## C.4 Clifford Algebra and Spinor Classification

Now that we have studied a little bit about spinors and their property under diffeomorphisms, it is high time to take a deeper look, especially at how the spinor bundle structure depends on dimensions and the signature. The starting point is again the Clifford algebra spanned by the Dirac matrices.

In Euclidean signature, we have $Cl_d$ algebra spanned by

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab} \tag{C.4.1}$$

Given these generators, more independent matrices may be constructed from completely antisymmetric products, i.e., sums

$$\gamma^{a_1 \cdots a_p} \equiv \frac{1}{p!} \sum_\sigma (-1)^{|\sigma|} \gamma^{a_{\sigma_1}} \cdots \gamma^{a_{\sigma_p}} \tag{C.4.2}$$

over all possible permutations $\sigma$ with the parity $(-1)^{|\sigma|}$. In particular, when $d = 2n$, there exists a special generator $\gamma^{1\cdots d}$ which, as we saw earlier, is related to the chirality operator,

$$\Gamma = (-\mathbb{i})^n \gamma^1 \cdots \gamma^{2n} \tag{C.4.3}$$

For $d = 2n + 1$, the same set of $\gamma$'s for $d = 2n$ may be used for the first $2n$ Dirac matrices, while for the last one, $\gamma^{2n+1}$, we would use either $\Gamma$ or $-\Gamma$.

In the Lorentzian signature, the Clifford algebra takes the form

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab} \tag{C.4.4}$$

and is denoted $Cl_{1,d-1}$. More generally, we may also consider more negative signs on the right hand side, say,

$$\text{diag}(\underbrace{-,\cdots,-}_{p},\underbrace{+,\cdots,+}_{d-p}) \tag{C.4.5}$$

denoted as $Cl_{p,d-p}$. The familiar spinors are acted on by multiplication by $\gamma$'s and $\Gamma$'s on the left, so form a representation. In this final section, we will explore these spinor representations, with emphasis on how the Lorentz group enter the discussion along the way. We do not really need a separate construction for these alternate signatures since we may as well start from $\gamma^a$'s and construct $\Gamma^{a\leq p} = -\mathbb{i}\gamma^{a\leq p}$'s and identify the rest intact, $\Gamma^{a>p} = \gamma^{a>p}$.

The questions here are how these spinor representations of Clifford algebras in various dimensions and signatures can be characterized and sometimes decomposed further when we dote on $\mathfrak{so}(p, d-p)$ subalgebra of $Cl_{p,d-p}$, generated by antisymmetric products of two distinct $\Gamma$'s. For instance, we have seen how even $d$ implies that the spinor splits into two distinct irreducible representations under $\mathfrak{so}(p, d-p)$, due to the existence of the chirality operator. Equally important are the charge conjugation operations, which can also halve the spinor by imposing reality conditions. Much of these detail played crucial roles for path integrals of fermions, as we have encountered numerously in the main text. One well-known phenomenon we will rediscover here is the so-called Bott periodicity, whereby the structure repeats itself under a shift of $d$ by 8.

### A Canonical Representation

We start with discussion for the first where all $\gamma$'s are Hermitian, and come to the other signatures later. As we saw in earlier discussion of spinors, $\gamma^{ab}$'s play a special role as the rotation generators, on spinors, of the underlying Lorentz group. As such, we have a sequence of algebras

$$\mathfrak{so}(d) \quad \subset \quad Cl_d^{\text{even}} \quad \subset \quad Cl_d \tag{C.4.6}$$

where the middle is a subalgebra spanned by even products of $\gamma$'s. The primary objects of interest are the first two, or more precisely the spin group $Spin(d)$, which is related to $SO(d)$ group by a $\mathbb{Z}_2$ division and can be constructed from $Cl_d^{\text{even}}$, and the representations thereof, but we will start with a canonical representation of $Cl_d$

which is useful for the rest of the discussions.

To construct the representation explicitly, it is convenient to define

$$\alpha_S \equiv \frac{1}{2}\left(\gamma^{n+S} + \mathbb{i}\gamma^S\right) \ , \qquad \alpha_S^\dagger = \frac{1}{2}\left(\gamma^{n+S} - \mathbb{i}\gamma^S\right) \tag{C.4.7}$$

such that the fermionic oscillators,

$$\{\alpha_S, \alpha_T^\dagger\} = \delta_{ST} \tag{C.4.8}$$

may be used to construct a $2^n$ dimensional Fock space,

$$|0\rangle \ , \qquad \alpha_S^\dagger|0\rangle \ , \quad \cdots \quad , \quad \alpha_1^\dagger\alpha_2^\dagger\cdots\alpha_n^\dagger|0\rangle \tag{C.4.9}$$

