Lee-Wick's Complex Ghost violates Unitarity in Quadratic Gravity

Taichiro Kugo
YITP, Kyoto University

November 18 – 23, 2024

The 12th KIAS Workshop on Particle Physics and Cosmology
and 2024 Korea-France STAR workshop

based on the collaboration with

Jisuke Kubo
Max Planck Institute für Kernphysik, Heidelberg and Univ. Toyama

PTEP 2023 (2023) 12, 123B02 [arXiv:2308.09006 [hep-th]]
PTEP 2024 (2024) 5, 053B01 [arXiv:2402.15956 [hep-th]]

1 Introduction

Standard model contains no scale (i.e., dimensionful) parameter except only the Higgs mass term.

This suggests that *the fundamental theory of our world may have no scale parameter*.

For instance, Salvio and Strumia (2017) proposed 'Agravity' theory (adimentional gravity theory), whose action looks like

$$\mathcal{L} = \underbrace{R^2 + R_{\mu\nu}^2}_{\mathcal{L}_{\text{Pure Quad.Gravity}}} + \underbrace{\mathcal{L}'_{\text{SM}}}_{m^2|H|^2\text{omitted}} + \xi|H|^2R. \tag{1}$$

(plus singlet scalar terms).

Such a theory is renormalizable and UV complete, and the scale of the world would be generated by spontaneous breaking of scale invariance owing to quadratic gravity dynamics which is asymptotically free and IR strong around Planck scale. So it is wonderful if it could give a fundamental theory for all the interactions.

But it contains fourth order derivative terms of gravity field, thanks to which the theory is made renormalizable but simultaneously causes the problem of negative metric ghost. We discuss this problem in this talk.

Fourth order derivative theories

Lee-Wick: Finite QED '69 $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}\left(1 - \frac{\Box}{m^2}\right)F^{\mu\nu}$

Quadratic Gravity Theory $\mathcal{L} = m^2 R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu}$

Propagator

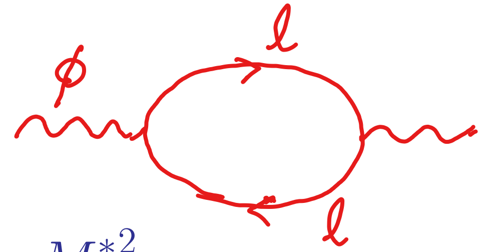
$$\langle\phi\,\phi\rangle = \frac{1}{k^2 + k^4/m^2} = \frac{1}{k^2} - \frac{1}{k^2 + m^2}$$

$\frac{1}{k^2}$	$\frac{1}{k^2+m^2}$
massless	massive (ghost)
positive metric	negative metric

$\Rightarrow$ Finite QED becomes finite.
Quadratic Gravity becomes renormalizable and UV complete!

But, Massive Ghost $\Longrightarrow$ Physical Unitarity is violated

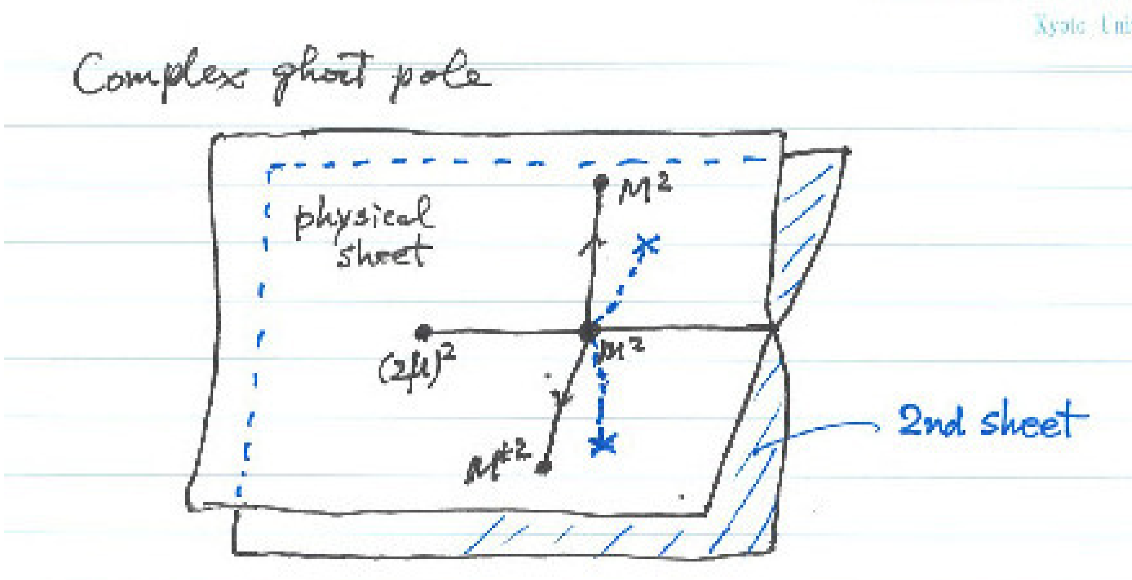
Lee-Wick noted:
massive ghost can "decay" into lepton pair
Pole at one-loop : lepton-loop graph $\Sigma(q)$



$$q^2 + q^4/m^2 - \Sigma(q) = 0 \quad \rightarrow \quad q^2 = M^2,\ M^{*2}$$

$$\boxed{M^2 = m^2 + i\gamma^2}$$

Complex poles on the physical sheet! for ghost



Complex energy $E_{\boldsymbol{q}} = \sqrt{\boldsymbol{q}^2 + M^2} \quad \Rightarrow$ Complex Ghost

$$\langle\phi\phi\rangle = \underset{\langle AA\rangle}{\frac{1}{q^2}} - \frac{1}{2}\Big(\underset{\langle\varphi\varphi\rangle}{\frac{1}{q^2+M^2}} + \underset{\langle\varphi^\dagger\varphi^\dagger\rangle}{\frac{1}{q^2+M^{*2}}}\Big) \qquad M^2 = m^2 + i\gamma^2$$

$$\longrightarrow \quad \boxed{\phi = A + \frac{1}{\sqrt{2}}(\varphi + \varphi^\dagger)}$$

Lee-Wick and their successors claim the Physical Unitarity based mainly on two grounds (reasoning):

1. Energy Conservation

   Complex ghosts will never be produced by collisions of physical particles possessing real energy $\quad\rightarrow\quad$ physical unitarity holds

2. Massive ghost is unstable

   Even if the ghosts are produced, they will decay into lighter ordinary particles and eventually disappear after sufficiently long time. $\quad\rightarrow\quad$ So physical unitarity holds in any case.

In this talk we show

$$\boxed{\text{These are totally wrong}}$$

and that

1. Complex ghost can actually be created by physical particle collisions with finite probability, consistently with energy conservation.
2. The ghost has strong stability, which may be called as "Anti-instability"; that is, the more it 'decays' into ordinary particles, the larger the probability it remains as itself becomes.

Strangely enough,

$$\begin{array}{rcl} \text{N. Nakanishi} & \rightarrow & \text{Lorentz inv. broken} \\ \text{S. Coleman} & \rightarrow & \text{Causality, broken} \end{array}$$

criticize, nevertheless seem to approve the reasoning of Physical Unitarity by Energy conservation law ("in particular, because of the conservation of the imaginary part of the energy" (Coleman)).

Even recently, in Quadratic Gravity (or in 'Lee-Wick standard model'), this Lee-Wick's complex ghost theory is revived:
Anselmi(2017,2018), Donoghue(2019, 2021),
(Grinstein, O'Connell and Wise (2008 – 2009))

If this is OK, Quadratic Gravity theory, in particular, a fascinating scale-invariant 'Agravity' theory, already gives perturbatively renormalizable and UV complete gravity theory!

$\rightarrow$ Unfortunately, however, this is not true.

