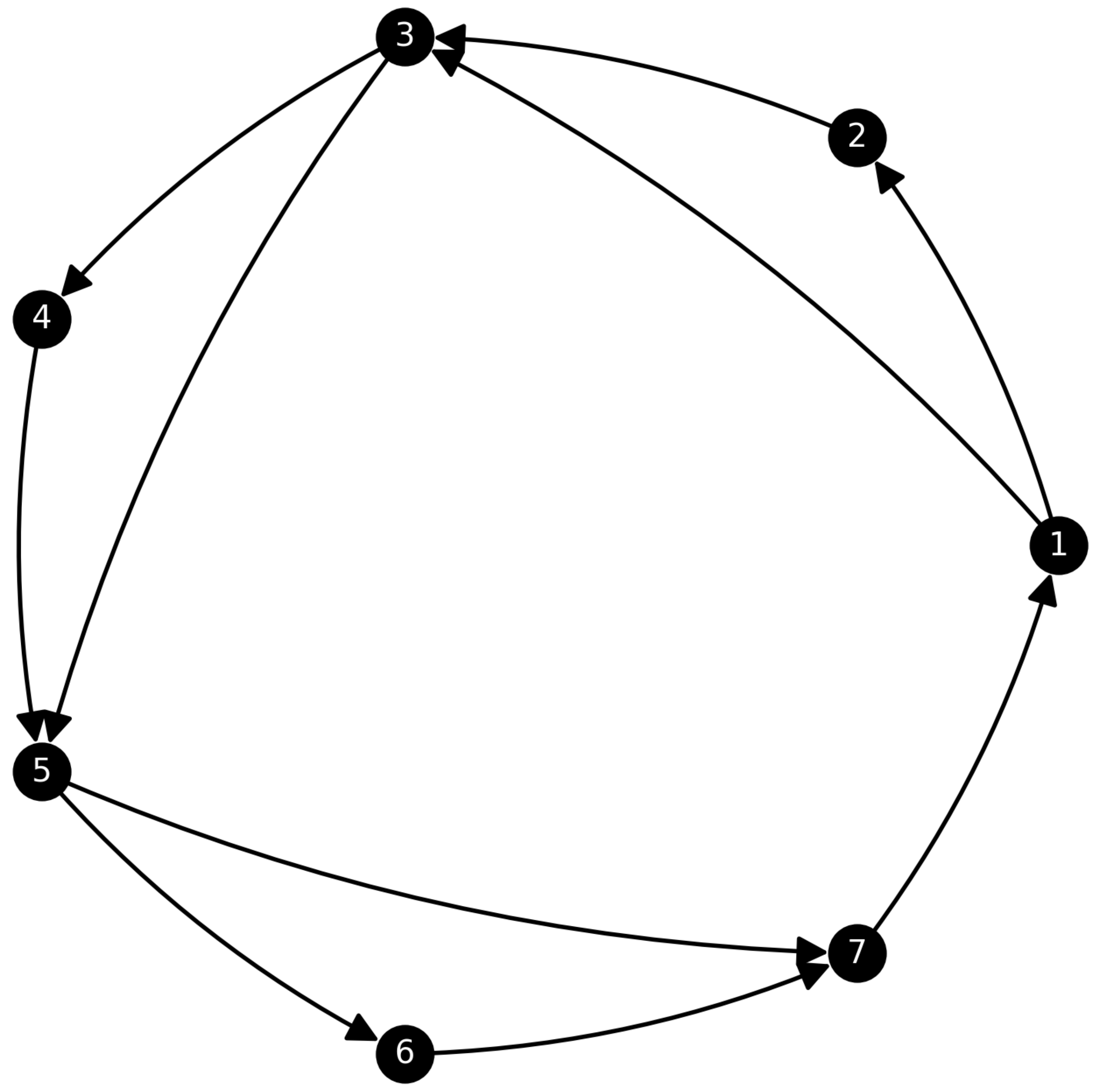
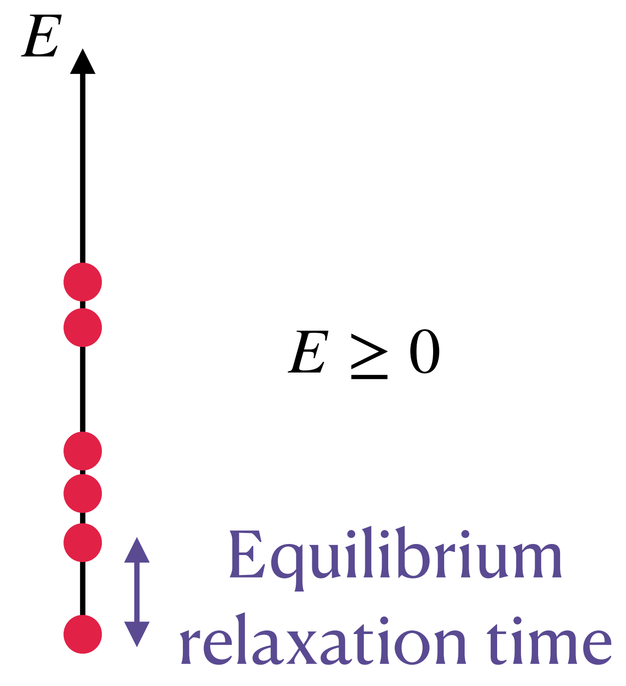
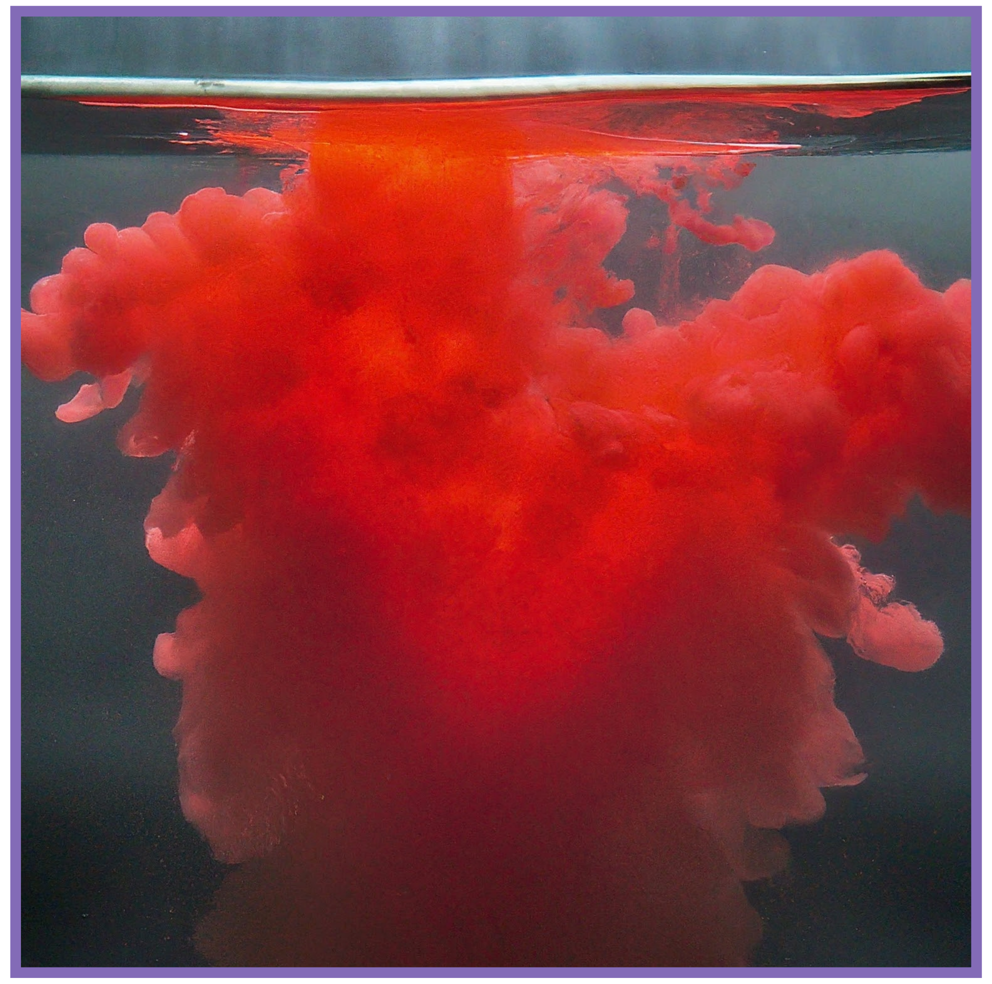


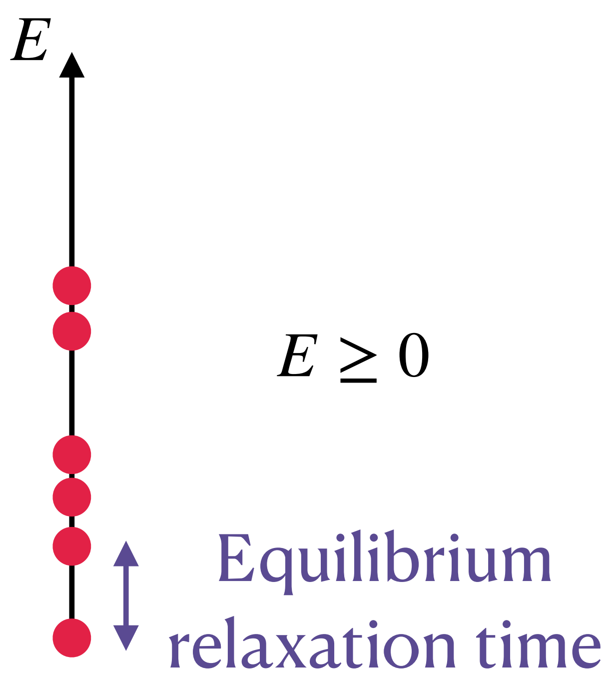


Motivation
why should we care to do it?