starting from the vacuum state, $\alpha_S|0\rangle = 0$, which serve as a basis that span the Dirac spinor. The oscillators merely shuffle a basis state to another, with $\pm 1$ coefficients, so $\gamma^S = -\mathbb{i}(\alpha_S - \alpha_S^\dagger)$ are pure imaginary and antisymmetric while $\gamma^{n+S} = \alpha_S + \alpha_S^\dagger$'s are real symmetric. Finally, since the Hermitian $\gamma^{2n+1} = \pm\Gamma = \pm(-\mathbb{i})^n\gamma^1\cdots\gamma^{2n}$ obeys

$$(\gamma^{2n+1})^* = (-1)^n(-1)^n\gamma^{2n+1} = \gamma^{2n+1} \tag{C.4.10}$$

$\gamma^{n+S}$'s are real symmetric all the way for $S = 1, \ldots, n+1$ in this representation.

Note that we can repeat the construction for other signatures; the only new element here is to replace some $\gamma$ by $-\mathbb{i}\gamma$ as the new Dirac matrices. For a more streamlined notation, let us introduce a different notation for these anti-Hermitian Dirac matrices as

$$\begin{aligned} \Gamma^a &\equiv -\mathbb{i}\gamma^a \quad , \qquad a = 1, \cdots, p \le n \\ \Gamma^b &\equiv \gamma^b \qquad , \qquad b = p+1, \cdots, d \end{aligned} \tag{C.4.11}$$

where we restricted the number of such anti-Hermitian $\Gamma^a$ to be no more than $n = \lfloor d/2 \rfloor$, the integer part of $d/2$.

The Fock space construction proceeds the same way as in the Euclidean case, since we may as well use $\gamma^a$'s and multiply $-\mathbb{i}$ for the first $p$ of them in the end.

$$-\mathbb{i}\gamma^1, \cdots, -\mathbb{i}\gamma^p; \gamma^{p+1}, \cdots, \gamma^n; \gamma^{n+1}, \cdots, \gamma^{2n} \tag{C.4.12}$$

The first $p$ are real antisymmetric, the middle $(n-p)$ are imaginary antisymmetric, and the last $n$ are real symmetric. In particular, when $p=n$, we see that all Dirac matrices are real. The chirality operator, or $\gamma^{2n+1}$ modulo sign if $d=2n+1$,

$$\Gamma^{2n+1} \equiv \Gamma = (-\mathbb{i})^n \gamma^1 \cdots \gamma^{2n} = (-\mathbb{i})^{n-p} \Gamma^1 \cdots \Gamma^{2n} \tag{C.4.13}$$

is real and symmetric, regardless of $p$ and $n$, in this representation.

### Complex Conjugations and Majorana Spinors

One may then construct

$$\mathbb{C} \equiv \Gamma^{p+1} \cdots \Gamma^n = \gamma^{p+1} \cdots \gamma^n \ , \qquad \mathbb{C}^{-1} = \mathbb{C}^\dagger \tag{C.4.14}$$

where one should note that we take product of pure imaginary ones among Γ's in the above Fock space representation of the Clifford algebra. With our choice $p \leq n$, and $d=2n$ or $d=2n+1$, these constitute no more than half of all Dirac matrices. Later we will come to the complimentary choice, $\mathcal{C}$, available for $d=2n$ as the product of $(n+p)$ real Γ's down to $\Gamma^{2n}$, playing a similar role. For $d=2n+1$, no such independent $\mathcal{C}$ exists since product of all Γ's is proportional to 1.

This $\mathbb{C}$ obey

$$\mathbb{C}^{-1}\gamma^{a\leq p}\mathbb{C} = -(-1)^{n-p}(\gamma^{a\leq p})^* \ , \qquad \mathbb{C}^{-1}\gamma^{a>p}\mathbb{C} = (-1)^{n-p}(\gamma^{a>p})^* \tag{C.4.15}$$

which can be used as a charge conjugation for $\Gamma^a$'s

$$\mathbb{C}^{-1}\Gamma^a\mathbb{C} = (-1)^{n-p}(\Gamma^a)^* \tag{C.4.16}$$

up to $a=2n+1$, from which we find

$$(\Gamma^{ab})^* = \mathbb{C}^{-1}\Gamma^{ab}\mathbb{C} \tag{C.4.17}$$

on $\mathfrak{so}(p,d-p)$ rotation generators for $d=2n,2n+1$. Therefore, the Dirac spinor $\Psi$ and its complex conjugate

$$\Psi_{\mathbb{C}} \equiv \mathbb{C}\Psi^* \tag{C.4.18}$$

transform the same way under $\mathfrak{so}(p, d-p)$.

