Microscopic heat engine subjected to stochastic resetting

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Korea Institute for Advanced Study

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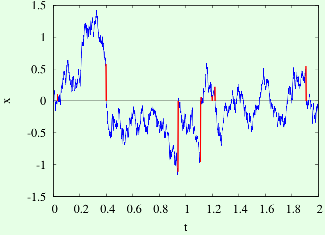
- The resetting actions, even if acting on an otherwise free particle, leads to a stationary distribution!



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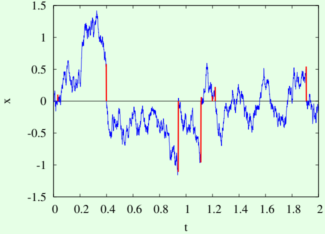

- Other than the infinitesimal time step where the particle position gets "reset", it evolves like a free particle.

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Dynamics

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$$\gamma \dot{x} = -k(t)x + \xi(t),$$

where the noise ξ follows a **Gaussian** distribution and has the auto-correlation $\langle \xi(t)\xi(t') \rangle = 2\gamma k_B T \delta(t - t')$.

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- It is now subjected to a **resetting rate** of r to a **location** x_{r}, such that the position gets updated as follows*:

$$x(t + \mathrm{d}t) = \begin{cases} x_{\mathrm{r}} \text{ with probability } r\mathrm{d}t, \\ x(t)\left(1 - \frac{k(t)}{\gamma}\mathrm{d}t\right) + \frac{1}{\gamma}\int_t^{t+\mathrm{d}t} \mathrm{d}t' \, \xi(t') \\ \text{ with probability } 1 - r\mathrm{d}t. \end{cases}$$

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- The time interval τ between two consecutive resets obeys the distribution $p(\tau) = re^{-r\tau}$.

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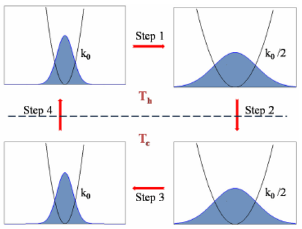
Thermodynamics

- We prepare a **Stirling engine** as shown*.
- The harmonic potential has the form $V(x) = k(t)x^2/2$, where the time-dependence of $k(t)$ is chosen to be **linear**.
- The total cycle time is $\mathcal{T}$.



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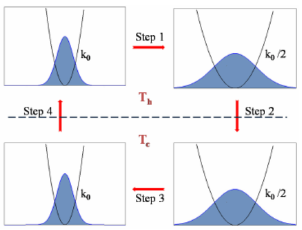

- **Step 1 (Isothermal expansion):** In contact with a heat bath at temperature T_H, the stiffness parameter changes as

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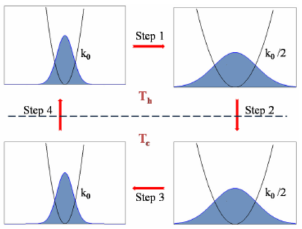

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- **Step 2 (Isochoric cooling):** At **constant** stiffness of $k_0/2$, the heat bath temperature is suddenly changed to $T_C < T_H$.

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- At the end of the first cycle, the final distribution acts as the initial distribution for the second one, and so on.

Thermodynamics

- The **total work** W_{tot} comes either from the reset (with probability rdt) or from the normal evolution (with probability $1 - rdt$):

$$W_{\text{tot}}(t + dt) - W_{\text{tot}}(t) = rdt[V(x) - V(x_0)] + (1 - rdt)\left(-\frac{\partial V}{\partial t}dt\right).$$

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$$\begin{aligned} W_{\text{tot}} &= -\frac{1}{2}\int_{t_i}^{t_f} \mathrm{d}t\ \dot{k}(t)x^2 \ +\ r\int_{t_i}^{t_f} \mathrm{d}t\ [V(x) - V(x_0)] \\ &= W_{\text{out}} + W_{\text{reset}}. \end{aligned}$$

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- The work extracted from the engine corresponds to W_{out} and not to W_{tot}*, as will be clear soon.
- **Sign convention:** work **extracted** and heat **absorbed** are positive.