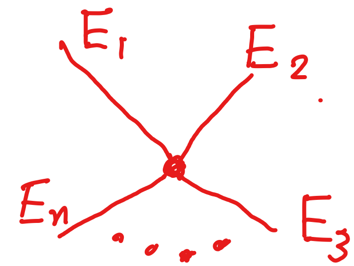
Contents

2 Complex ghost can be created

2.1 Energy Conservation and Complex Delta Function

"What is Energy Conservation Law?"

$$\int_{-\infty}^{\infty} e^{-iEt} dt = 2\pi\delta(E), \qquad E = \sum_i E_i$$



where E_i: energy of i-th particle coming into a interaction vertex.

If some E_i are complex energies, we need regularization

$$\mathcal{L}_{\text{int}}(t) \quad \rightarrow \quad e^{-a^2t^2}\mathcal{L}_{\text{int}}(t)$$

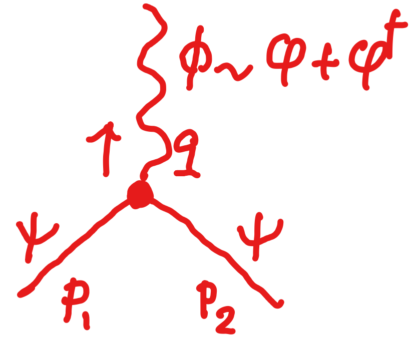
and take $\lim_{a\rightarrow 0}$ finally.

Then, $\delta(E)$ replaced by

$$\Delta_a(z) := \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{-a^2t^2} e^{-izt} = \frac{1}{2\sqrt{\pi}\,a} e^{-z^2/a^2} \,. \tag{1}$$

Its $a \rightarrow 0$ limit defines Complex Delta function (distribution)

$$\boxed{\lim_{a\rightarrow 0} \Delta_a(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{-izt} =: \delta_c(z) \,,} \tag{2}$$



The support of this distribution is not localized at $z = 0$!

Consider the single ghost production in Lee's scalar model[1]

$$\psi(\boldsymbol{p}_1) + \psi_{(}\boldsymbol{p}_2) \to \phi(\boldsymbol{q}), \qquad \text{by} \quad \mathcal{L}_{\rm int} = \frac{f}{2}\psi^2\phi$$

. Then, the invariant amplitude square (Ghost production probability) is proportional to

$$|\mathcal{M}|^2 \sim f^2\left(\frac{1}{\omega_{\boldsymbol{q}}}\delta_c(E-\omega_{\boldsymbol{q}}) + \frac{1}{\omega^*_{\boldsymbol{q}}}\delta_c(E-\omega^*_{\boldsymbol{q}})\right) \qquad \omega_{\boldsymbol{q}} = \sqrt{\boldsymbol{q}^2 + M^2}$$

and $E = \sqrt{\boldsymbol{p}_1^2 + \mu^2} + \sqrt{\boldsymbol{p}_2^2 + \mu^2}$ is the incident Energy (real).

So let us examine

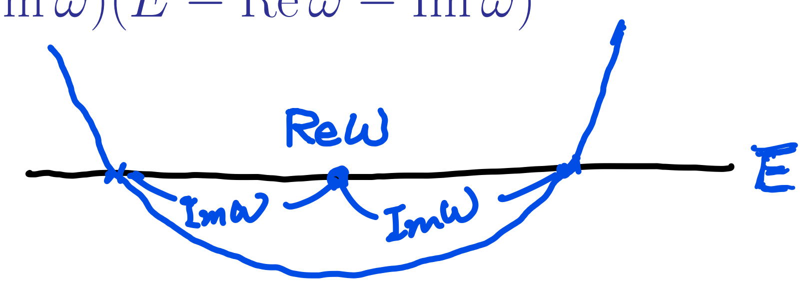
$$\Delta_a(E-\omega) \propto \frac{1}{a}e^{-(E-\omega)^2/a^2} = \frac{1}{a}\exp\left[-\frac{(E-\mathrm{Re}\,\omega)^2 - (\mathrm{Im}\,\omega)^2}{a^2}\right]\cdot e^{i\Theta}$$

$$\text{with} \quad \Theta = \frac{2}{a^2}(E-\mathrm{Re}\,\omega)\mathrm{Im}\,\omega.$$

Since

$$(E-\mathrm{Re}\,\omega)^2 - (\mathrm{Im}\,\omega)^2 = (E-\mathrm{Re}\,\omega + \mathrm{Im}\,\omega)(E-\mathrm{Re}\,\omega-\mathrm{Im}\,\omega)$$

[1]next section



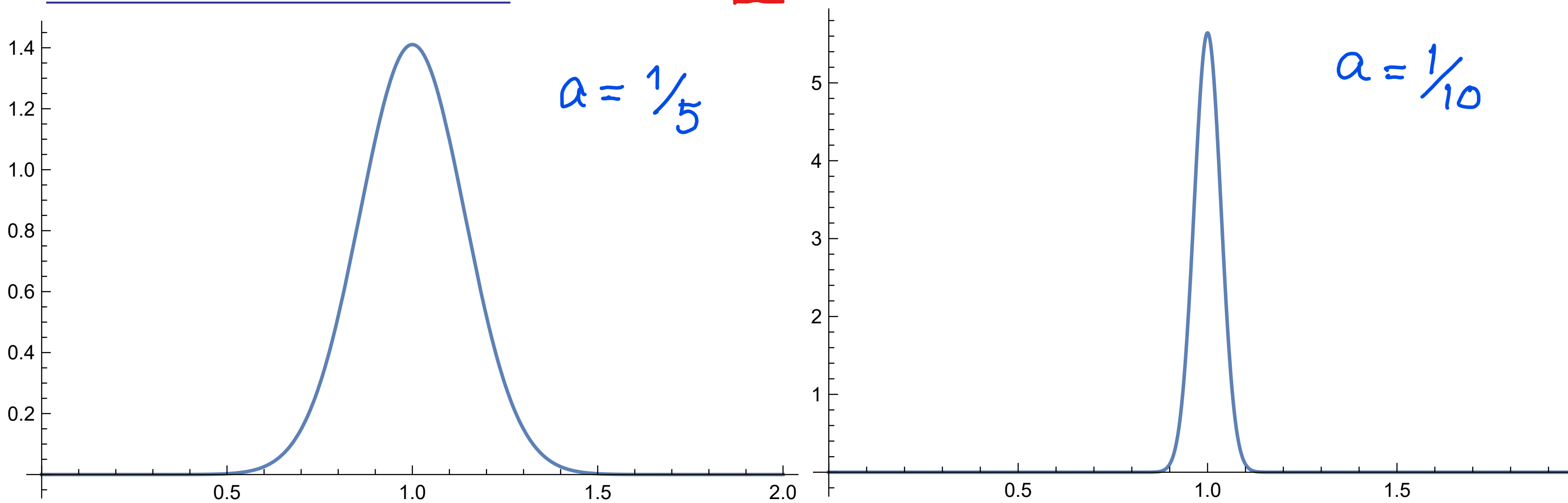
we have

$$\delta_c(E-\omega) = \lim_{a\to 0} \Delta_a(E-\omega) = 0 \quad \text{for} \quad \begin{cases} E < \operatorname{Re}\omega - \operatorname{Im}\omega \\ \operatorname{Re}\omega + \operatorname{Im}\omega < E \end{cases} . \tag{3}$$