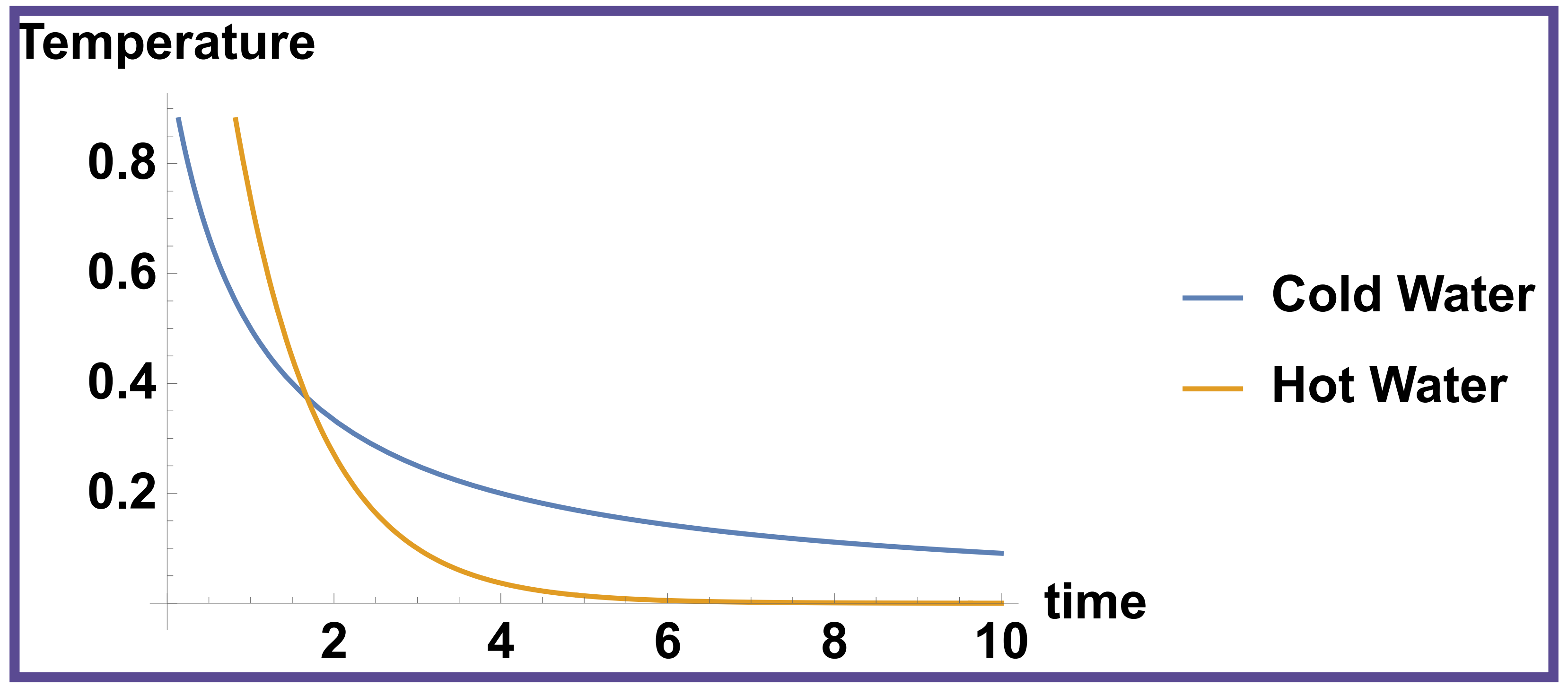
Motivation

Thermalization

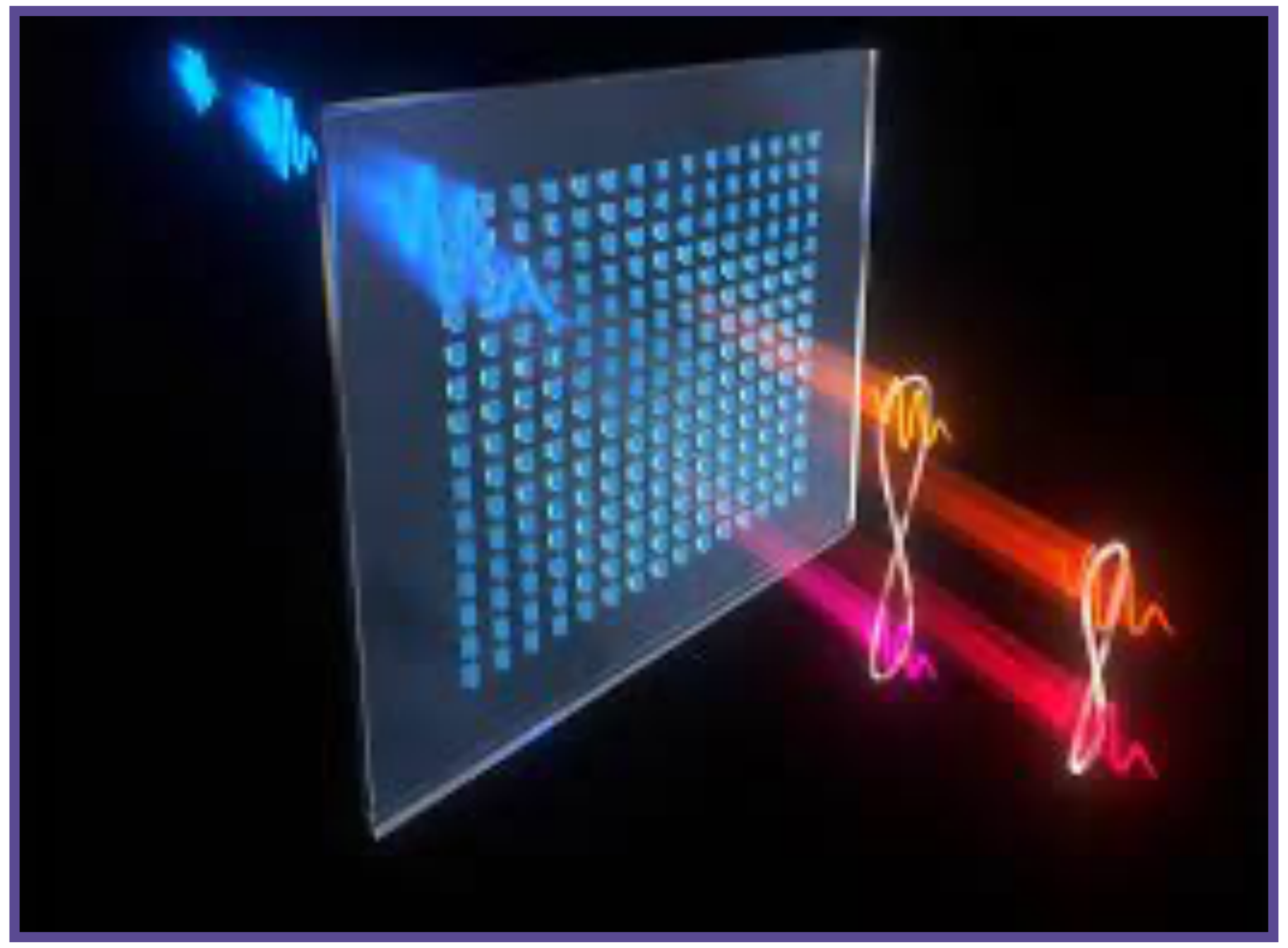




Cooling strategies (Mpemba effect)



(Interaction induced) metastable state engineering

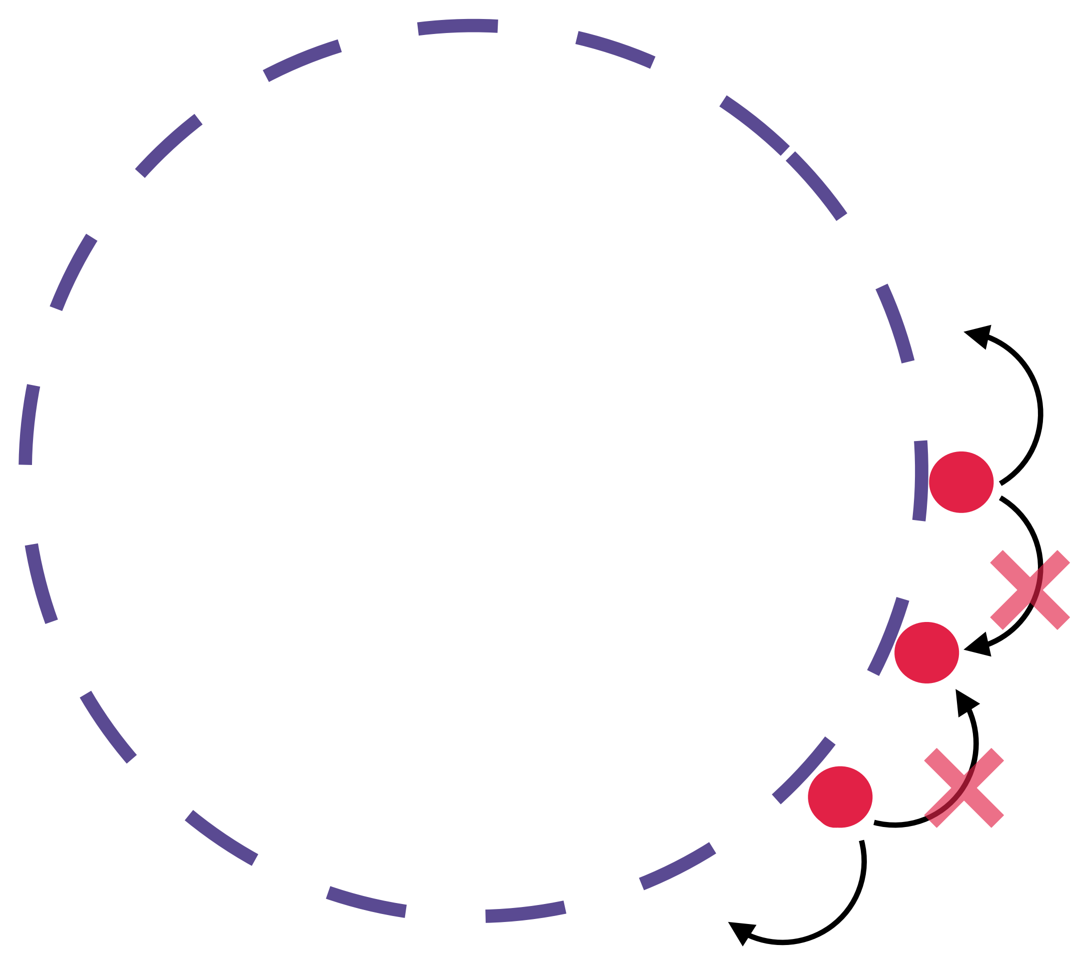


What's the challenge?

Handling many-body systems is hard

<u>The simple exclusion process</u>:

- Each lattice site is either occupied or not
- Particles jump to nearest vacant neighbors with some rate
- Process may not be symmetric, or homogenous.