If we perform this conjugation operation twice, the spinors come back to itself, generally modulo a sign,

$$(\Psi_{\mathbb{C}})_{\mathbb{C}} = \mathbb{C}(\mathbb{C}\Psi^*)^* = \begin{cases} \Psi & n-p = 0, 3 \bmod 4 \\ \\ -\Psi & n-p = 1, 2 \bmod 4 \end{cases} \tag{C.4.19}$$

since

$$\mathbb{C}\mathbb{C}^* = \gamma^{p+1} \cdots \gamma^n (-1)^{n-p} \gamma^{p+1} \cdots \gamma^n = (-1)^{(n-p)(n-p+1)/2} \tag{C.4.20}$$

Since $\Psi$ and $\Psi_c$ transform the same way under $\mathfrak{so}$, one may use this conjugation operation to project the Dirac spinor to real and imaginary halves. With $n-p = 0, 3$ mod 4, for which $\mathbb{C}\mathbb{C}^* = 1$,

$$\left[\frac{1}{2}(\Psi + \Psi_{\mathbb{C}})\right]_{\mathbb{C}} = \frac{1}{2}(\Psi + \Psi_{\mathbb{C}}) \ , \qquad \left[\frac{\mathbb{i}}{2}(\Psi - \Psi_{\mathbb{C}})\right]_{\mathbb{C}} = \frac{\mathbb{i}}{2}(\Psi - \Psi_{\mathbb{C}}) \tag{C.4.21}$$

which split the Dirac spinor into real and imaginary part.

If we restrict our attention to $d = 2n$, there is one more choice of the charge conjugation operator,

$$\mathcal{C} = \mathbb{C}\Gamma = \mathbb{C}\Gamma^{2n+1} \tag{C.4.22}$$

again with $\mathcal{C}\mathcal{C}^\dagger = 1$, and

$$\mathcal{C}^{-1}\Gamma^a\mathcal{C} = (-1)^{n-p+1}(\Gamma^a)^* \ , \qquad (\Gamma^{ab})^* = \mathcal{C}^{-1}\Gamma^{ab}\mathcal{C} \tag{C.4.23}$$

The charge conjugation under $\mathcal{C}$,

$$\Psi_{\mathcal{C}} \equiv \mathcal{C}\Psi^* \tag{C.4.24}$$

has the property,

$$(\Psi_{\mathcal{C}})_{\mathcal{C}} = \mathcal{C}(\mathcal{C}\Psi^*)^* = \begin{cases} \Psi & n-p = 0, 1 \bmod 4 \\ \\ -\Psi & n-p = 2, 3 \bmod 4 \end{cases} \tag{C.4.25}$$

since

$$\mathcal{C}\mathcal{C}^* = (-1)^{(n-p)(n-p+1)/2+(n-p)} = (-1)^{(n-p)(n-p+3)/2} \tag{C.4.26}$$

which equals 1 for $n-p = 0, 1 \bmod 4$ and allows the split

$$\left[\frac{1}{2}(\Psi + \Psi_{\mathcal{C}})\right]_{\mathcal{C}} = \frac{1}{2}(\Psi + \Psi_{\mathcal{C}}) \ , \qquad \left[\frac{\mathbb{i}}{2}(\Psi - \Psi_{\mathcal{C}})\right]_{\mathcal{C}} = \frac{\mathbb{i}}{2}(\Psi - \Psi_{\mathcal{C}}) \tag{C.4.27}$$

in the same manner as above.

Although the nomenclatures on this varies, we will call the projected spinors, possible with the help of $\mathbb{C}\mathbb{C}^* = 1$ or $\mathcal{C}\mathcal{C}^* = 1$, Majorana. The other case, with $\mathbb{C}\mathbb{C}^* = -1$ for odd dimensions or $\mathcal{C}\mathcal{C}^* = \mathbb{C}\mathbb{C}^* = -1$ for even dimensions, are called symplectic Majorana. For such symplectic Majorana spinors, the charge conjugation extends the global symmetry algebra $\mathfrak{u}(1)$ that act on a single Dirac spinor, to the $\mathfrak{sp}(1) = \mathfrak{usp}(2)$, even though the above split of Dirac spinor into "real" and "imaginary" parts is not possible

Majorana spinors, with truly half the degree of freedom relative to Dirac spinors, are possible in odd dimensions under $\mathbb{C}$ if $n - p = 0, 3 \bmod 4$. These are $d = 2n+1 = 7, 9$ etc for $\mathfrak{so}(2n+1)$'s and $d = 2n+1 = 3, 9, 11$ etc for $\mathfrak{so}(1, 2n)$'s. In even dimensions, on the other hand, we can use either of $\mathbb{C}$ or $\mathcal{C}$, so the Majorana spinor is possible provided that $n - p = 0, 1, 3$. For $\mathfrak{so}(2n)$, these are $d = 2, 6, 8$ etc while for $\mathfrak{so}(1, 2n-1)$ these are $d = 2, 4, 8, 10$ etc.