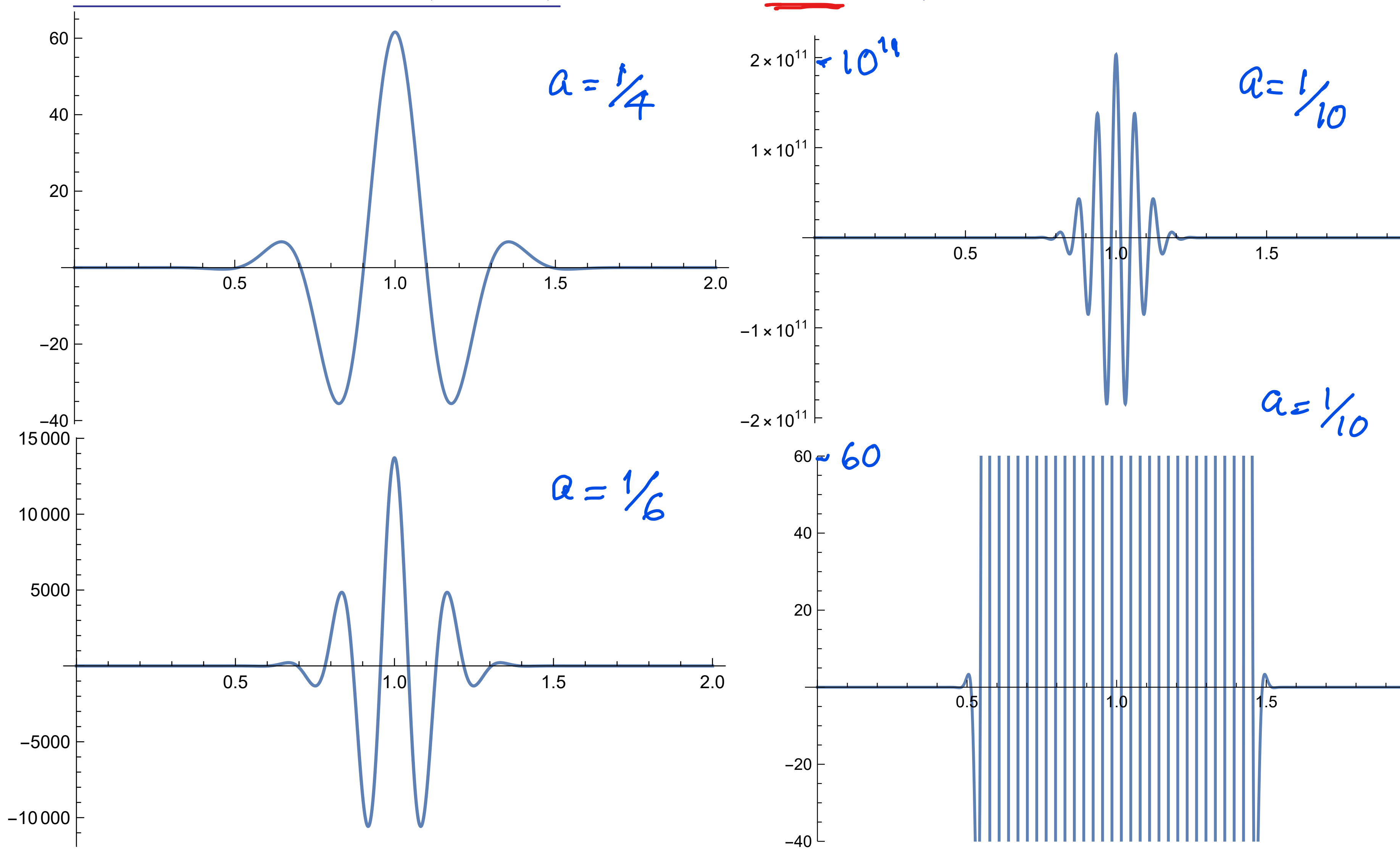
So, in the limit $a \to 0$, it has the support only in $|E - \operatorname{Re}\omega| \leq \operatorname{Im}\omega$. On the support, it has no definite limit; since it is divergent and rapidly oscillating. It gives a well-defined distribution. It is as usual for the distribution.

Figure of $\delta_c(E-\omega) = \lim_{a\to 0} \Delta_a(E-\omega)$

- **Dirac delta** $\delta(E-\omega)$: $\omega = 1 + 0i$, $1/a = 5, 20$



- **Complex delta** $\delta_c(E-\omega)$: $\quad \omega = 1 + 0.5i, \quad 1/a = 4, 10$



The Lower Left: $1/a=6$, $\qquad E_{\rm Lth}=\mathrm{Re}\,\omega-\mathrm{Im}\,\omega = 0.5$, $\; E_{\rm Uth}=\mathrm{Re}\,\omega+\mathrm{Im}\,\omega=1.5$

Property of δ_c

For test fn $\forall f(E)$: analytic in a rectangular strip D,

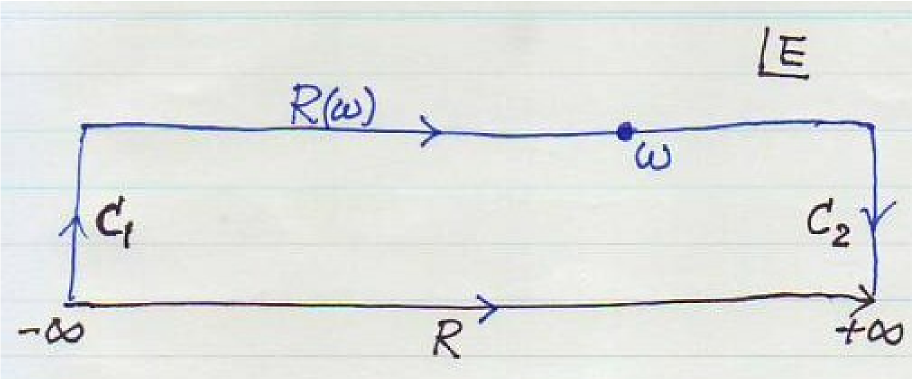
$$\boxed{\int_R dE\, \delta_c(E-\omega)\, f(E) = f(\omega)} \tag{4}$$

($\because$)

$$\int_{-\infty}^{\infty} dE\, \delta_c(E-\omega)\, f(E) = \lim_{a\to 0} \int_{-\infty}^{\infty} dE\, \frac{1}{2a\sqrt{\pi}} \exp\left[-\frac{(E-\omega)^2}{a^2}\right] f(E)$$

is evaluated by deforming the contour

$E \to E' + i\,\mathrm{Im}\,\omega$ $\qquad R \;\Rightarrow\; C_1 + R(\omega) + C_2$



If $f(E)$ is analytic in the rectangular domain D (surrounded by $[C_1 + R' + C_2 - R]$)

$$\int_{R(\omega)} dE\, \delta_c(E-\omega) f(E) = \int_{-\infty}^{\infty} dE'\, \delta(E' - \mathrm{Re}\,\omega) f(E' + i\mathrm{Im}\,\omega)$$

$$= f(\mathrm{Re}\,\omega + i\mathrm{Im}\,\omega) = f(\omega)\,. \text{ q.e.d.}$$

But, in actual Feynman graph calculations, $f(E)$ is meromorphic.

$\delta_{\rm c}$: Well-defined distribution?

In the actual experiment, there is a finite width σ in the incident energy:

$$f_{P^0}(E) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2}\Big(\frac{E-P^0}{\sigma}\Big)^2\right]$$

Then, the ghost production probability becomes

$$\begin{aligned} P_\varphi &= -\mathrm{Re}\left[f^2 2\pi \int_{-\infty}^{\infty} dE\, f_{P^0}(E)\delta_{\rm c}(E-\omega)\frac{1}{2\omega}\right] \\ &= -\mathrm{Re}\left[f^2\frac{\pi}{\omega}f_{P^0}(\omega)\right] = -\frac{f^2}{\sqrt{2\pi\sigma^2}}\mathrm{Re}\left[\frac{\pi}{\omega}\exp\Big[-\frac{1}{2}\Big(\frac{\omega-P^0}{\sigma}\Big)^2\Big]\right]. \end{aligned}$$

This is finite.

We conclude:

Complex Ghost is produced with a finite probability for the energy range $\mathrm{Re}\,\omega - \mathrm{Im}\,\omega < E < \mathrm{Re}\,\omega + \mathrm{Im}\,\omega$.