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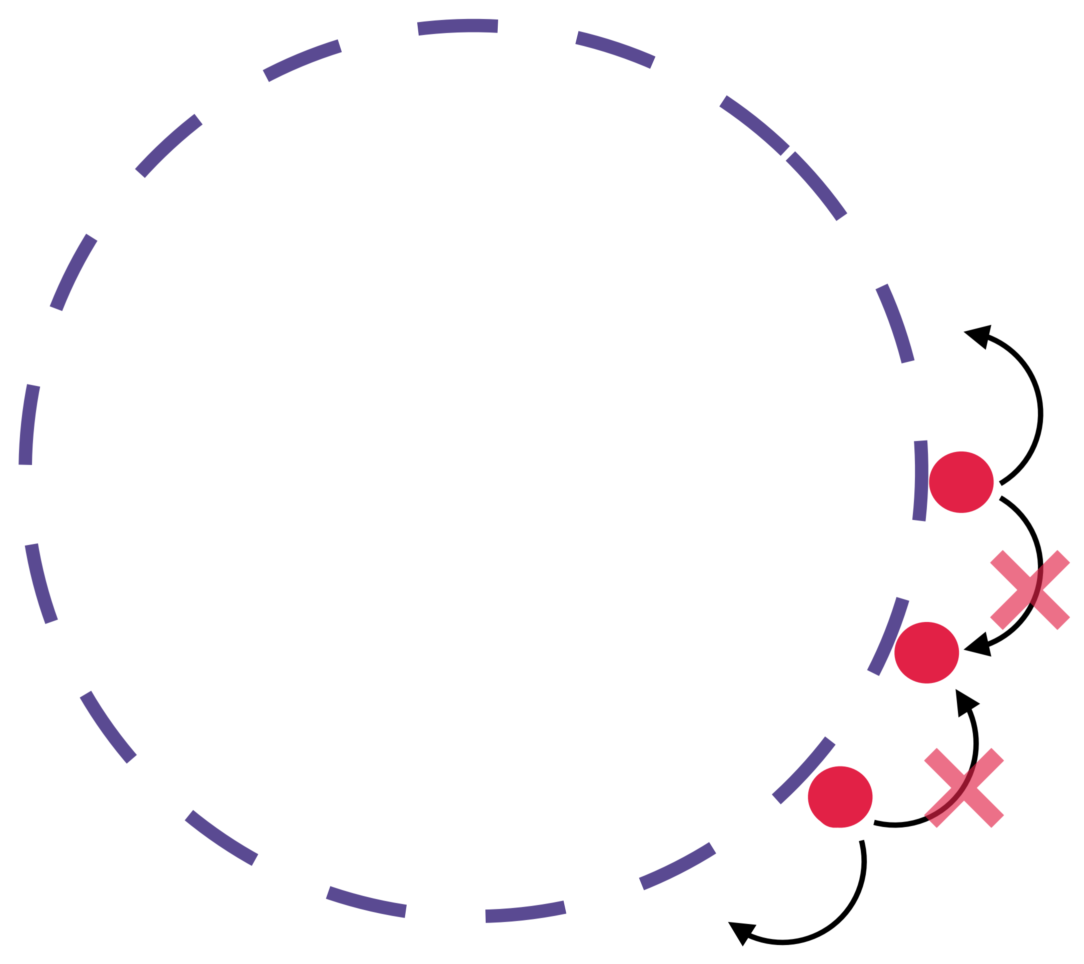

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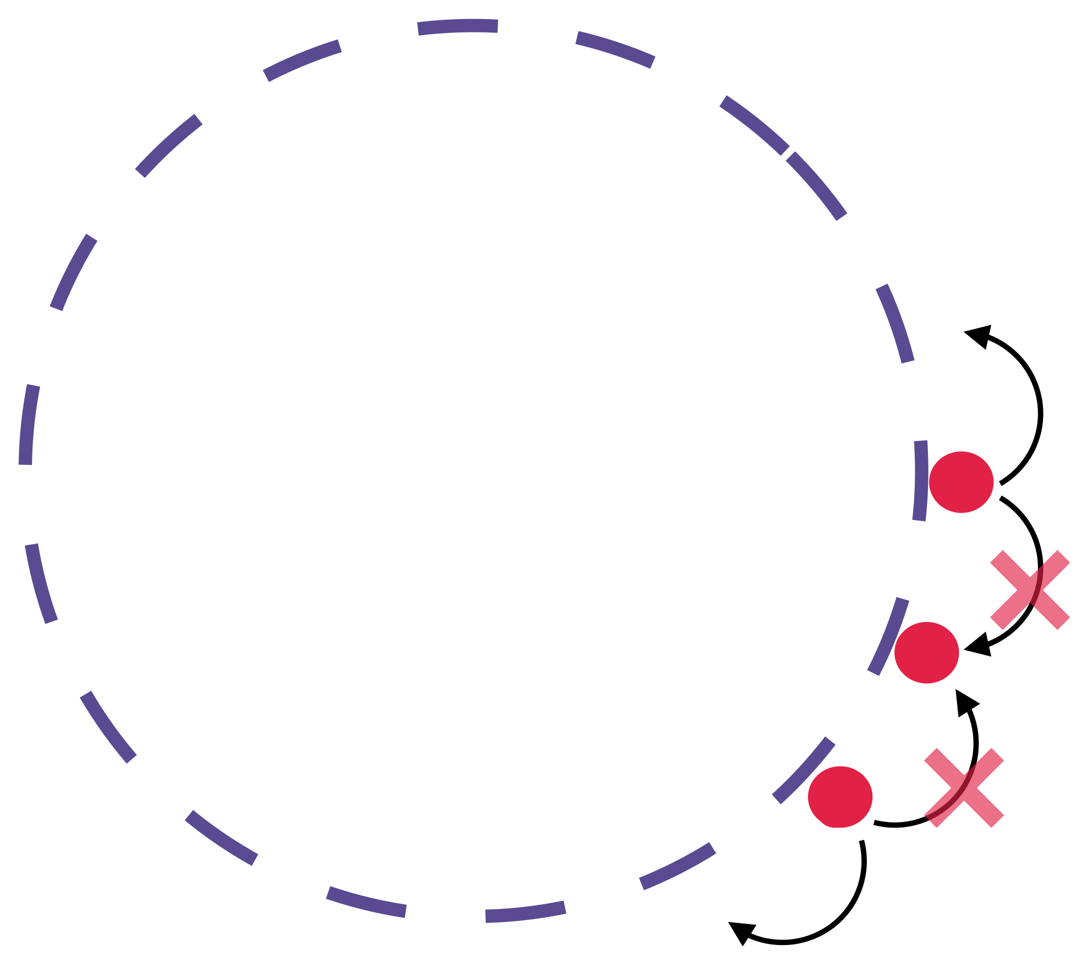

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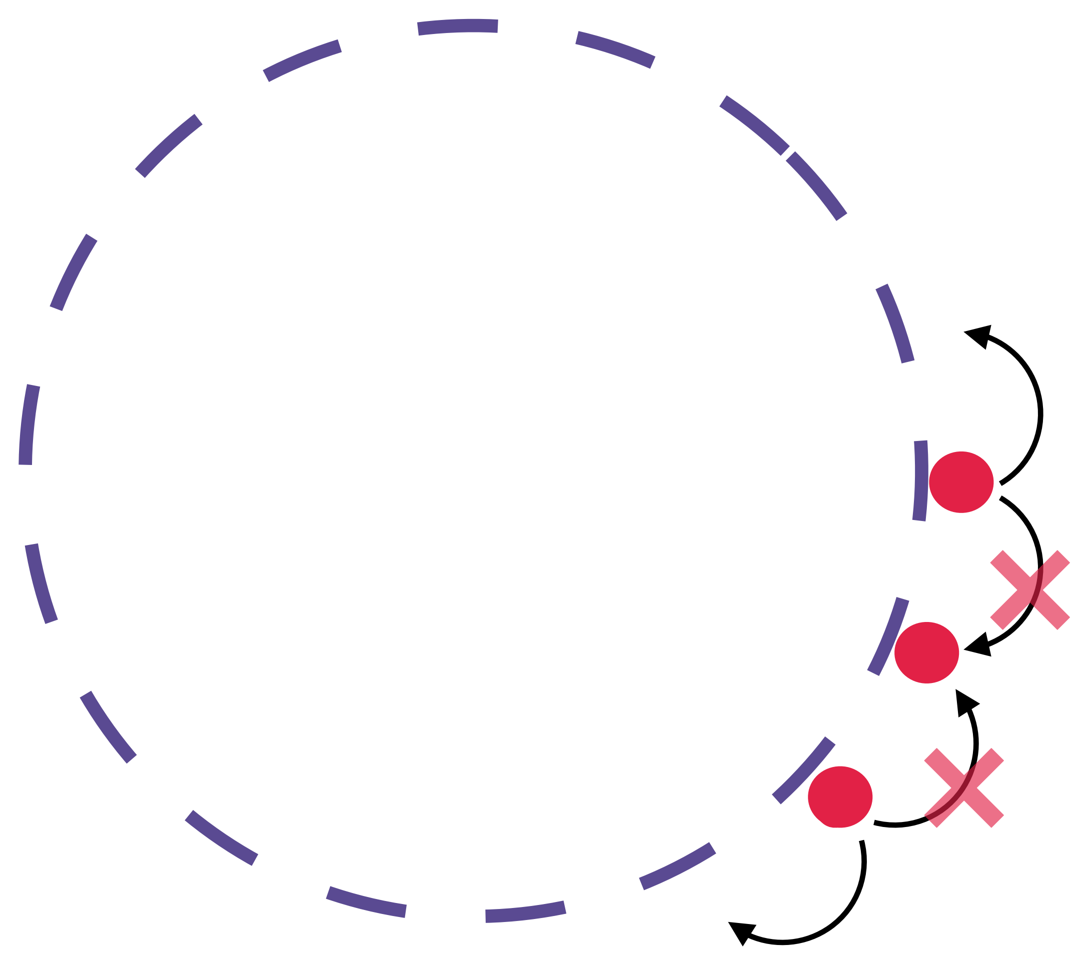

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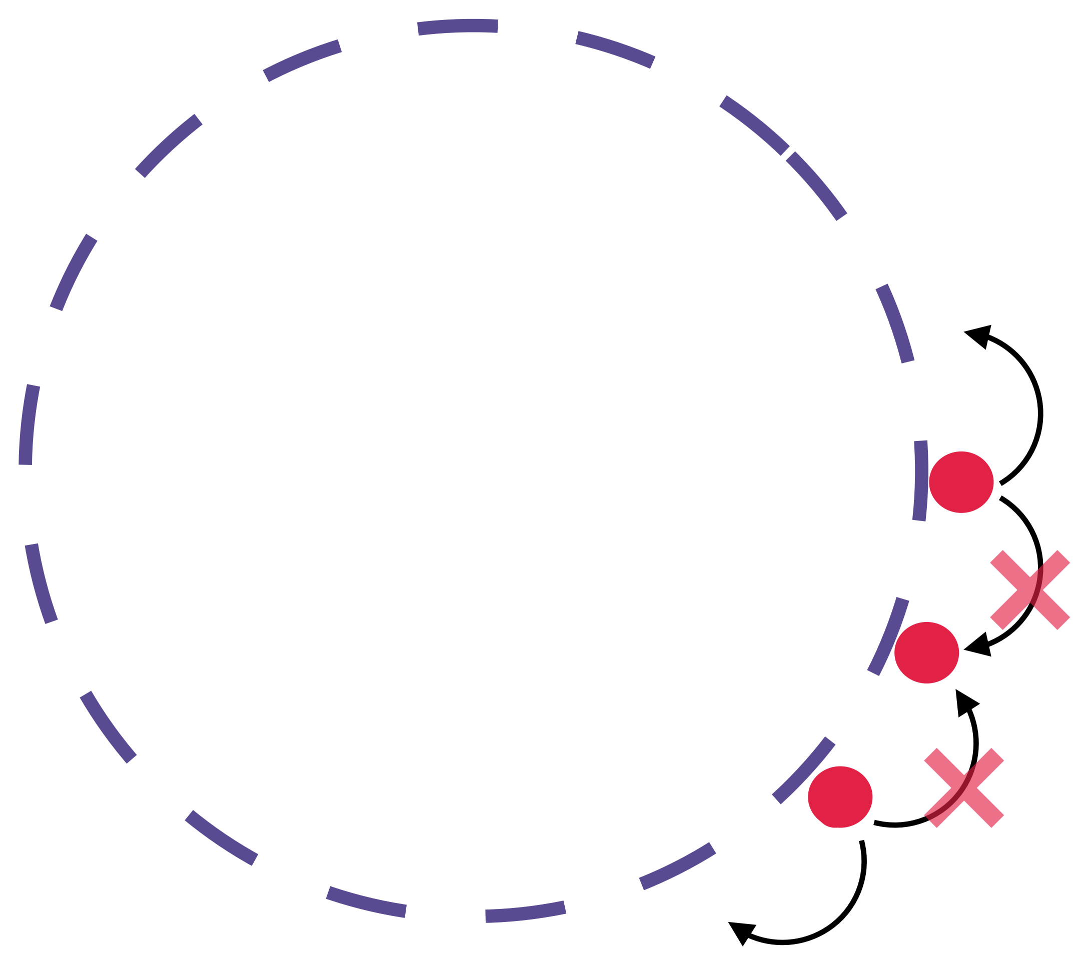

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- Even numerically, we need a trick!

