8.2.1 Hamilton-Jacobi

One alternate version of the classical dynamics goes by the name of the Hamilton-Jacobi, which has profound implications toward quantum mechanics. Let us dwell on this, with the relativistic particle motion in black hole spacetimes as an excuse.

The main object of interest is the Hamilton-Jacobi function, $S_{\rm HJ}$, whose definition looks superficially similar to that of the action in that it is obtained by integrating the Lagrangian L. Unlike the action S, $S_{\rm HJ}$ is something we evaluate on actual solutions. Given an initial time τ_i and a point x_i, one defines $S_{\rm HJ}(\tau,\, x)$ by integrating over the trajectories that connect the initial time and the initial postion, to their final counterparts τ and x,

$$S_{\rm HJ}(\tau, x) = \int_{\tau_0}^{\tau} d\tau' L(x_{\rm sol}(\tau'), \dot{x}_{\rm sol}(\tau')) \Big|_{x(\tau_0)=x_0}^{x(\tau)=x} . \tag{8.2.1}$$

rendering it a function rather than a functional.

One important assumption here is that given the initial and the final points, and also, given the time span $\tau - \tau_0$ specified, there is generically a unique solution $x_{\rm sol}(\tau')$ that interpolates the two. Instead of specifying two sets of initial conditions, say the position and the momentum, we fix initial and final points. This may not actually hold. Take for instance a free particle on a circle. Even if we declare the initial and the final point, for example take them to be identical, there are infinite number of Newtonian trajectories, each of which is characterized by how many times the trajectory move around the circle. Therefore, $S_{\rm HJ}$ are in principle classified by additional labels, which in this last example, are the winding numbers, even if we fix the inial and the final condition. The point is that what follows is applicable to one such class each at a time.

The key property of $S_{\rm HJ}$ is that the derivatives with respect to its argument computes the conjugate momenta and the energy of the classical trajectory in question. Suppose we shift the final point slightly so that one of coordinate values, say x^μ, is shifted slightly as $x^\mu \to x^\mu + \delta$. This of course shift the solution to $x_{\rm sol} + \Delta x_{\rm sol}$ which also obeys the equation of motion with the initial position at τ_0 fixed, $\Delta x_{\rm sol}^{\nu\neq\mu}(\tau) = 0$, and $\Delta x^\mu_{\rm sol}(\tau) = \delta$.

Taking a derivative with respect to x, therefore, we have

$$\begin{aligned}
\frac{\partial S_{\rm HJ}}{\partial x^\mu} &= \lim_{\delta\to 0}\frac{1}{\delta}\int_{\tau_0}^{\tau} ds\,\left(L(x_{\rm sol}+\Delta x_{\rm sol}) - L(x_{\rm sol})\right) \\
&= \lim_{\delta\to 0}\frac{1}{\delta}\int_{\tau_0}^{\tau} ds\left(\Delta x^\lambda\frac{\delta L(x,\dot{x})}{\delta x^\lambda} + \Delta\dot{x}^\lambda\frac{\delta L(x,\dot{x})}{\delta \dot{x}^\lambda}\right)\Bigg|_{x\to x_{\rm sol}} \\
&= \lim_{\delta\to 0}\frac{1}{\delta}\int_{\tau_0}^{\tau} ds\frac{d}{ds}\left(\Delta x^\lambda\frac{\delta L}{\delta\dot{x}^\lambda}\right)\Bigg|_{x\to x_{\rm sol}} \\
&= p_\mu(x(\tau),\tau)\Bigg|_{x\to x_{\rm sol}}\,,
\end{aligned} \tag{8.2.2}$$

where the last is meant to be the value of the conjugate momenta p_μ at τ for the solution $x_{\rm sol}(s)$. We emphasize that in the last step the lower end of the integral does not contribute since $\Delta x_{\rm sol}(\tau_0) = 0$.