Although we worked with mostly plus sign of the signature, it would be immediately clear that these classifications is symmetric under $\mathfrak{so}(p, d-p) \to \mathfrak{so}(d-p, p)$ since all we need to do is to map $\Gamma^a \to \mathbb{i}\Gamma^a$, under which the rotation generator change signs at most. In even dimensions, this flip exchanges $\mathbb{C}$ and $\mathcal{C}$, for example. Note that the above discussion of reality and pseudo-reality refers to the representation under $\mathfrak{so}(p, d-p)$, rather than those of the Clifford algebra. The (pseudo-)reality of $Cl_{p,d-p}$ representations are another matter, since the much-smaller algebra $\mathfrak{so}(p, d-p)$ resides in $Cl^{\text{even}}_{p,d-p}$ consisting of even number of antisymmetrized product of $\Gamma^a$'s, inside $Cl_{p,d-p}$. For this reason, the (pseudo-)reality classification of the Clifford algebra $Cl_{p,d-p}$ itself, often found in mathematics literature, looks different from that of $\mathfrak{so}$ spinors above.

### (Symplectic) Majorana-Weyl

On the other hand, the Dirac spinor is not irreducible under $\mathfrak{so}(p, 2n-p)$ given how $\Gamma^a\Gamma^{2n+1}$ are no longer rotation generators. As we have seen earlier, $\Gamma^{2n+1} = \Gamma$ plays the role of a chirality operator instead, and split the Dirac spinor into a pair of Weyl spinors

$$\Psi_\pm = \frac{1}{2}(1 \pm \Gamma)\Psi \tag{C.4.28}$$

With this, we need to check whether the two types of the above projections can be simultaneously imposed. The rotation generators for these are respectively,

$$\Sigma^{ab}_\pm = \Gamma^{ab}\frac{(1 \pm \Gamma)}{2} \tag{C.4.29}$$

and their properties under the charge conjugation are

$$\mathbb{C}^{-1}(\Sigma^{ab}_\pm)^*\mathbb{C} = \mathbb{C}^{-1}(\Gamma^{ab})^*\frac{(1 \pm \Gamma^*)}{2}\mathbb{C} = \Gamma^{ab}\frac{(1 \pm (-1)^{n-p}\Gamma)}{2} \tag{C.4.30}$$

and the same with $\mathcal{C}$,

$$\mathcal{C}^{-1}(\Sigma^{ab}_\pm)^*\mathcal{C} = \mathcal{C}^{-1}(\Gamma^{ab})^*\frac{(1 \pm \Gamma^*)}{2}\mathcal{C} = \Gamma^{ab}\frac{(1 \pm (-1)^{n-p}\Gamma)}{2} \tag{C.4.31}$$

Thus Weyl projection is compatible with either of the charge conjugation if and only if $n-p$ is even.

Let us first concentrate on $\mathfrak{so}(1, d-1)$. Recall that, with $p = 1$, the Majorana projection was possible for $d = 2, 3, 4, 8, 9, 10, 11$, respectively with $n = 1, 1, 2, 4, 4, 5, 5$. Among even dimensions, therefore, we find the Majorana projection and the Weyl projections are compatible only in $d = 2, 10$ dimensions. Spinors projected twice this way is called Majorana-Weyl. On the other hand, in $d = 6$ dimensions, we have $n - p = 2$ so that the charge conjugation associated with the symplectic Majorana property there does preserve the Weyl projection. We call the Weyl spinor in such cases symplectic Majorana-Weyl whose net effect is merely enlargement of the global symmetry associated with the Weyl spinor.

With the Euclidean signature $\mathfrak{so}(d = 2n)$, the Majorana projection is available for $d = 6, 8$ with $n = 3, 4$, respectively, so the Majorana-Weyl spinor is possible only for $d = 8$.