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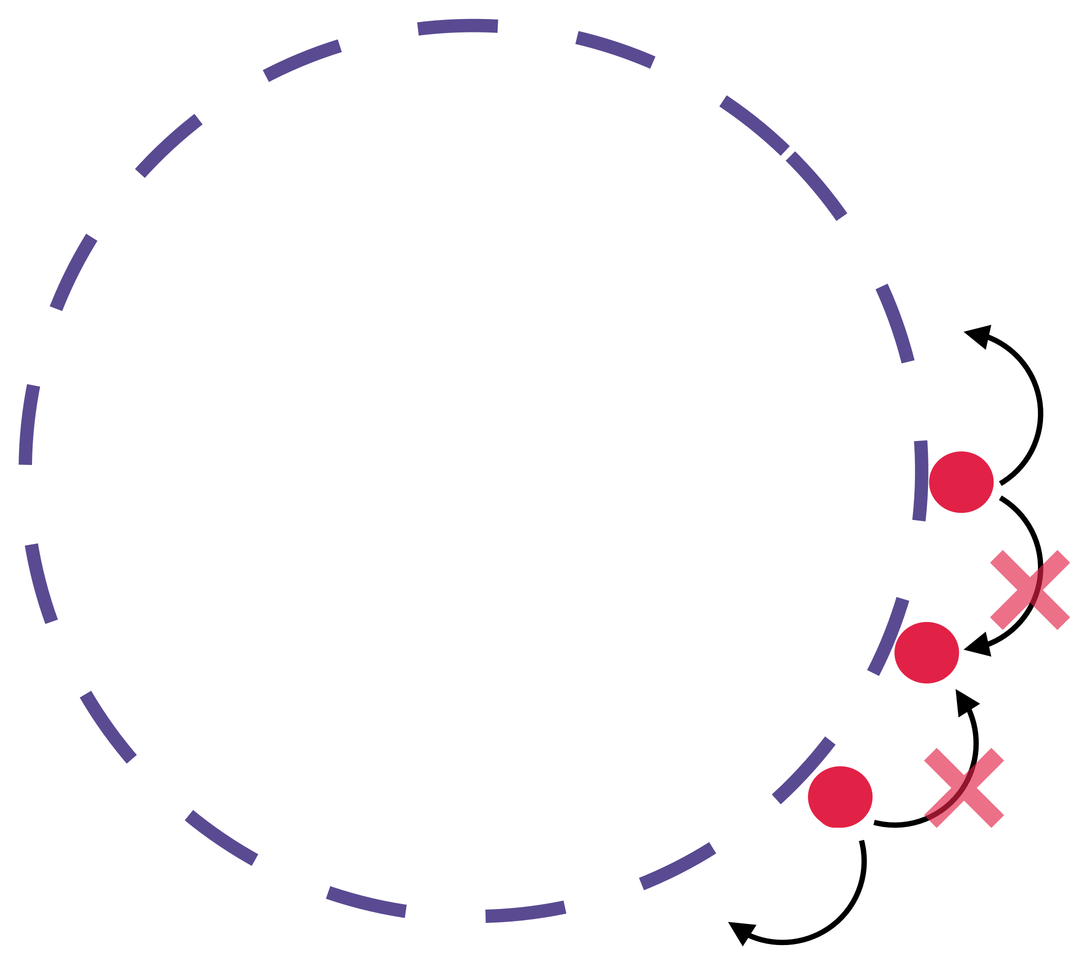
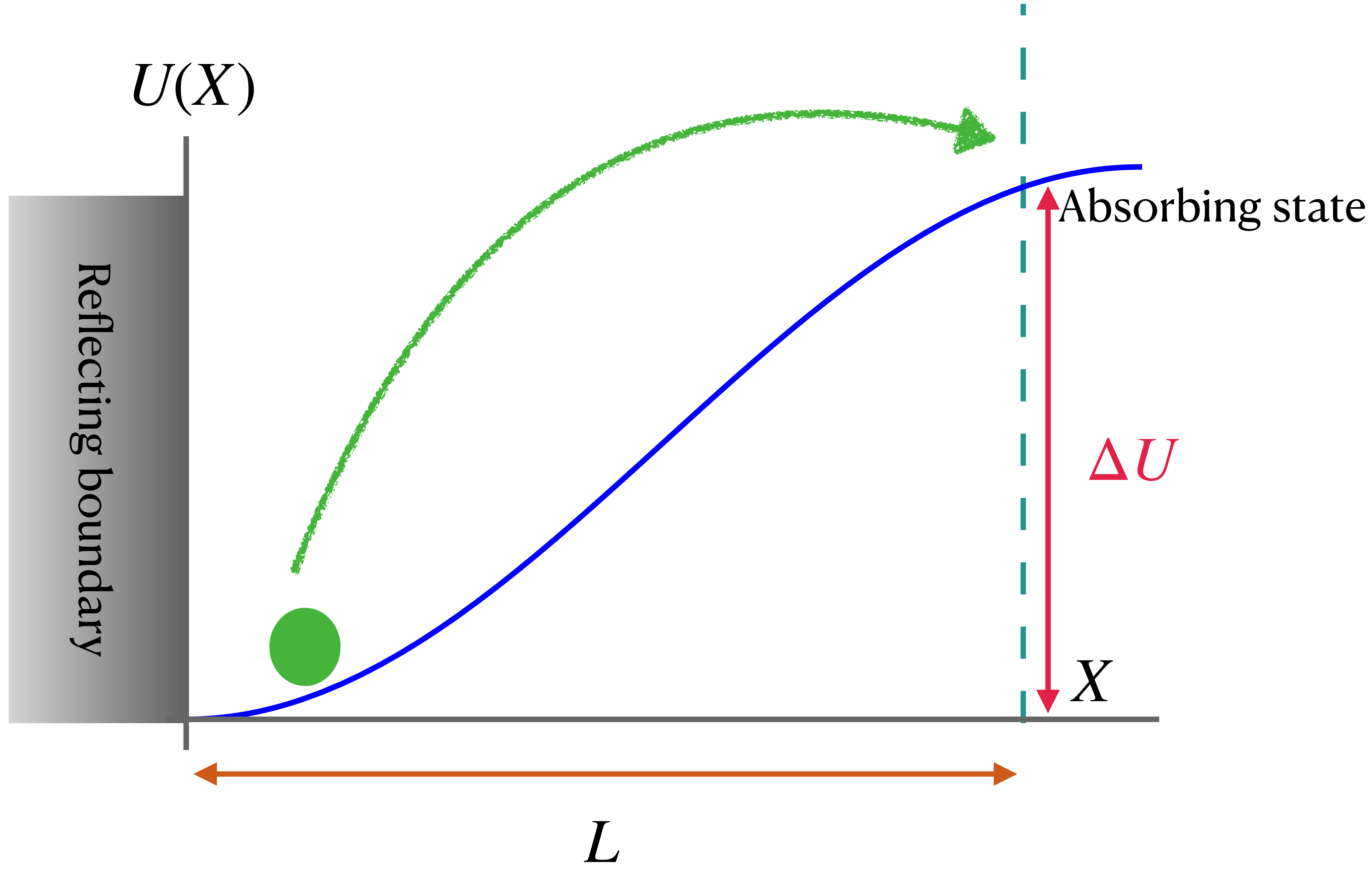


Can we do it then?

Escape time of interacting systems in a deep trap

Finding the escape time of a particle from a deep trap is equivalent to finding the first excited energy in the relaxation spectrum



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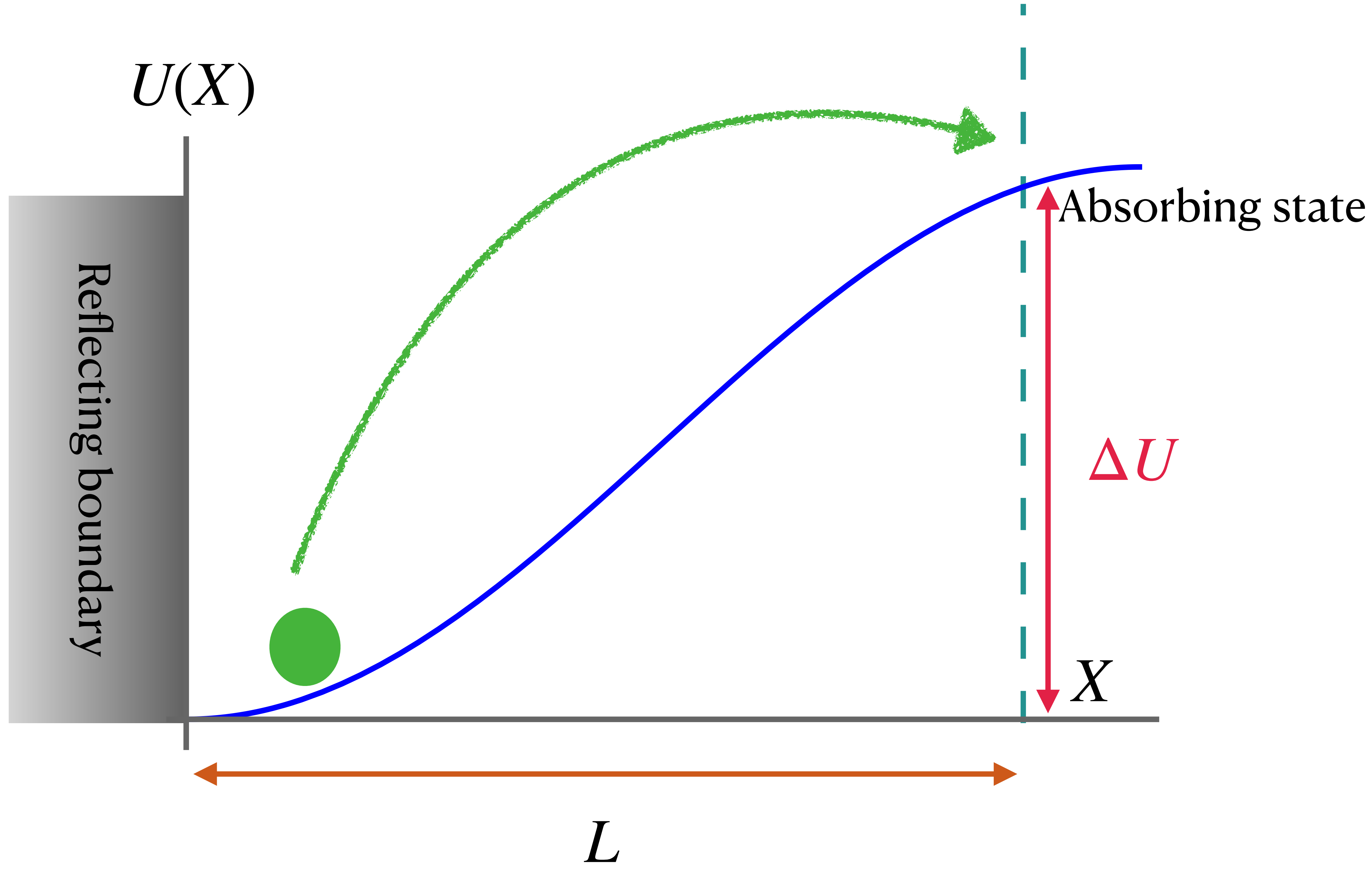
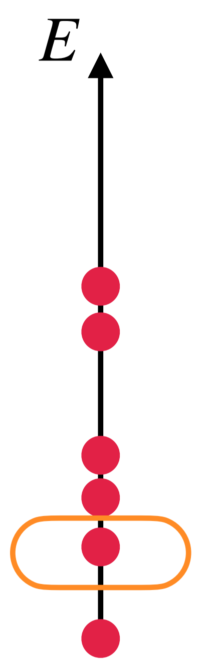
We were able to capture the escape time using the macroscopic fluctuation theory, for diffusive systems.