We can also take a derivative with respect to τ by keeping the final point x fixed but rather requires the trajectory to take a slightly longer time $(\tau - \tau_0) + \delta\tau$. Again this shifts the trajectory to something nearby $x_{\rm sol} + \tilde{\Delta} x_{\rm sol}$,

$$\frac{\partial S_{\rm HJ}}{\partial \tau} = \lim_{\delta\to 0}\frac{1}{\delta\tau}\left(\int_{\tau_0}^{\tau+\delta\tau} dsL(x_{\rm sol}+\tilde{\Delta}x_{\rm sol}) - \int_{\tau_0}^{\tau} dsL(x_{\rm sol})\right)\,. \tag{8.2.3}$$

There are two types of terms on the right. The first is from $\delta\tau$ shift in the integration range, which simply gives

$$L(x(\tau), \dot{x}(\tau))\bigg|_{x\to x_{\rm sol}} . \tag{8.2.4}$$

The latter is more subtle. Since we are interested in the linear deviation due to $\tilde{\Delta}x$ inside the Lagrangian, this second contribution become quite similar to the above spatial derivative,

$$\begin{aligned} &\lim_{\delta\tau\to 0}\frac{1}{\delta\tau}\int_{\tau_0}^{\tau} ds\left(L(x_{\rm sol}+\tilde{\Delta}x_{\rm sol}) - L(x_{\rm sol})\right) \\ = &\lim_{\delta\tau\to 0}\frac{1}{\delta\tau}\int_{\tau_0}^{\tau} ds\frac{d}{ds}\left(\tilde{\Delta}x^\lambda\frac{\delta L(x_{\rm sol})}{\delta\dot{x}^\lambda}\right) \\ = &\lim_{\delta\tau\to 0}\frac{1}{\delta\tau}\int_{\tau_0}^{\tau} ds\frac{d}{ds}\left(\tilde{\Delta}x^\lambda\frac{\delta L(x_{\rm sol})}{\delta\dot{x}^\lambda}\right) . \end{aligned} \tag{8.2.5}$$

On the other hand, $x_{\rm sol}+\tilde{\Delta}x_{\rm sol}$ is supposed to arrive at x at $s=\tau+\delta\tau$, so at $s=\tau$, it falls short by small amount,

$$\tilde{\Delta}x^\lambda_{\rm sol}\bigg|_{\tau'=\tau} = -\delta\tau\,\dot{x}^\lambda_{\rm sol} , \tag{8.2.6}$$

whereas $x_{\rm sol}+\tilde{\Delta}x_{\rm sol}$ is still anchored at x_0 at τ_0. Therefore the second contribution gives

$$-\dot{x}^\lambda(\tau)p_\lambda(\tau)\bigg|_{x\to x_{\rm sol}} . \tag{8.2.7}$$

Combining, we find the time derivative gives the minus of the conserved energy,

$$\frac{\partial S_{\rm HJ}}{\partial\tau} \;=\; -\left(\dot{x}^\lambda(\tau)p_\lambda(\tau) - L(x(\tau),\dot{x}(\tau))\right)\bigg|_{x\to x_{\rm sol}} \;=\; -E(x_{\rm sol}) . \tag{8.2.8}$$

where the extra sign on the right should be noted.

There is actually a simpler way to see these facts, if somewhat formal. Recall how the action may be written in the canonical form as,

$$S = \int d\tau\,(p_\mu\dot{x}^\mu - H(x,p)) = \int (p_\mu dx^\mu - d\tau\,H(x,p)) \tag{8.2.9}$$

Applying this to actual trajectories, $x_{\rm sol}(s)$ and $p_{\rm sol}(s)$, we find

$$S_{\rm HJ}(\tau, x) = \left(\int_{x_0}^{x} dy^\mu \, p_\mu - \int_{\tau_0}^{\tau} ds \, H(y(s), p(s)) \right) \Bigg|_{y = x_{\rm sol}(s),\, p = p_{\rm sol}(s)} \tag{8.2.10}$$

or

$$dS_{\rm HJ} = dx^\mu \, p_\mu - d\tau \, E(x_{\rm sol}, p_{\rm sol}) \tag{8.2.11}$$

which gives the same relations as above.