2.2 Lee's Model in "Quanta"

$$\text{Asymptotic field:} \qquad \text{ghost} \quad \phi = \underset{\text{'photon'}}{A} + \underbrace{\frac{1}{\sqrt{2}}\left(\varphi + \varphi^\dagger\right)}_{=B}$$

$$\text{matter} \quad \psi$$

$$\mathcal{L} = \mathcal{L}_\phi + \mathcal{L}_\psi + \mathcal{L}_{\text{int}}$$

$$\mathcal{L}_\phi = -\frac{1}{2}\left[(\partial_\mu A)^2 + \delta^2 A^2\right] + \frac{1}{2}\left[(\partial_\mu B)^2 + m^2 B^2\right] - \frac{1}{2}\left[(\partial_\mu C)^2 + m^2 C^2\right]$$

$$\underline{-\gamma^2 BC}$$

$$= \boxed{-\frac{1}{2}\left[(\partial_\mu A)^2 + \delta^2 A^2\right] + \frac{1}{2}\left[\partial_\mu\varphi\,\partial^\mu\varphi + M^2\varphi^2 + \partial_\mu\varphi^\dagger\,\partial^\mu\varphi^\dagger + M^{*2}\varphi^{\dagger 2}\right]},$$

$$\mathcal{L}_\psi = -\frac{1}{2}(\partial_\mu\psi)^2 - \frac{1}{2}\mu^2\psi^2$$

where $M^2 = m^2 + i\gamma^2$ and

$$\varphi = \frac{1}{\sqrt{2}}(B - iC) \qquad \text{or} \qquad \begin{cases} B = (\varphi + \varphi^\dagger)/\sqrt{2} \\ C = i(\varphi - \varphi^\dagger)/\sqrt{2} \end{cases} .$$

$$\mathcal{L}_{\rm int}(\phi,\psi) = f\psi^2\phi, \quad f\psi^2\phi^2, \cdots \tag{5}$$

'Photon' A and ghost φ , anti-ghost$\varphi^\dagger$ interact only through $\mathcal{L}_{\rm int}(\phi,\psi)$

This system is canonically quantized.

Free fields:

$$A(x) = \int \frac{d^3\boldsymbol{q}}{\sqrt{(2\pi)^3 2\nu_{\boldsymbol{q}}}} \left(a(\boldsymbol{q}) e^{i\boldsymbol{q}\boldsymbol{x} - i\nu_{\boldsymbol{q}} x^0} + a^\dagger(\boldsymbol{q}) e^{-i\boldsymbol{q}\boldsymbol{x} + i\nu_{\boldsymbol{q}} x^0} \right), \quad \nu_{\boldsymbol{q}} = \sqrt{\boldsymbol{q}^2 + \delta^2}$$

$$\psi(x) = \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3 2E_{\boldsymbol{p}}}} \left(d(\boldsymbol{p}) e^{i\boldsymbol{p}\boldsymbol{x} - iE_{\boldsymbol{p}} x^0} + d^\dagger(\boldsymbol{p}) e^{-i\boldsymbol{p}\boldsymbol{x} + i\nu_{\boldsymbol{p}} x^0} \right), \quad E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + \mu^2} \ .$$

$$[a(\boldsymbol{p}), a^\dagger(\boldsymbol{q})] = [d(\boldsymbol{p}), d^\dagger(\boldsymbol{q})] = +\delta^3(\boldsymbol{p} - \boldsymbol{q}),$$

Ghost

$$\boxed{\varphi(x) = \int \frac{d^3\boldsymbol{q}}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{q}}}} \left(\alpha(\boldsymbol{q}) e^{i\boldsymbol{q}\boldsymbol{x} - i\omega_{\boldsymbol{q}} x^0} + \beta^\dagger(\boldsymbol{q}) e^{-i\boldsymbol{q}\boldsymbol{x} + i\omega_{\boldsymbol{q}} x^0} \right)}$$

where $\omega_{\boldsymbol{q}}$ is the complex energy

$$\omega_{\boldsymbol{q}} = \sqrt{\boldsymbol{q}^2 + M^2} = \sqrt{\boldsymbol{q}^2 + m^2 + i\gamma^2} \tag{6}$$

CCR

$$\begin{aligned} [\alpha(\boldsymbol{p}), \beta^\dagger(\boldsymbol{q})] = [\beta(\boldsymbol{p}), \alpha^\dagger(\boldsymbol{q})] &= -\delta^3(\boldsymbol{p} - \boldsymbol{q}), \\ [\alpha(\boldsymbol{p}), \alpha^\dagger(\boldsymbol{q})] = [\beta(\boldsymbol{p}), \beta^\dagger(\boldsymbol{q})] &= 0. \end{aligned}$$

1-ghost states

$$|\alpha(\boldsymbol{p})\rangle := \alpha^\dagger(\boldsymbol{p}) |0\rangle , \quad |\beta(\boldsymbol{p})\rangle := \beta^\dagger(\boldsymbol{p}) |0\rangle$$

have off-diagonal innerproduct structure

$$\begin{aligned} \langle \alpha(\boldsymbol{p}) | \alpha(\boldsymbol{q}) \rangle &= 0, \qquad & \langle \beta(\boldsymbol{p}) | \alpha(\boldsymbol{q}) \rangle &= -\delta^3(\boldsymbol{p} - \boldsymbol{q}), \\ \langle \beta(\boldsymbol{p}) | \beta(\boldsymbol{q}) \rangle &= 0, \qquad & \langle \alpha(\boldsymbol{p}) | \beta(\boldsymbol{q}) \rangle &= -\delta^3(\boldsymbol{p} - \boldsymbol{q}). \end{aligned}$$

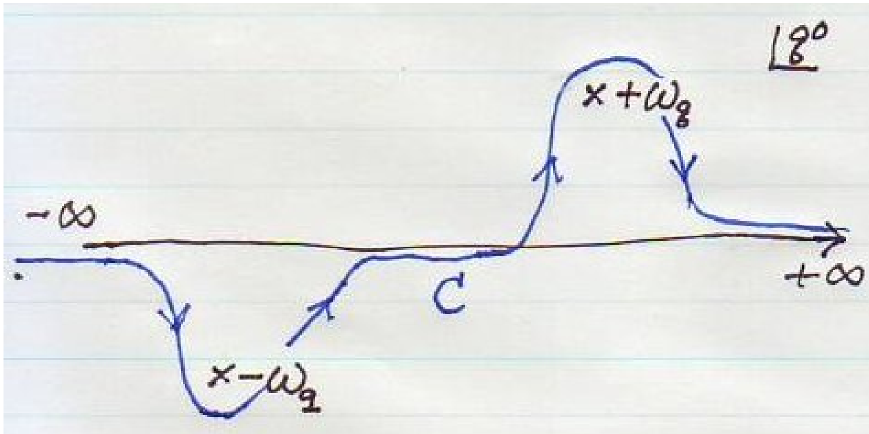
Propagator:

$$\begin{aligned}
&\langle 0|\, \mathrm{T}\varphi(x)\,\varphi(y)\,|0\rangle \\
&= \int \frac{d^3\boldsymbol{q}d^3\boldsymbol{p}}{(2\pi)^3\sqrt{2\omega_{\boldsymbol{q}}2\omega_{\boldsymbol{p}}}}\Big\{\theta(x^0-y^0)e^{i(\boldsymbol{qx}-\omega_{\boldsymbol{q}}x^0)-i(\boldsymbol{py}-\omega_{\boldsymbol{p}}y^0)}\,\langle 0|\,\alpha(\boldsymbol{q})\beta^\dagger(\boldsymbol{p})\,|0\rangle \\
&\qquad + \theta(y^0-x^0)e^{i(\boldsymbol{py}-\omega_{\boldsymbol{p}}y^0)-i(\boldsymbol{qx}-\omega_{\boldsymbol{q}}x^0)}\,\langle 0|\,\alpha(\boldsymbol{p})\beta^\dagger(\boldsymbol{q})\,|0\rangle\Big\} \\
&= - \int \frac{d^3\boldsymbol{q}}{(2\pi)^3 2\omega_{\boldsymbol{q}}}\Big\{\theta(x^0-y^0)e^{i\boldsymbol{q}(\boldsymbol{x}-\boldsymbol{y})-i\omega_{\boldsymbol{q}}(x^0-y^0)} + \theta(y^0-x^0)e^{-i\boldsymbol{q}(\boldsymbol{x}-\boldsymbol{y})+i\omega_{\boldsymbol{q}}(x^0-y^0)}\Big\}
\end{aligned}$$

Note that the over-all minus sign implying negative norm.