Kumar, Pal, and OS, *PRE* 24'
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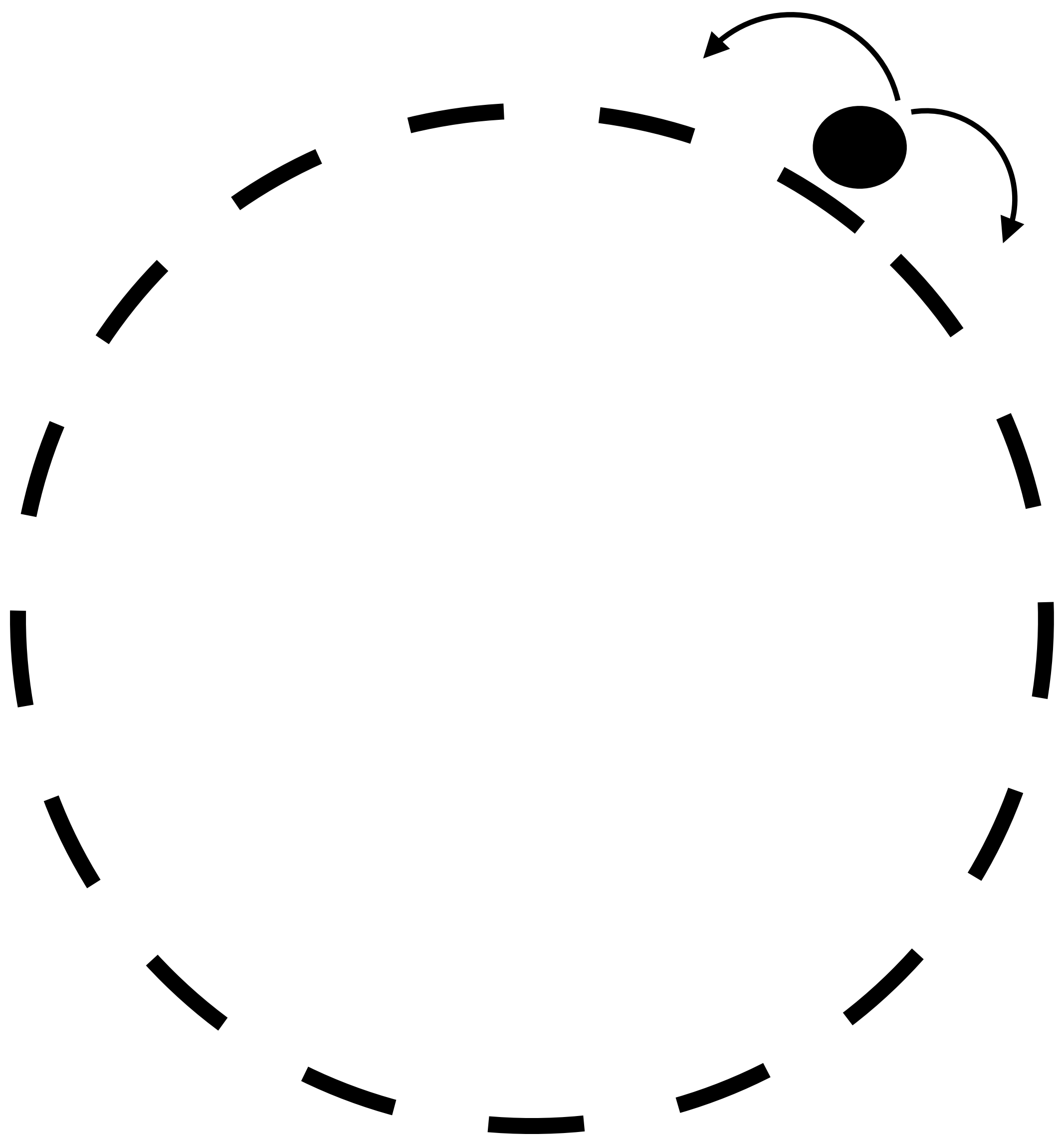
What can and cannot be done with hydrodynamics

An over damped particle on a ring

Lattice model

- A jump process of a particle on a periodic lattice, with unbiased jump rates.
- The master equation $\partial_t P_i(t) = P_{i-1}(t) - 2P_i(t) + P_{i+1}(t)$, with $P_i(t) = P_{i+L}(t)$.
- The relaxation spectrum $E = 2 - 2\cos(2\pi n/L), n \in \mathbb{Z}$



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Continuum Fokker-Planck eq.

- Fokker-Planck formalism $\partial_t P(x, t) = D\partial_{xx} P(x, t)$, with boundary conditions $P(x, t) = P(x + L, t)$
- The relaxation spectrum $E = (2\pi n/L)^2$ for $n \in \mathbb{Z}$

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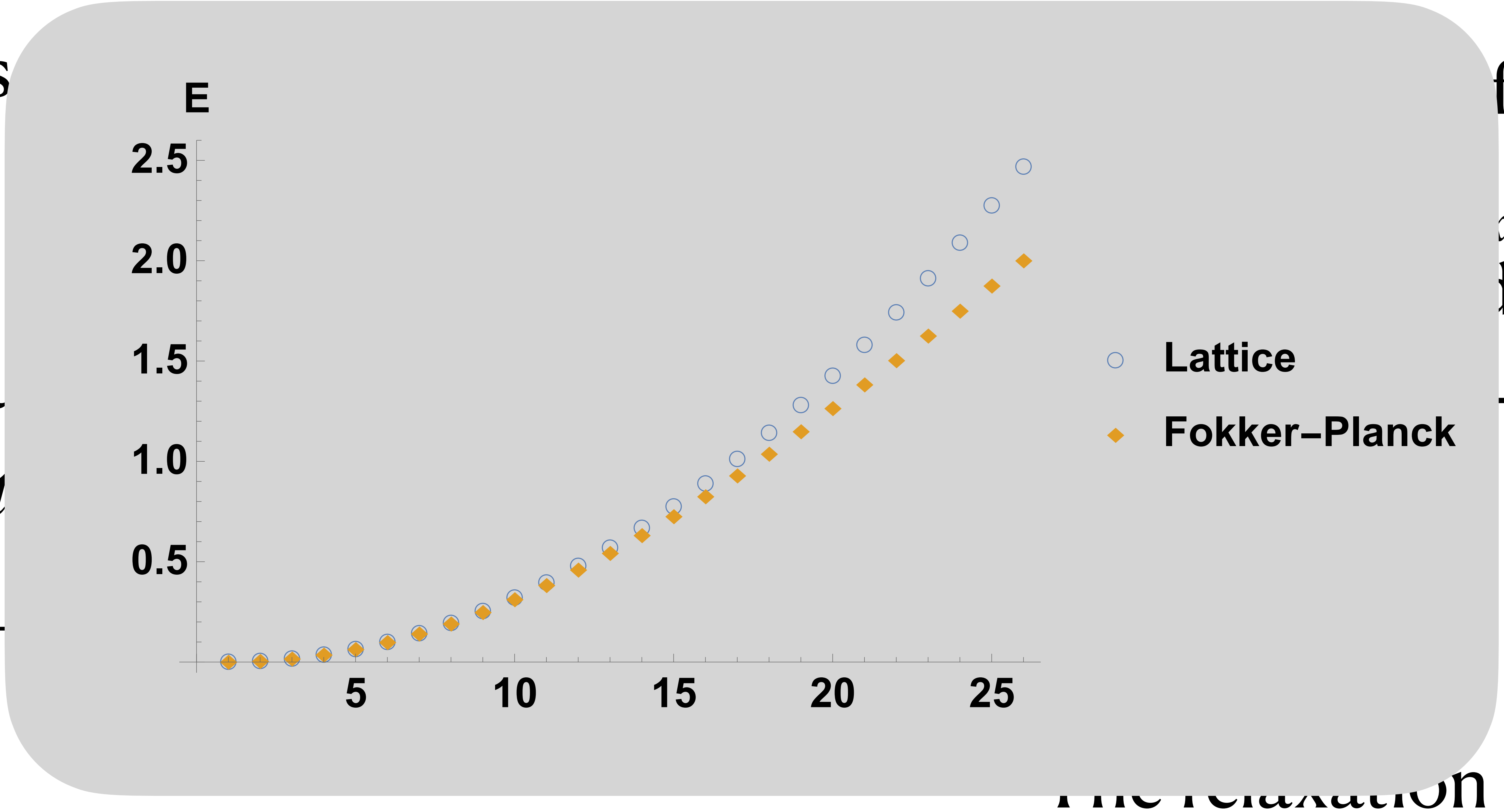