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Analytical Expressions

- $\langle W_{\text{out}} \rangle$ and $\langle W_{\text{tot}} \rangle$ have the following expressions:

$$\langle W_{\text{out}}(\mathcal{T}) \rangle = -\frac{1}{2}\int_0^{\mathcal{T}/2} \mathrm{d}t\ \dot{k}_{\text{exp}}(t)\sigma_r^{\text{exp}}(t) - \frac{1}{2}\int_{\mathcal{T}/2}^{\mathcal{T}} \mathrm{d}t\ \dot{k}_{\text{com}}(t)\sigma_r^{\text{com}}(t);$$

$$\begin{aligned}\langle W_{\text{tot}}(\mathcal{T}) \rangle =& \langle W_{\text{out}}(\mathcal{T}) \rangle + \frac{r}{2}\int_0^{\mathcal{T}/2} \mathrm{d}t\ [k_{\text{exp}}(t)\sigma_r^{\text{exp}}(t) - k_0\sigma_r^{\text{exp}}(0)] \\ &+ \frac{r}{2}\int_{\mathcal{T}/2}^{\mathcal{T}} \mathrm{d}t\ [k_{\text{com}}(t)\sigma_r^{\text{com}}(t) - (k_0/2)\sigma_r^{\text{com}}(\mathcal{T}/2)],\end{aligned}$$

where

$$\sigma_{\text{r}}^{\text{exp}}(t) = \int_{-\infty}^{+\infty} \mathrm{d}x_0 \int_{-\infty}^{+\infty} \mathrm{d}x\ x^2 P_{\text{r}}^{\text{exp}}(x,t|x_0,0)P_i(x_0),$$

$$\sigma_{\text{r}}^{\text{com}}(t) = \int_{-\infty}^{+\infty} \mathrm{d}x_{\mathcal{T}^+/2} \int_{-\infty}^{+\infty} \mathrm{d}x\ x^2 P_{\text{r}}^{\text{com}}(x,t|x_{\mathcal{T}^+/2},\mathcal{T}^+/2)P(x_{\mathcal{T}^+/2}).$$

- The averages have been taken over the **modified position distributions**, $P_{\text{r}}^{\text{exp}}$ and $P_{\text{r}}^{\text{com}}$, in presence of resetting.

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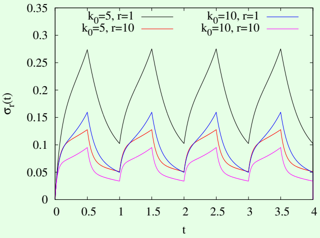
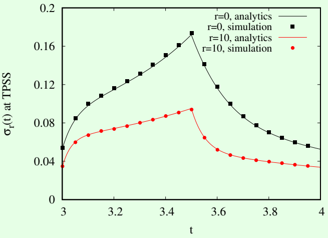
- We thus begin our investigation from the fourth cycle onwards.
- We have plotted $\sigma_{\rm r}(t)$ in the **left figure** to ensure that we indeed reach the steady state.
- The agreement between analytics and simulations has been shown in the **right figure**.





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- If a weight is attached to the piston, then the work done **by the piston** is the output work.
- The reset operation due to the electric field only **indirectly affects** this work by changing the phase space configuration of the particles.

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- To verify this, we first "incorrectly" use $\langle W_{\mathrm{tot}} \rangle / \langle Q_H \rangle$ as the definition of efficiency.
- The value of $\langle Q_H \rangle$ can be obtained by using the **First Law**:

$$\langle Q_H \rangle_{\mathrm{exp}} = \langle W_{\mathrm{out}} \rangle_{\mathrm{exp}} + \langle \Delta E \rangle_{\mathrm{exp}},$$

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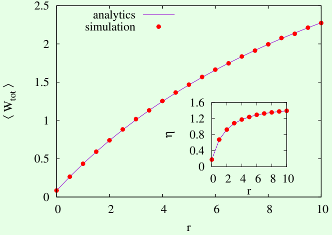

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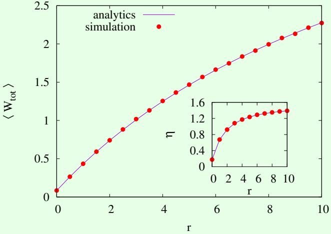

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- The inset in the figure shows that the efficiency **exceeds 100%**!

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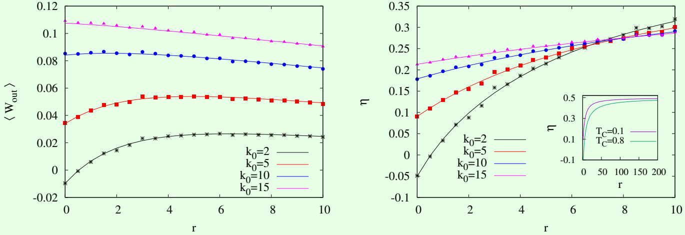
- Let us now use the “correct” definition: $\eta = \langle W_{\text{out}} \rangle / \langle Q_H \rangle$.