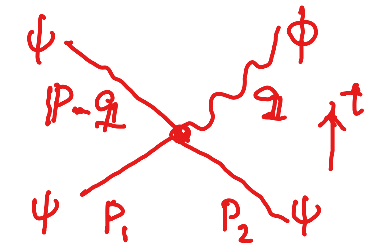
This 3d expression over $d^3\boldsymbol{q}$ can be rewritten into the usual 4d form over $d^4q = d^3\boldsymbol{q}dq^0$

$$= -\int \frac{d^3q}{(2\pi)^3}e^{i\boldsymbol{q}(\boldsymbol{x}-\boldsymbol{y})}\left[\int_C \frac{dq^0}{2\pi i}\frac{e^{-iq^0(x^0-y^0)}}{q^2+M^2}\right]$$



2.3 Lee-Wick's mistake: property of complex delta function

To compute the cross section of "single ghost ϕ + matter ψ" production

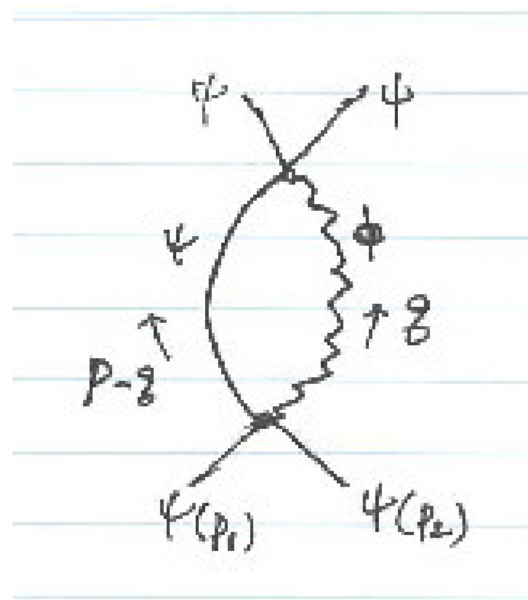


$$\psi(\boldsymbol{p}_1) + \psi(\boldsymbol{p}_2) \quad \rightarrow \quad \phi(\boldsymbol{q}) + \psi(\boldsymbol{p}-\boldsymbol{q})$$

($\boldsymbol{p}_1 + \boldsymbol{p}_2 \equiv \boldsymbol{p}$), they calculate the forward scattering amplitude

$$\psi(\boldsymbol{p}_1) + \psi(\boldsymbol{p}_2) \quad \xrightarrow[(\phi(\boldsymbol{q})+\psi(\boldsymbol{p}-\boldsymbol{q})\ \text{Loop})]{} \quad \psi(\boldsymbol{p}_1) + \psi(\boldsymbol{p}_2)$$

whose imaginary part gives the desired cross section.



$$\mathcal{L}_{\text{int}} = \frac{1}{3!}\psi^3\phi$$

$$\phi = A + \frac{1}{\sqrt{2}}(\varphi + \varphi^\dagger)$$

Lee wrote $\quad [\ \phi_i = (A, \varphi, \varphi^\dagger),\ M_i^2 = (\delta^2, M^2, M^{*2})\ ]$

$$\boxed{\Sigma(p) = -f^2 \sum_j \int_{C_j} \frac{d^4q}{i(2\pi)^4} \overbrace{\frac{1}{q^2 + M_j^2}}^{D_\phi(q)} \overbrace{\frac{1}{(p-q)^2 + \mu^2 - i\varepsilon}}^{D_\psi(p-q)}} \tag{7}$$

and computed it by contour integration ($\boldsymbol{k} \equiv \boldsymbol{p} - \boldsymbol{q}$) to find

$$= \frac{f^2}{32\pi^2} \int d^3\boldsymbol{q}$$

$$\times \Bigg\{ \frac{1}{\nu_{\boldsymbol{q}} E_{\boldsymbol{k}}} \left(\frac{1}{p^0 - \nu - E + i\varepsilon} - \frac{1}{p^0 + \nu + E - i\varepsilon} \right) \qquad \leftarrow A\,\psi$$

$$- \frac{1}{2} \Bigg[\frac{1}{\omega_{\boldsymbol{q}} E_{\boldsymbol{k}}} \left(\frac{1}{p^0 - \omega - E} - \frac{1}{p^0 + \omega + E} \right) \qquad \leftarrow \varphi\,\psi$$

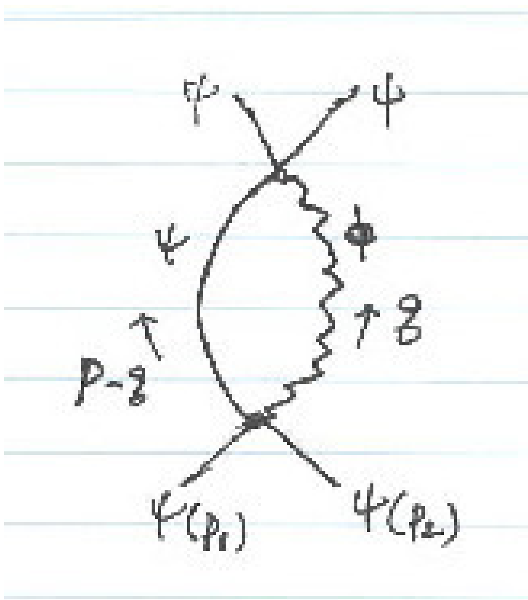
$$+ \frac{1}{\omega^*_{\boldsymbol{q}} E_{\boldsymbol{k}}} \left(\frac{1}{p^0 - \omega^* - E} - \frac{1}{p^0 + \omega^* + E} \right) \Bigg] \Bigg\} \qquad \leftarrow \varphi^\dagger\,\psi$$

Sum of the last two terms is real, has no imaginary part! He concluded Ghost is not produced.

What's wrong?

Shocking fact!

Naive Feynman rule is incorrect.



$$\int_C d^4q\, D_\psi(p-q)\, D_\phi(q)$$ ← This is wrong!

$$\neq \int_C d^4q \int_R d^4k\, \underbrace{\delta^4_c(k+q-p)}\, D_\psi(k)\, D_\phi(q)$$

↑ This is correct

The latter is the correct one.

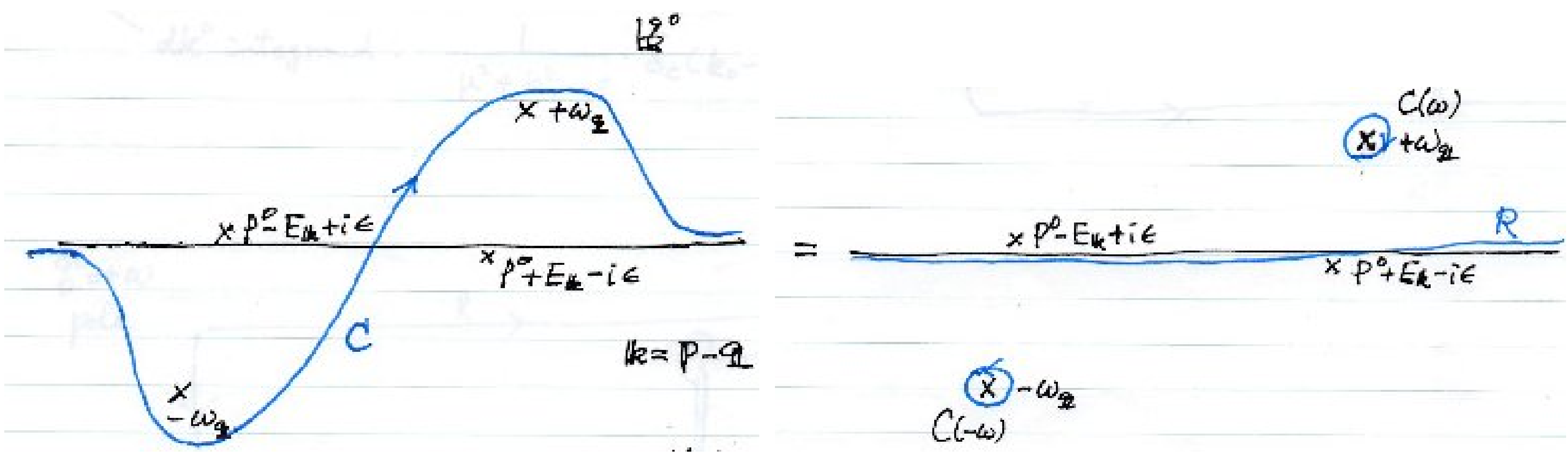
Starting Feynman rule is wrong. Correct one is:

$$-\Sigma(p) = f^2 \int \frac{d^3\boldsymbol{q}}{i(2\pi)^4} \int d^3\boldsymbol{k}\, \delta^3(\boldsymbol{k}+\boldsymbol{q}-\boldsymbol{p})$$
$$\times \int_C dq^0 \int_R dk^0\, \delta_c(k^0+q^0-p^0)\, \overset{D_\varphi(q)}{\frac{1}{q^2+M^2}}\, \overset{D_\psi(k)}{\frac{1}{k^2+\mu^2-i\varepsilon}}$$