**Minimal Spinor Representations for $\mathfrak{so}(1, d-1)$**

Recall how the two relevant sign factors that entered the above discussion are determined by the combination, $n-p$, as

$$(-1)^{n-p} \,, \qquad (-1)^{(n-p)(n-p+1)/2} \tag{C.4.32}$$

The former repeat itself in $n-p$ mod 2 while we have seen that the latter repeat itself in $n-p$ mod 4. With $\mathfrak{so}(p, d-p) = \mathfrak{so}(p, q)$, on the other hand, $n-p = (q-p)/2 + \cdots$, so the first and the second repeat themselves in $q-p$ mod 4 and mod 8 respectively.

Combined, this implies that the pattern repeats itself in $d$ mod 8 and is invariant under the shift $(p, q) \to (p+1, q+1)$. With this understood, it suffices to list the minimal representation for a particular signature, say, $p = 1$. In the table, we list the smallest spinor representations for $\mathfrak{so}(1, d-1)$ for $d \leq 11$, with the resulting minimal number of components displayed in the second column. The last column represents the largest possible global symmetry when $N$ such spinors are simultaneously present; $N_\pm$ refers to the numbers of chiral and anti-chiral spinors, respectively, when applicable.

For $d = 4, 8$, where one can choose either Weyl or Majorana, we displayed the Weyl spinor; Weyl is more versatile than Majorana in that it can more naturally accommodate more diverse gauge representations. We should emphasize again that "symplectic Majorana" has the same content as a Dirac; the difference is how "symplectic" case admit maximal global symmetry algebra $\mathfrak{sp}(N) = \mathfrak{usp}(2N)$ instead of $\mathfrak{u}(N)$ for a collection of $N$ such spinors. With $N_\pm$ referring to the number of chiral and anti-chiral Weyl spinors as above, "symplectic Majorana-Weyl" may have the symmetry $\mathfrak{sp}(N_+) \oplus \mathfrak{sp}(N_-)$.

Note how these spinors decompose under the reduction $\mathfrak{so}(1, d-1) \to \mathfrak{so}(1,1) \oplus \mathfrak{so}(d-2)$. Starting from the former's spinor $\Psi$, there is a universal decomposition,

$$\Psi \qquad \to \qquad \Psi_{1/2} + \Psi_{-1/2} \tag{C.4.33}$$

where $\pm 1/2$ refers to the charge under $\mathfrak{so}(1,1)$. Since the latter's smallest spinor is a single real component and since the type of the minimal spinor representation is common between $\mathfrak{so}(1, d-1)$ and $\mathfrak{so}(d-2)$, the reality property of $\Psi$ is inherited by $\Psi_{1/2}$ and $\Psi_{-1/2}$, each carrying exactly half the component of $\Psi$. For instance, with $d = 10$, a Majorana-Weyl $\Psi$ carries 16 real components while $\Psi_{\pm 1/2}$ is again Majorana-Weyl with 8 real components each, whose chiralities are determined as $\pm 1$ times that of $\Psi$. This exercise is closely tied to how the dynamical contents of fields transforming covariantly under $\mathfrak{so}(1, d-1)$ are classified by the little group, which for massless cases is effectively $\mathfrak{so}(d-2)$.

| | # of components | minimal spinor | global symmetry |
|---|---|---|---|
| $\mathfrak{so}(1,1)$ | 1 real | Majorana-Weyl | $\mathfrak{so}(N_+) \oplus \mathfrak{so}(N_-)$ |
| $\mathfrak{so}(1,2)$ | 2 real | Majorana | $\mathfrak{so}(N)$ |
| $\mathfrak{so}(1,3)$ | 2 complex | Weyl (or Majorana) | $\mathfrak{su}(N) \oplus \mathfrak{u}(1)$ |
| $\mathfrak{so}(1,4)$ | 4 complex | symplectic Majorana | $\mathfrak{sp}(N)$ |
| $\mathfrak{so}(1,5)$ | 4 complex | symplectic Majorana-Weyl | $\mathfrak{sp}(N_+) \oplus \mathfrak{sp}(N_-)$ |
| $\mathfrak{so}(1,6)$ | 8 complex | symplectic Majorana | $\mathfrak{sp}(N)$ |
| $\mathfrak{so}(1,7)$ | 8 complex | Weyl (or Majorana) | $\mathfrak{su}(N) \oplus \mathfrak{u}(1)$ |
| $\mathfrak{so}(1,8)$ | 16 real | Majorana | $\mathfrak{so}(N)$ |
| $\mathfrak{so}(1,9)$ | 16 real | Majorana-Weyl | $\mathfrak{so}(N_+) \oplus \mathfrak{so}(N_-)$ |
| $\mathfrak{so}(1,10)$ | 32 real | Majorana | $\mathfrak{so}(N)$ |