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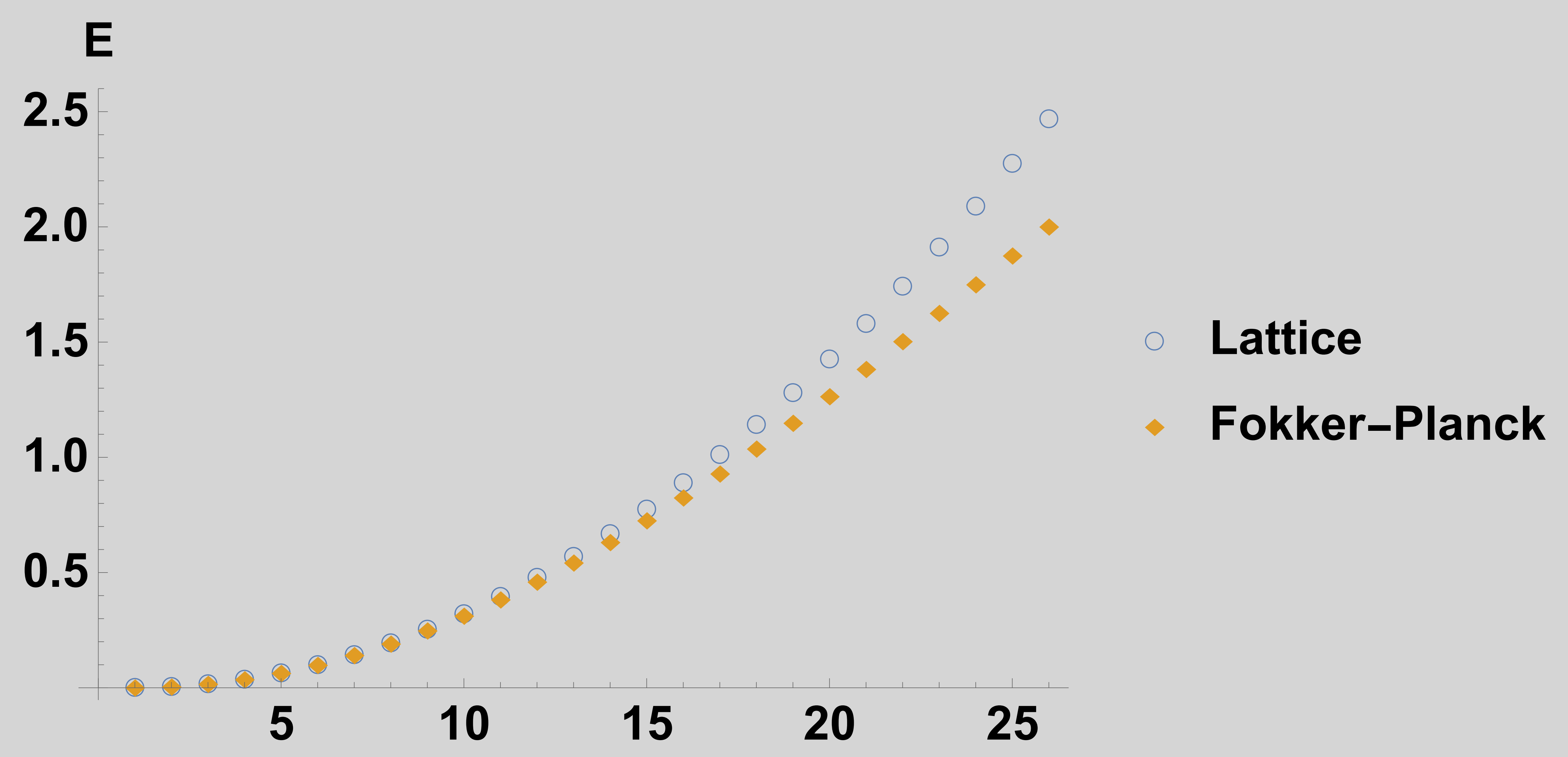

The relaxation spectrum

- The rel

$E = 2$

Fast modes are traced out

only the long time scales survive the hydrodynamic limit

A different formalism
Path probability

Path probability

Instead of Fokker-Planck, go over all paths

(*) $P(x_f, t) = \int dx_i \, P(x_i, 0) \mathbb{T}(x_i \to x_f; t)$

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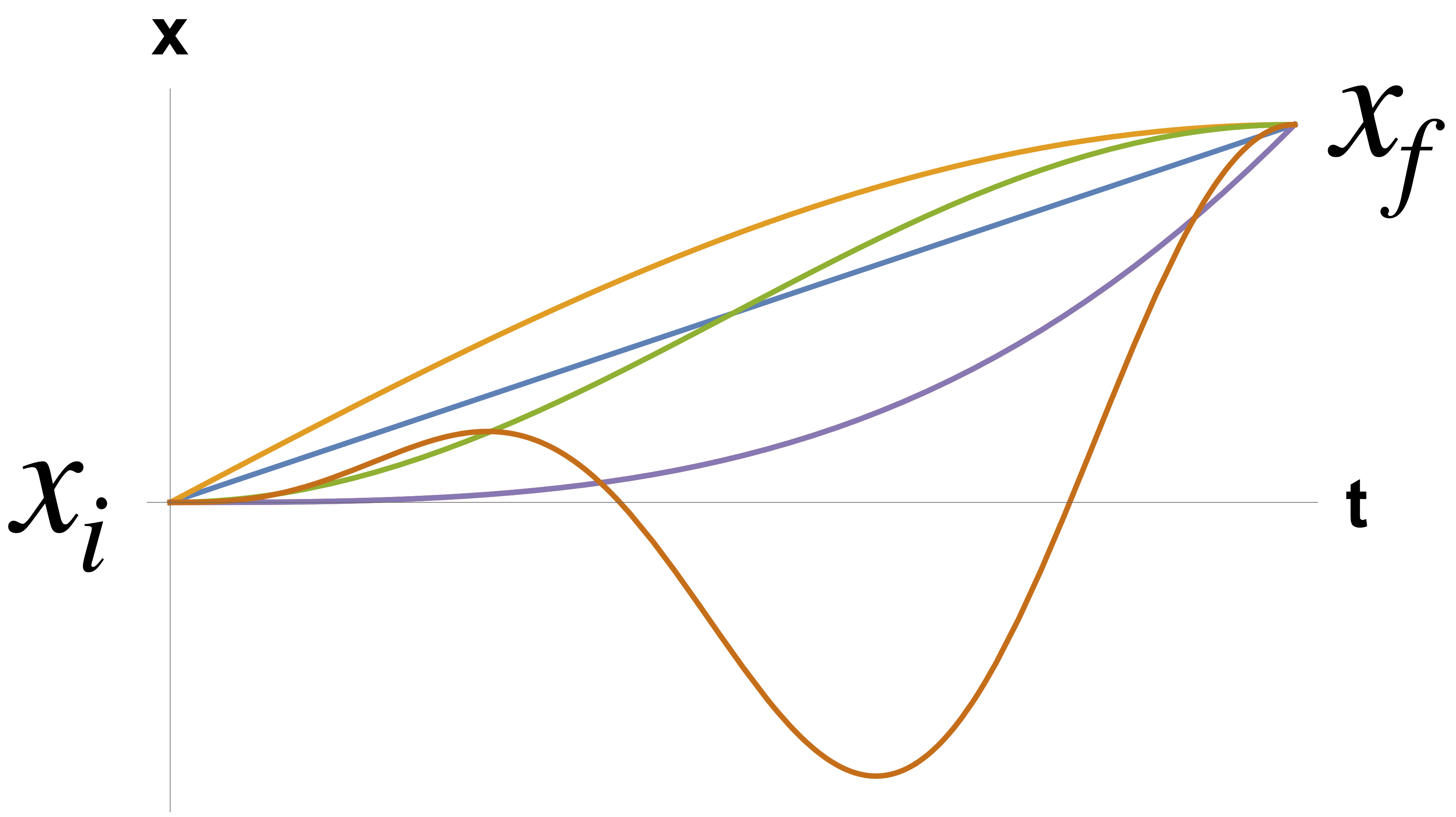

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$$S = \frac{1}{D} \int dt \, \hat{p}\dot{\hat{x}} - H(\hat{x}, \hat{p}), \qquad H(x, p) = -p\partial_x U + p^2.$$

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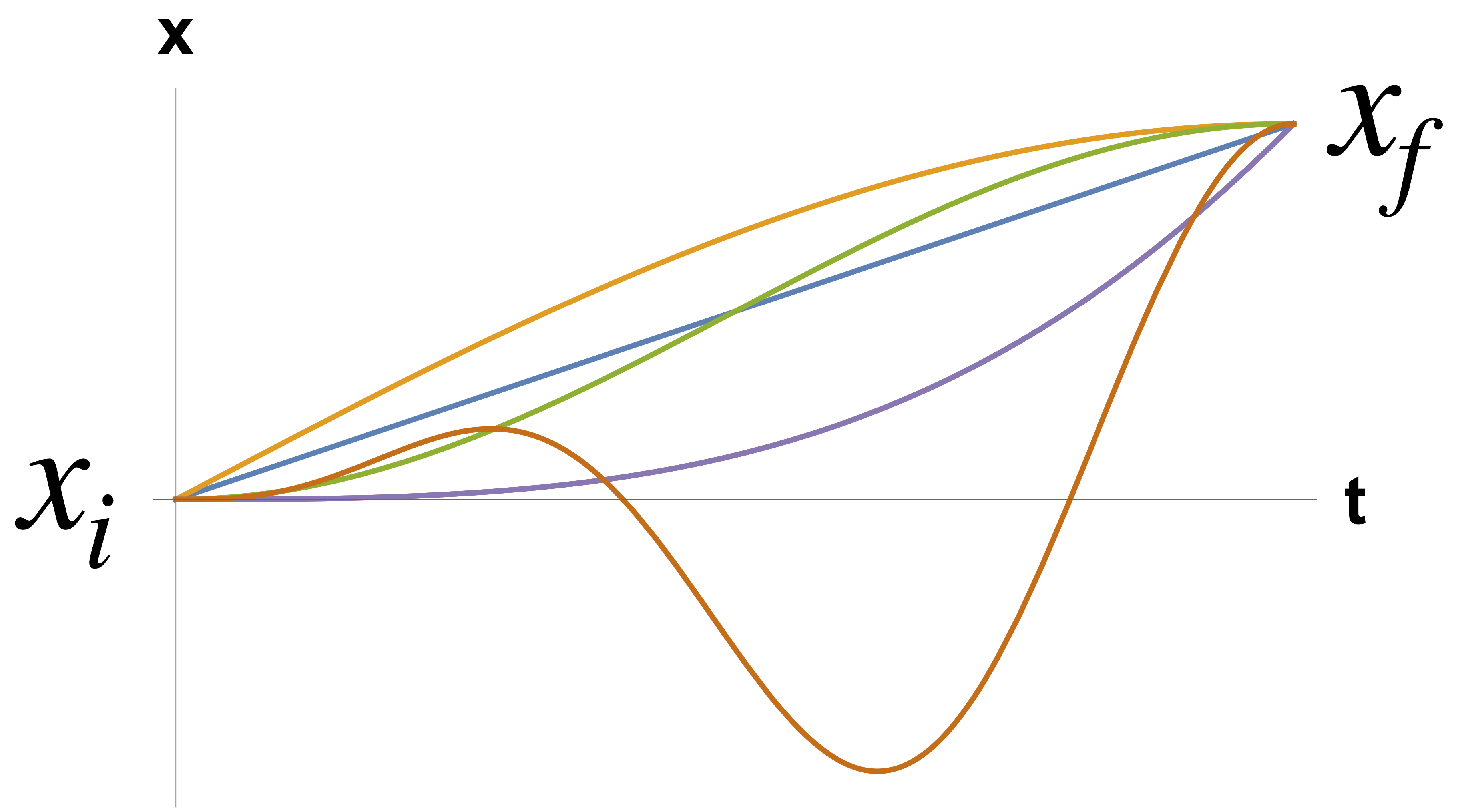