The usual substitution rule $k^0 \to p^0 - q^0$ does not apply for k^0 integration! because q^0 is complex on the contour C.

So perform q^0 integration first and then k^0 integration

$$\int_C dq^0 = \int_R dq^0 + \int_{C(-\omega)} dq^0 + \int_{C(+\omega)} dq^0$$

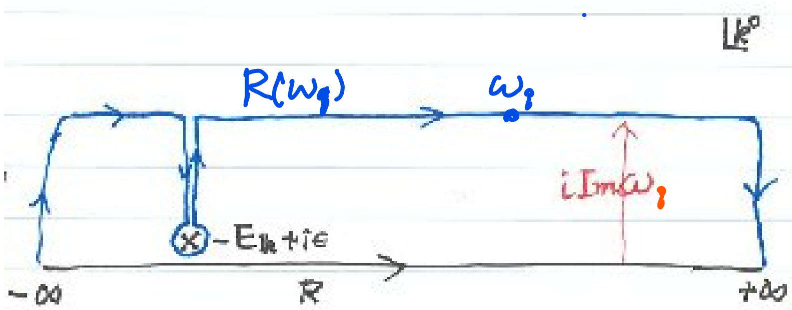


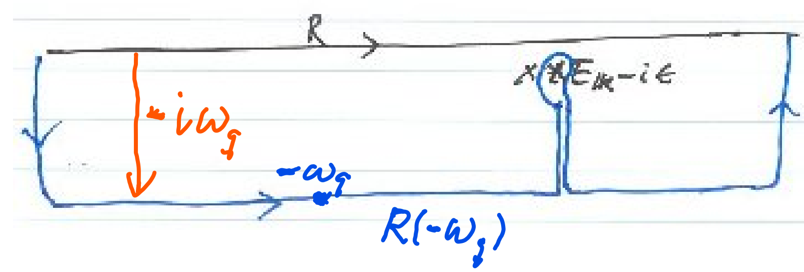
The latter two pole contributions read

$$\int d^3\boldsymbol{q}\left(\frac{2\pi i}{2\omega_{\boldsymbol{q}}}\right)\int_R dk^0\left[\delta_{\rm c}(p^0+\omega_q-k^0)+\delta_{\rm c}(p^0-\omega_q-k^0)\right]\frac{1}{E_{\boldsymbol{k}}^2-k^{0\,2}-i\varepsilon}$$

When performing k^0 integration, the contour R should be lifted (lowered) by $\mathrm{Im}\,\omega$ for the first (second) term in order to make the argument of $\delta_{\rm c}$ real so that they reduces to the usual Dirac delta function. All the terms for which the $\delta_{\rm c}$ reduced to the usual delta function rproduce the naive Feynman rule term.

But, on the way to lift (lower) the k^0 integration contour R to $R(\omega_{\boldsymbol{q}})$ $(R(-\omega_{\boldsymbol{q}}))$, we encounter the k^0 poles:





The two extra contributions are

$$\mathrm{EXTRA}=-f^2\int\frac{d^3\boldsymbol{q}}{i(2\pi)^4}\left\{\frac{(2\pi i)^2}{2\omega_{\boldsymbol{q}}\,2E_{\boldsymbol{k}}}\left(\delta_{\rm c}(p^0+\omega_{\boldsymbol{q}}+E_{\boldsymbol{k}})+\delta_{\rm c}(p^0-\omega_{\boldsymbol{q}}-E_{\boldsymbol{k}})\right)\right\}$$

The usual delta function terms gives the naive Feynman rule terms which gave no imaginary part for the ghost production. The imaginary part for the ghost production solely come from the EXTRA terms:

$$\mathrm{Im}\,\Sigma(p) = f^2 \frac{-\pi^2}{(2\pi)^4} \int d^3\boldsymbol{q}\, \mathrm{Re}\left[\frac{1}{\omega_{\boldsymbol{q}}\, E_{\boldsymbol{k}}}\Big(\delta_{\mathrm{c}}(p^0 + \omega_{\boldsymbol{q}} + E_{\boldsymbol{k}}) + \delta_{\mathrm{c}}(p^0 - \omega_{\boldsymbol{q}} - E_{\boldsymbol{k}})\Big)\right]$$

This agrees with the result obtained from the direct calculation of the production amplitude.

We thus have found where Lee (and Wick) made an error, and has proven that the complex ghost is actually created by the collision of ordinary particles.

Let us next see that the produced complex ghost state is really very stable ('Anti-Unstable'), and exist as an asymptotic state.

3 Complex ghost is anti-unstable

3.1 The ghost 2-point vertex function $\Gamma_\phi^{(2)}(p)$

To examine the ghost (in-)stability, consider an $O(N)$ scalar model:

$$\boxed{\mathcal{L} = -\epsilon_g \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi + m_0^2\phi^2) - \sum_{i=1}^{N}\frac{1}{2}(\partial_\mu\psi_i\partial^\mu\psi_i + \mu^2\psi_i^2) + \sum_{i=1}^{N}\frac{1}{2}\frac{g}{\sqrt{N}}\phi\psi_i\psi_i\,.}$$

ϕ : heavy (m) ghost with metric $\epsilon_g = \pm 1$, ψ : light (μ) normal particle

In the leading order in $1/N$-expansion, ϕ's 2-point vertex $\Gamma_\phi^{(2)}(p)$ is given as

$$\boxed{\Gamma_\phi^{(2)}(p) = -\epsilon_g\,(p^2 + m^2) + \Sigma(p).} \tag{7}$$

Here $\Sigma(p)$ is the self-energy diagram in Fig.1 ($+\,\delta m^2 = m_0^2 - m^2$):

$$\Sigma(p) = \frac{g^2}{32\pi^2}\left[\bar{\varepsilon}^{\,-1} + \delta m^2 + 2 - \ln\mu^2 + f(s)\right],$$

$$f(s) = \sqrt{1 - \frac{4\mu^2}{s}}\ln\left(\frac{\sqrt{1-4\mu^2/s}-1}{\sqrt{1-4\mu^2/s}+1}\right)$$

$$s \equiv -p^2, \qquad \bar{\varepsilon}^{-1} \equiv \frac{2}{4-n} - \gamma + \ln 4\pi.$$

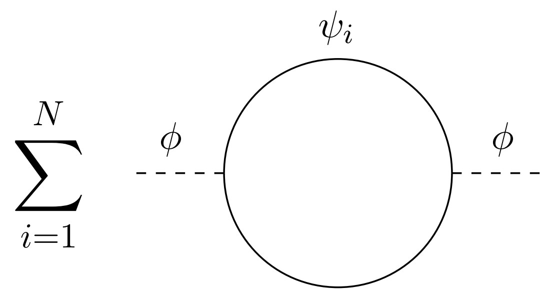


Figure 1: ϕ's self-energy diagram

$\Sigma(s = -p^2)$ develops an imaginary part on the real axis for $s > 4\mu^2$:

$$\lim_{\varepsilon \to +0} \operatorname{Im} \Sigma(s \pm i\varepsilon) = \pm\pi\sqrt{1 - \frac{4\mu^2}{s}}\,\theta(s - 4\mu^2).$$