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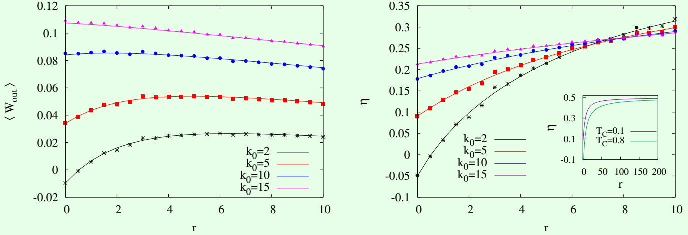

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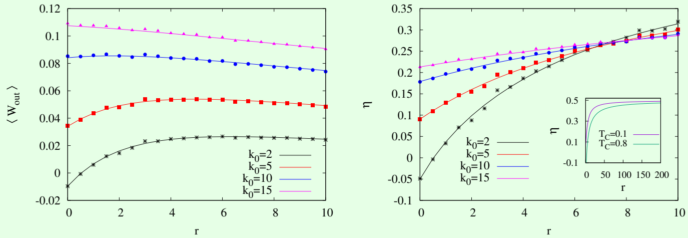

- We find that the values of η at large times **do not** exceed 100%.
- This result demonstrates that W_{out} is the more reasonable definition of output work.

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- Clearly, this contradicts our results. So the above definition for effective temperature is **not valid** for our system.

Efficiency at Maximum Power (η_{MP})

- For thermal baths, η_{MP} depends on the temperature ratio as follows*:

$$\eta_{\mathrm{CA}} = 1 - \sqrt{\frac{T_C}{T_H}}.$$

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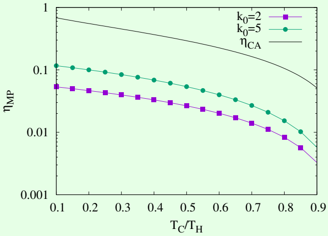
- Using the $\langle W_{\text{out}} \rangle$ vs r curves that are non-monotonic, we could plot the functional dependence of η_{MP} on T_C/T_H.
- Our results show substantial deviation of the system from the Curzon-Ahlborn relation.



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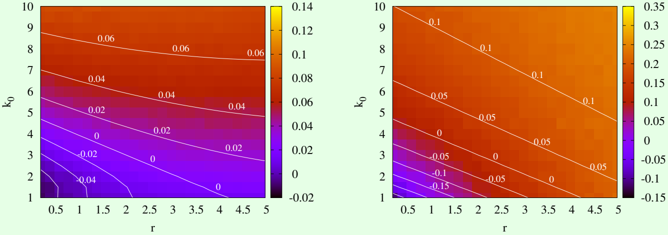
Phase Plots

- The **output work** (left) and **efficiency** (right) as functions of the stiffness parameter and resetting rate are shown in the heat maps.



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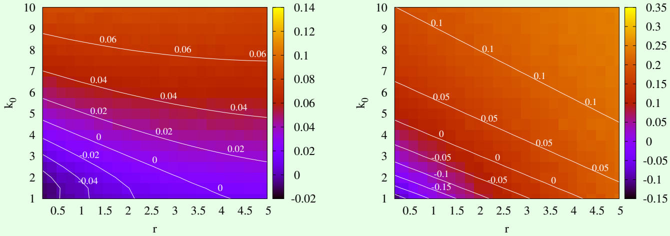

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- The region in which $\langle W_{\text{out}} \rangle < 0$ but $\eta > 0$ corresponds to work done **on** the system and heat **dissipated** into the hot bath, i.e., the **refrigerator** regime.

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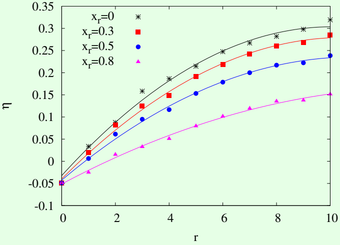

$k_0 = 2.$

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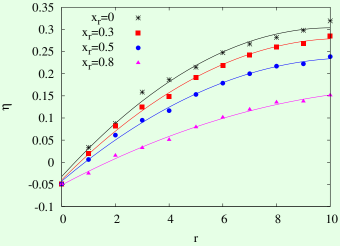
- What happens when $x_\mathrm{r} \neq 0$?
- Since $\frac{\partial V}{\partial t}$ is **larger** at $x_\mathrm{r} \neq 0$, the value of $\langle W_{\mathrm{out}} \rangle = -\int \frac{\partial \langle V \rangle}{\partial t} dt$ is **smaller**. This decreases η.



$k_0 = 2.$

- The plots are obtained from **simulations**, and the solid lines are the quadratic fits.

Summary and Conclusions

- We have studied a **Brownian Stirling engine**, where the particle undergoes stochastic resetting.
- The effect of the resetting rate r on the **work output** and **efficiency** of the engine has been explored.
- The efficiency is found to **increase** with increase in r. The work output may be **non-monotonic**, depending on k_0.
- Defining an effective temperature in terms of the position variance is ruled out.
- $\eta_{\rm MP}$ deviates appreciably from $\eta_{\rm CA}$.
- For $x_{\rm r} \neq 0$, the efficiency gets **reduced**, for a given r.
- By tuning k_0 and r, the device can be converted to a **refrigerator**.

Thank You