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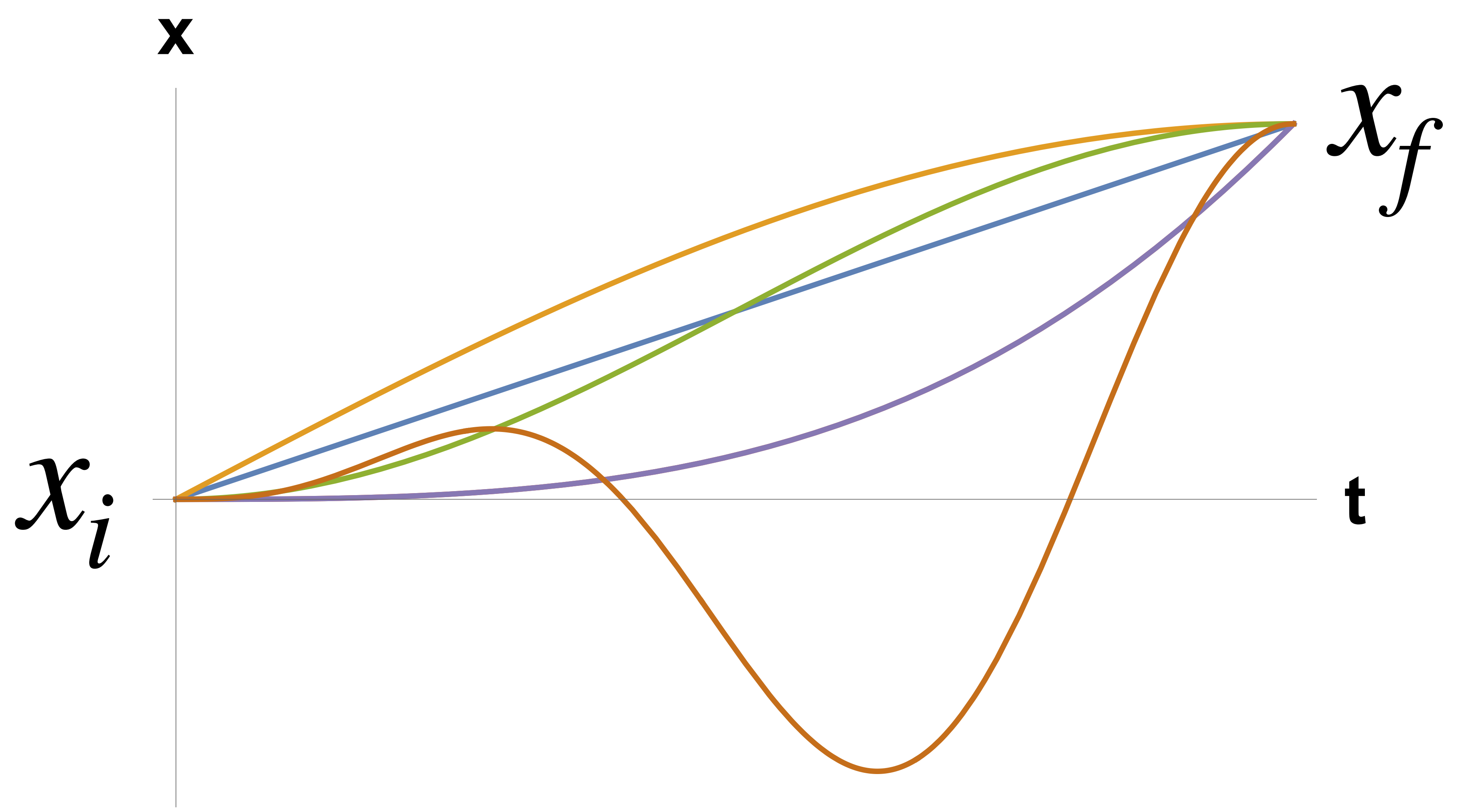

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Path lives on a constant H manifold

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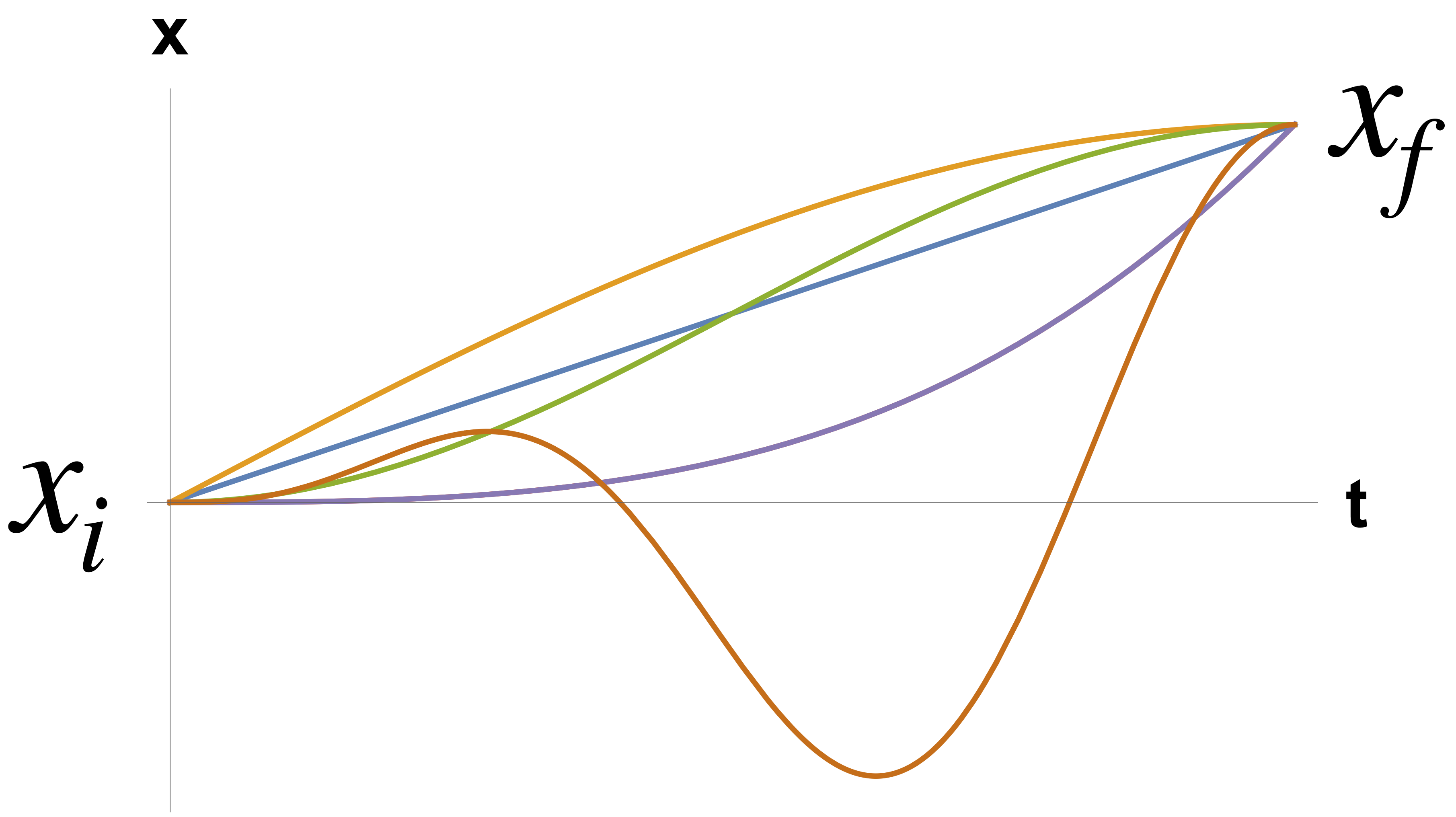

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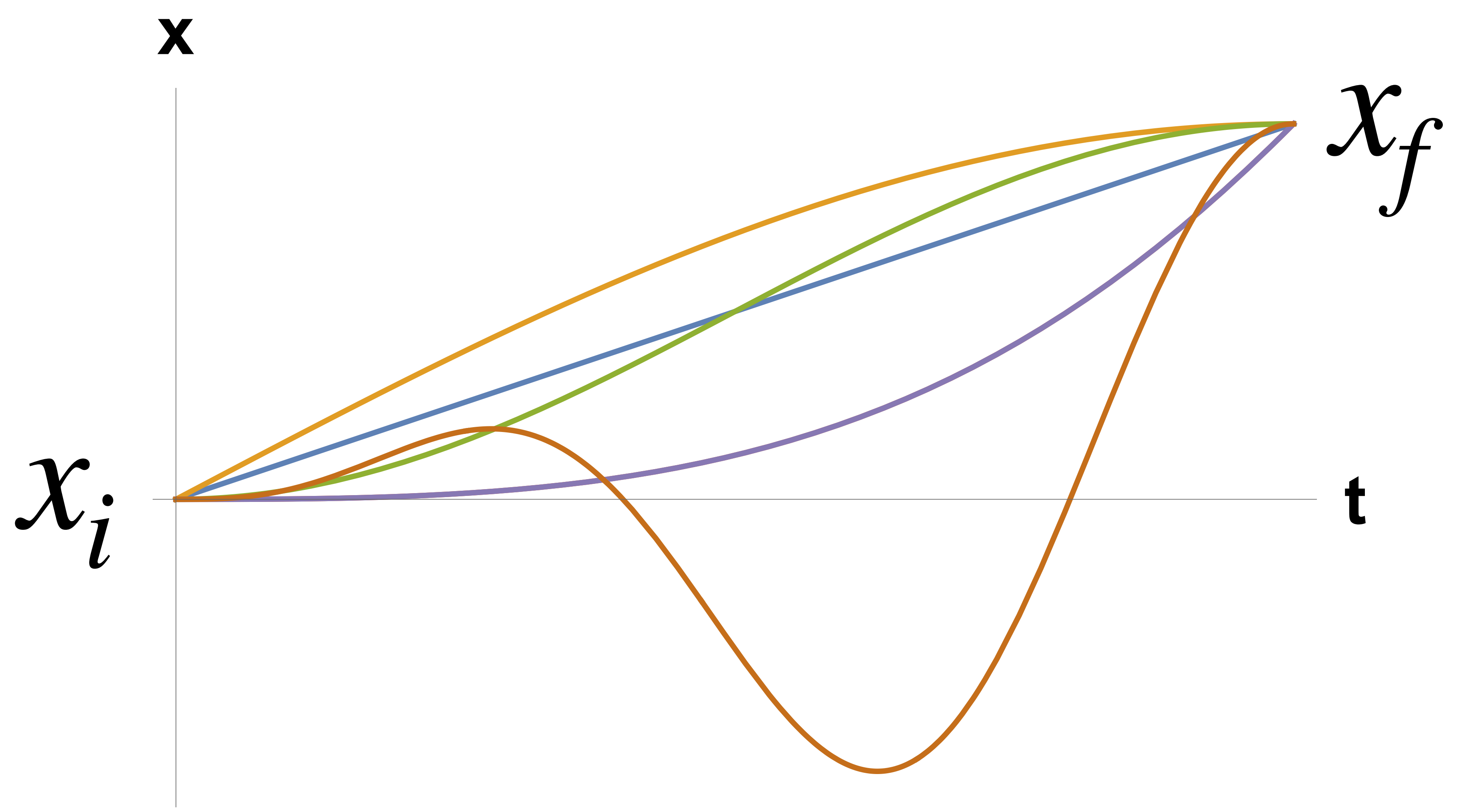
a) Following Hamilton's equations

b) A constant energy manifold $H = -DE$.