So the zero(s) of $\Gamma_\phi^{(2)}(s)$, pole(s) of propagator $D_\phi(s) = i/\Gamma_\phi^{(2)}(s)$, is at

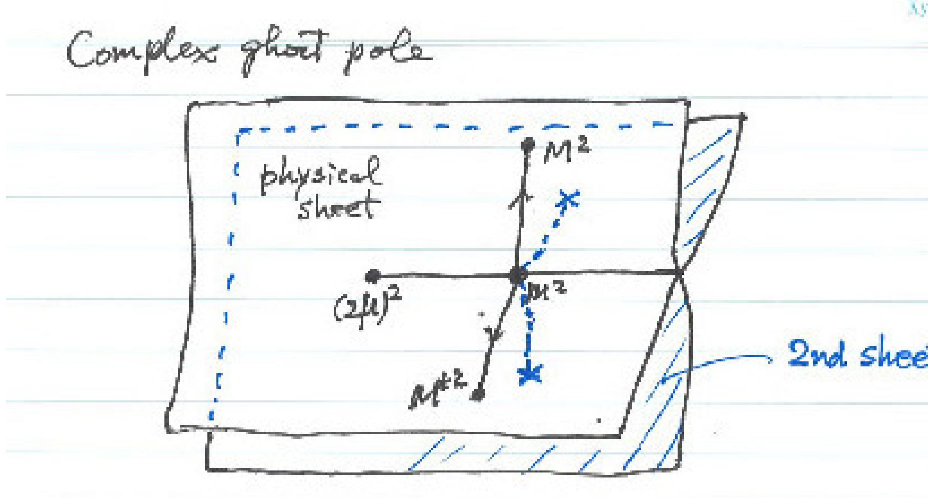
$$\begin{cases} \text{a real point } s = m^2 & \text{if } m^2 < 4\mu^2, \\ \text{complex conjugate points } s = M^2 = m^2 + i\gamma m \text{ and } M^{*2} & \text{if } m^2 > 4\mu^2. \end{cases}$$

The imaginary part γm is determined by $\gamma m = -\epsilon_g \operatorname{Im} \Sigma(s = m^2 + i\gamma m)$ from Eq.(1).

$\gamma m > 0$ for the ghost case $\epsilon_g = -1$

$\rightarrow$

Complex ghost poles appear on the physical sheet!



3.2 Dispersion relation for the ϕ propagator

Consider the following contour integration

$$I \equiv \frac{1}{2\pi i} \int_C ds \, \frac{D_\phi(s)/i}{s+p^2} \tag{9}$$

of the propagator $D_\phi(s=-p^2) = \frac{i}{\Gamma_\phi^{(2)}(p)}$ (for a general complex value of $-p^2$) along the closed contour on the physical sheet, $C = C_1 + C_R + C_2 + C_r$, depicted in Fig. 2. The integrand function has poles and a cut as shown there for the ghost ϕ case $\epsilon_g = -1$.

This integral can be evaluated

by summing the pole residues at $s = -p^2$ and complex poles $s = M^2$ and M^{*2},

or,

by the integral of the discontinuity $\rho(s) = \mathrm{Im}\,(-D_\phi(s)/i)$ from $s = 4\mu^2$ to ∞ on C_1.

Equating both evaluation,

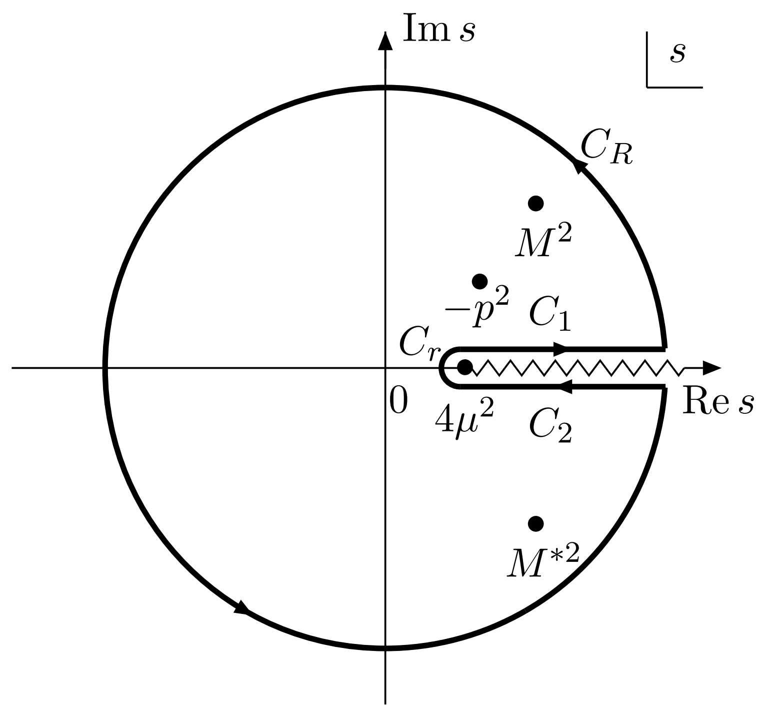


Figure 2: Contour $C = C_1 + C_R + C_2 + C_r$ on the physical sheet.

we obtain a dispersion relation for our ϕ propagator $D_\phi(-p^2)$:

$$D_\phi(-p^2) = \frac{iZ}{M^2+p^2} + \frac{iZ^*}{M^{*2}+p^2} + \frac{1}{i\pi}\int_{4\mu^2}^{\infty} ds\, \frac{\rho(s)}{s+p^2}. \tag{10}$$

This dispersion relation takes the form of Källen-Lehman's spectral representation, so that we can understand the meaning of each term:

· 3rd integral term: Contrib. to Im part from 2-body continuum spectrum state of physical two $\psi_i$particles.

· 1st and 2nd pole terms: Contrib. of complex ghost 1-particle states with complex squared masses $s = M^2$ and $s = M^{*2}$.

Note that *the poles appearing on the physical sheet mean the existence of the corresponding one-particle asymptotic states in the complete set of states of the theory.*

Indeed, this can be easily understood if we consider the same propagator $D_\phi(-p^2)$ for the other parameter value cases in the present system.

Consider, first, the case $m^2 < 4\mu^2$ for ordinary stable particle $(\epsilon_g = +1)$: This case, $D_\phi(s)$ has only a single particle pole at $s = m^2$ on the real axis, so that the above dispersion relation (10) reads

$$D_\phi(-p^2) = \frac{\epsilon_g}{i}\frac{Z}{m^2+p^2} + \frac{1}{i\pi}\int_{4\mu^2}^{\infty} ds\, \frac{\rho(s)}{s+p^2} \qquad \text{for } m^2 < 4\mu^2 \text{ case}.$$

This is the usual spectral representation for stable particle; the 1-particle pole term implied the existence of an asymptotic field $(\Box - m^2)\phi^{\rm as}(x) = 0$.

Next, the case $m^2 > 4\mu^2$ with positive metric $\epsilon_g = +1$: This case, the complex pole M^2 and its conjugate pole M^{*2} move into the second sheet and disappear from the physical sheet, as seen above. So these poles do not exist inside the integration contour C in Fig. 2, so that the above dispersion relation (10) is now replaced by

$$D_\phi(-p^2) = +\frac{1}{i\pi}\int_{4\mu^2}^{\infty} ds\, \frac{\rho(s)}{s+p^2} \quad \text{(Unstable particle)}\,.$$

This has no 1-particle pole term, which agrees with the fact that there is no asymptotic fields corresponding to an unstable particle. This is because, however small the decay probability is, any unstable particle decays out into lighter stable particles and eventually disappears in sufficiently long time.