So what do we do with these? Suppose we have a simple one-dimensional Newtonian problem,

$$L = \frac{1}{2} m \dot{x}^2 - V(x) \tag{8.2.12}$$

for example, we always have at least the Hamiltonian as one conserved quantity,

$$\mathcal{H}(x, p) = \frac{1}{2m} p^2 + V(x) = E \ , \tag{8.2.13}$$

where we insert the above to find a partial differential equation for $S_{\rm HJ}$

$$\frac{1}{2m} \left(\frac{\partial S_{\rm HJ}}{\partial x} \right)^2 + V(x) = -\frac{\partial S_{\rm HJ}}{\partial t} = E \ . \tag{8.2.14}$$

Solving this for $S_{\rm HJ}$, we find

$$S_{\rm HJ} = -E\,(t - t_0) \pm \int_{x_0}^{x} dx' \sqrt{2m\,(E - V(x'))} \ . \tag{8.2.15}$$

defining the Hamilton-Jacobi function for this simple one-dimensional mechanics.

The assertion, which can be verified through standard Lagrangian dynamics, is that the classical trajectories emerge by extremizing $S_{\rm HJ}$ with respect to the parameters, namely the values of the conserved quantities. In this simple case, E is the parameter in question, so the trajectories obey

$$0 = \frac{\partial S_{\rm HJ}}{\partial E} = -(t - t_0) \pm \int_{x_0}^{x} dx' \sqrt{\frac{m/2}{E - V(x')}} \ . \tag{8.2.16}$$

One can see that if one takes a derivative with respect to t, this gives the energy conservation $\mathcal{H} = E$ precisely, so this is nothing but the integrated conservation law.

Clearly, the same works if the number of conserved quantities is the same as the number of dynamical variables. For Newtonian central force problems, where we may set the spherical coordinates such that the trajectory lies along the plane, $\theta = \pi/2$, we find

$$\frac{1}{2m}\left(\frac{\partial S_{\rm HJ}}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial S_{\rm HJ}}{\partial \phi}\right)^2 + V(r) = -\frac{\partial S_{\rm HJ}}{\partial t} \tag{8.2.17}$$

with

$$\frac{\partial S_{\rm HJ}}{\partial \phi} = L\ , \qquad -\frac{\partial S_{\rm HJ}}{\partial t} = E\ , \tag{8.2.18}$$

leading us to

$$S_{\rm HJ} = -E\,(t - t_0) + L(\phi - \phi_0) \pm \int_{r_0}^{r} dr' \sqrt{2m\,(E - V(r')) - \frac{L^2}{r^2}}\ , \tag{8.2.19}$$

and again extremizing with respect to E and L produces the classical trajectories.

The fact that $S_{\rm HJ}$ should be extremized for actual trajectories is, of course, the mathematical consequence of the classical dynamics, which ultimately connects to the usual action principle. As is well known, the path integral representation of quantum mechanics has the measure

$$e^{\mathbb{i}S/\hbar}\ , \tag{8.2.20}$$

where S is the action. The path integral sum over all possible trajectories, not just the classical ones, but at the same time, the classical trajectories are still special in that the phase is extremized along such paths. Since the path integral is essentially sum over all possible waves between the starting point and the end point, and since extremization means constructive interference of such nearby wave forms, the emergence of classical path from the condition of extremization of S is quite natural from the quantum mechanics viewpoint.

Taking one more step, then, where we constrain our infinite-dimensional possibilities of paths down to those parameterized by those few conserved quantities, the extremization of $S_{\rm HJ}$ follows from the same principle of constructive interference. Recall that $S_{\rm HJ}$ is a rather special quantity from the path integral viewpoint as well, since it represents the value of the action in a semi-classical approach of the saddle point approximation.

Another important observation along the same line of reasoning tells us why the motion should be geodesics on the curved spacetime once we apply the same principle of constructive interference. By the way, although we say the trajectories are geodesics, which usually means the shortest path, the time-like trajectories can be seen to maximize the proper time accumulated relative to infinitesimally nearby paths.