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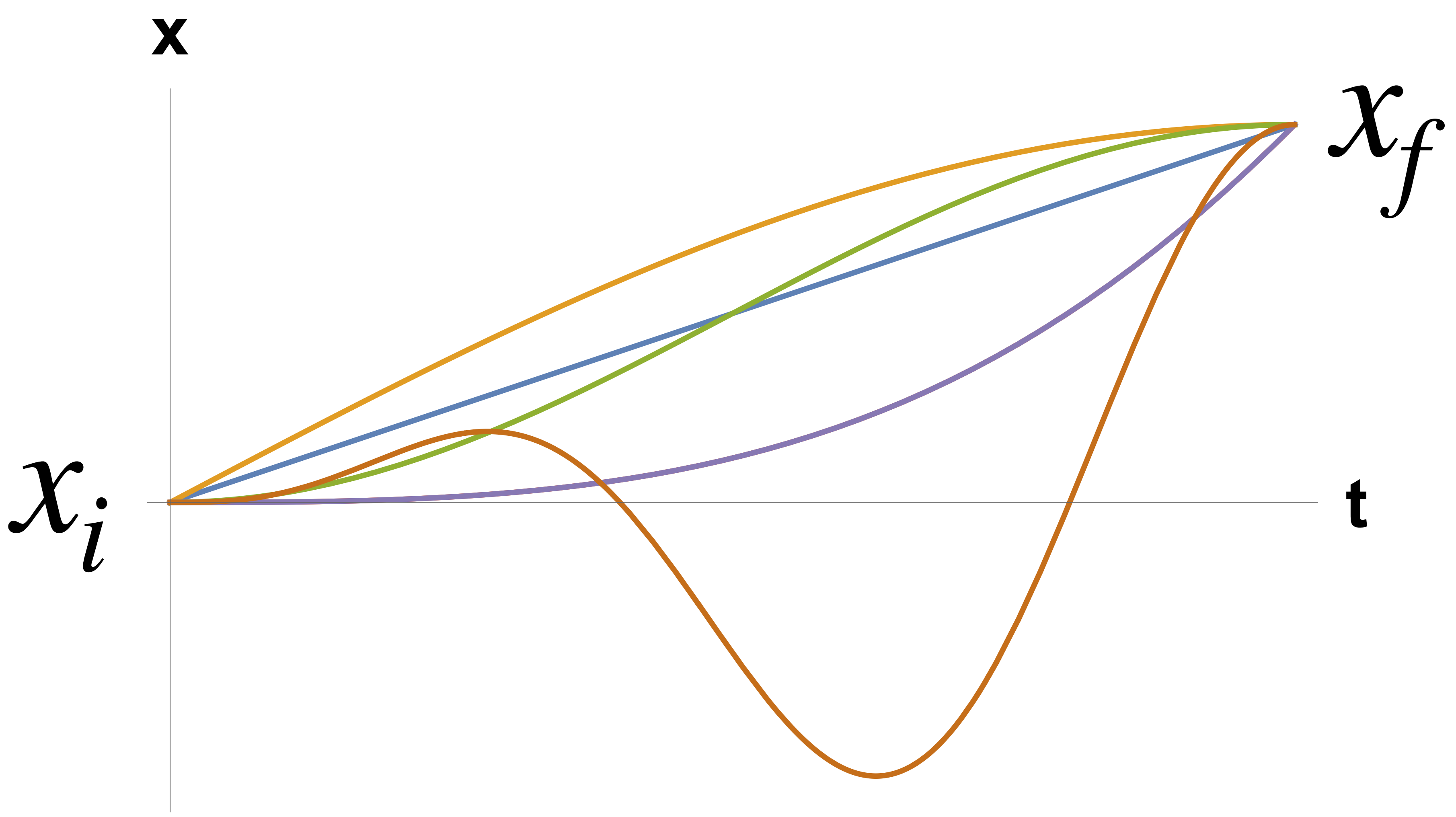

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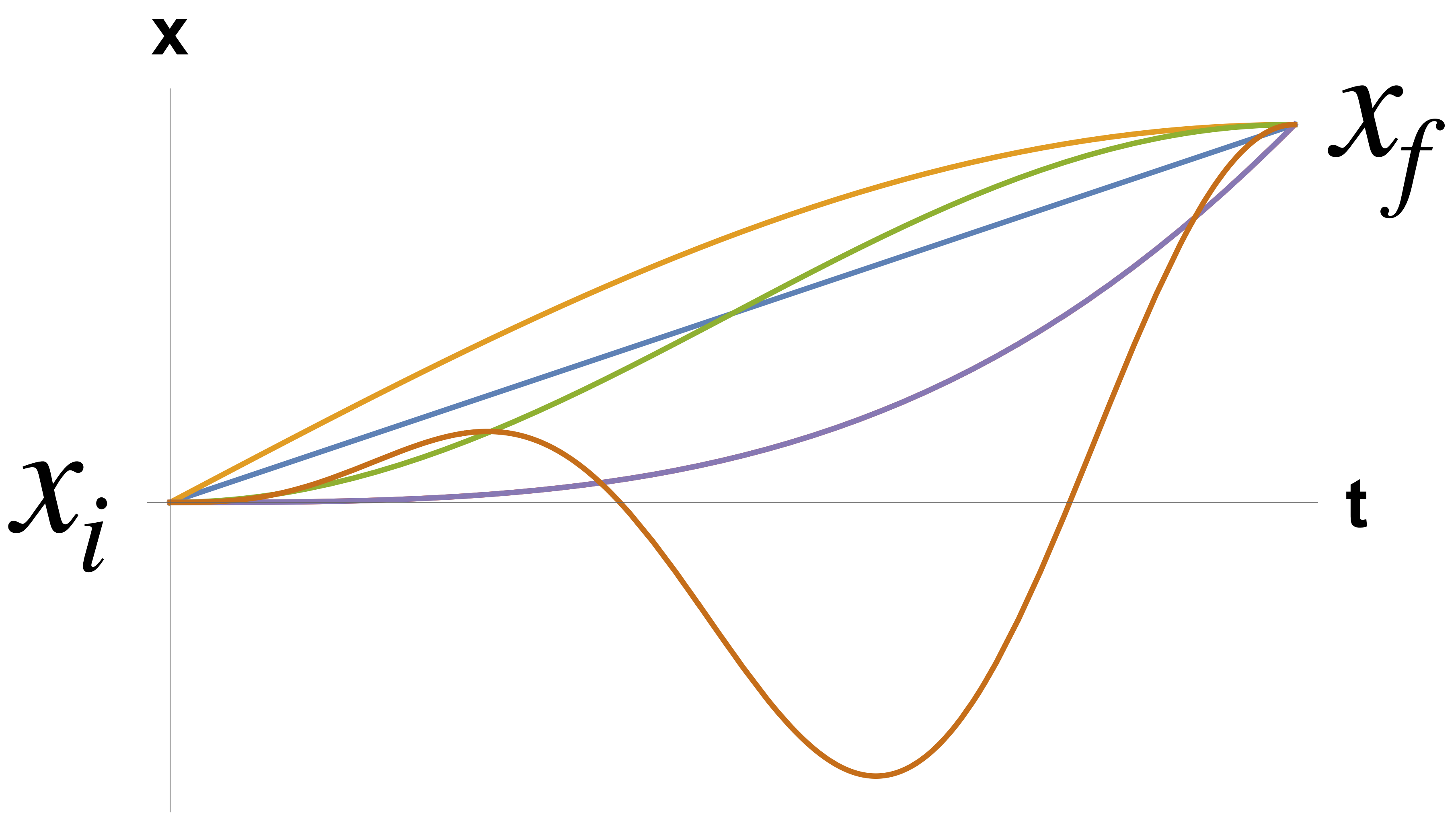
Notice, $H(x, p)$ defines a continuum of energy manifolds. **The inclusion of BC determines the spectrum**.



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Path probability approach to find the relaxation spectrum

- Only works as a saddle approximation.
- Need to handle boundary conditions carefully.
- Hard to go beyond the single particle.
- Need to solve eigenstates to infer the eigenvalues. The problem is coupled.

The macroscopic fluctuation theory

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A closed system of interacting particles with the density $\rho(x, t)$

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The probability to observe the density profile is given by

$$\mathscr{P}(\rho_f, t) \sim \int \mathscr{D}\rho_i \, \mathscr{P}(\rho_i, 0) \mathscr{T}(\rho_i, \rho_f; t)$$

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<u>This still looks challenging</u>

Saddle point comes naturally.

<u>Path probability</u>

$$\mathscr{T}(\rho_i,\rho_f;t) = \int \mathscr{D}\hat{\rho}\mathscr{D}\hat{\pi}\mathrm{e}^{-S}$$

$$S = L\int dxdt \, \hat{\pi}\partial_t\hat{\rho} - \mathscr{H}(\hat{\rho},\hat{\pi}).$$

Identifying the spectrum

Assume there exists a fixed point $\mathscr{V}_E[\rho(x)] = 0$

The Hamilton equations lead imply $\partial_t \rho = \partial_t \pi = 0$.

With these assumptions one can find E as an ODE of $\rho(x)$.

* BC apply directly on the ODE

* Disentangles the eigenvalues from the eigenfunctions !

A closed system of interacting particles with the density $\rho(x, t)$

Path probability

$$\mathscr{T}(\rho_i, \rho_f; t) = \int \mathscr{D}\hat{\rho}\mathscr{D}\hat{\pi} \mathrm{e}^{-S}$$

$$S = L \int dxdt\, \hat{\pi}\partial_t \hat{\rho} - \mathscr{H}(\hat{\rho}, \hat{\pi}).$$

Limitations

- The formalism, as of yet, works only for closed systems that relax to a unique equilibrium. That is, we need a local and global particle conservation.

- There is a huge degeneracy in the spectrum's eigenfunctions. That is, there are multiple quasi-potentials $\mathscr{V}_E$ for each energy value.

- While there is a "formula" for inferring the spectrum, finding the associated quasi-potential requires some luck and skill.

- Still much work to be done!