The present dispersion relation (10) for the ghost field ϕ with negative metric $\epsilon_g = -1$ implies that the Heisenberg field ϕ has the complex conjugate pair of asymptotic ghost fields, φ and $\varphi^\dagger$, $(\Box - M^2)\varphi(x) = 0$:

$$\phi(x) \quad \xrightarrow[x^0\to\infty]{} \quad Z^{1/2}\varphi(x) + Z^{*1/2}\varphi^\dagger(x),$$

3.3 Spectral representation for $\langle 0|[\phi(x), \phi(0)]|0\rangle$

Dispersion relation for the propagator (10) is rewritten via Fourier-trf into the spectral representation for the propagator $\langle 0|\,\mathrm{T}\phi(x)\phi(0)\,|0\rangle$ in x-space:

$$\langle 0|\,\mathrm{T}\phi(x)\phi(0)\,|0\rangle = -Z\Delta_{\mathrm{F}}(x;M^2) - Z^*\Delta_{\mathrm{F}}(x;M^{*2}) + \int_{4\mu^2}^{\infty} ds\,\frac{\rho(s)}{\pi}\Delta_{\mathrm{F}}(x;s)\,,$$

where $\Delta_{\mathrm{F}}(x;m^2)$ denotes the Feynman propagator function for the (positive metric) free field with mass squared m^2 including also the complex m^2 case.

Any type of 2-point function other than propagator also, can be written by using the same spectral function. So we can immediately write down the VEV of the ϕ commutator also as

$$\langle 0|\,[\phi(x), \phi(0)]\,|0\rangle = -Zi\Delta(x;M^2) - Z^*i\Delta(x;M^{*2}) + \int_{4\mu^2}^{\infty} ds\,\frac{\rho(s)}{\pi}i\Delta(x;s)\,.$$

in terms of the invariant commutator function $i\Delta(x;m^2) = [\psi(x), \psi(0)]$ for the positive metric free field ψ with (generally complex) mass squared m^2.

This eq, gives an important relation which we want. Take a time derivative $\partial/\partial x^0$ and set $x^0 = 0$ on both sides, Then, the LHS is reduced to the equal time commutator between the Heisenberg field operator $\phi(0)$ and its conjugate momentum operator $\pi(x) \equiv \partial\mathcal{L}/\partial\dot{\phi}(x) = \epsilon_g\,\dot{\phi}(x)$ at $x^0 = 0$:

$$\mathrm{LHS} = \langle 0|\,[\epsilon_g\,\pi(\boldsymbol{x},0),\ \phi(0)]\,|0\rangle = -i\epsilon_g\,\delta^3(\boldsymbol{x})\,.$$

The RHS can also be evaluated by using free field CCR $i\dot{\Delta}(\boldsymbol{x},0;m^2) = -i\delta^3(\boldsymbol{x})$. Since both sides $\propto\ -i\epsilon_g\,\delta^3(\boldsymbol{x})$, the coefficients leads to

$$-1 = -(Z+Z^*) + \int_{4\mu^2}^{\infty} ds\,\frac{\rho(s)}{\pi} \qquad \text{for ghost field } \epsilon_g = -1 \text{ case.} \quad (11)$$

If we apply the same procedure to to the dispersion relations for ordinary stable and unstable particle cases, respectively, we obtain

$$+1 = Z + \int_{4\mu^2}^{\infty} ds\,\frac{\rho(s)}{\pi} \qquad \text{for stable particle case with } \epsilon_g = +1, \quad (12)$$

$$+1 = \int_{4\mu^2}^{\infty} ds\,\frac{\rho(s)}{\pi} \qquad \text{for unstable particle case.} \quad (13)$$

Eq. (12) for stable particle has the usual interpretation:

· Z: probability $\phi(x)\,|0\rangle$ contains the one-particle state $\left|\boldsymbol{p};m^2\right\rangle$,

· $\int_{4\mu^2}^{\infty} ds\ \rho(s)/\pi\ =:\ c > 0$: probability $\phi(x)\,|0\rangle$ contains continuum many particle states. So, the Eq. (12) says that the total probability that $\phi(x)\,|0\rangle$ contains one-particle and many-particle states adds up to 1.

Similarly, Eq. (13) for unstable particle case shows that $\phi(x)\,|0\rangle$ contains no one-particle asymptotic state and the total probability is saturated only by the contribution $c = \int_{4\mu^2}^{\infty} ds\ \rho(s)/\pi$ from the continuum many particle states consisting of lighter particles produced by decays.

Finally, the Eq. (11) for the ghost field case is interpreted as follows:
· $Z + Z^*$ represents the probability $\phi(x)\,|0\rangle$ contains the complex ghost asymptotic 1-particle state, the *superposition* of $\varphi(x)\,|0\rangle$ and $\varphi^\dagger(x)\,|0\rangle$.
· The state $\phi(x)\,|0\rangle$ also contains continuum many particle states which appear as the 'decay products' of the original ghost ϕ. This probability of the 'decay products' state, $\int_{4\mu^2}^{\infty} ds\ \rho(s)/\pi = c > 0$, is the same as the previous two cases and hence positive. Then, the relation (11) tells us a very interesting but counter-intuitive relation:

$$\boxed{Z + Z^* = 1 + c\,.} \tag{14}$$

Surprisingly, the probability $Z + Z^*$ that the ghost remains as itself becomes even larger as c increases; that is, *the more the ghost 'decays' into lighter ordinary particles, the larger the probability the ghost remains as itself becomes*.

Coleman once suggested to call the complex ghosts "antistable particles" noticing *the radical difference* of situations from the ordinary unstable particles. But it would be more appropriate to call this strange property of ghost "Anti-Instability".

But, actually, this Anti-Instability property of ghost was already pointed out in our first paper by a brief argument based on the norm conservation of the total S-matrix: There we essentially wrote as

Since Hamiltonian is hermitian, the norm is conserved. When the parent particle is a complex ghost, initial state is of negative norm, so that it cannot decay into the sum of states which consist of positive norm particles alone. Negative norm particles have to remain among those final states.

4 Conclusion

• Complex ghosts are created by the collisions of ordinary (positive metric, real energy) particles with finite (non-zero) probability consistently with the energy-momentum conservation law.

• Complex ghost is NOT an unstable particle, but rather an anti-unstable particle which cannot totally decay into ordinary lighter particles.

• Once the complex ghosts are created, therefore, they will not disappear by themselves. Since they carry negative norms, the physical S-matrix unitarity (i.e., unitarity of physical particles alone) is violated.

• Complex ghost theory is mathematically a consistent theory.
But it is an inconsistent theory as the physical theory.

• In the low energy region with energy E below Lower threshold,
$E < \mathrm{Re}\sqrt{M^2} - \mathrm{Im}\sqrt{M^2}$, the theory is a good effective quantum field the-

ory satisfying unitarity and renormalizablity, since no ghosts are produced.

To save the fascinating 'Agravity' theory, or other quadratic gravity theories, as a renormalizable UV complete theory, we need more drastic new ideas.

– Since the coupling constants related with Quadratic Curvature terms are asymptotically free, we need totally novel non-perturbative understanding of their dynamics in the energy region around $E \sim M_{\text{Planck}}$.

– Below the Planck energy $E < M_{\text{Planck}}$, Einstein(-Hilbert) theory will emerge as the low-energy effective theory, so that the metric field $g_{\mu\nu}$ there may be different from that appearing in the Quadratic gravity theory.

– We should probably discard 'vierbein postulate', (expressing the spin-connection in terms of vierbein)

$$D_{[\mu}e_{\nu]}{}^{a} = 0 \quad \rightarrow \quad \omega_{\mu ab} = e_a^\nu \partial_{[\mu}e_{\nu]b} - e_b^\nu \partial_{[\mu}e_{\nu]a} - e_a^\rho e_b^\sigma e_\mu^c \partial_{[\rho}e_{\sigma]c}$$

and treat vierbein e_μ^a and spin-connection as quite independent variables. Then, the quadratic curvature terms are no longer higher derivative terms but merely second order derivative terms in the spin-connection variable $\omega_\mu{}^{ab}$. The theory is a Yang-Mills theory of the Local Lorentz group $SO(3,1) = SL(2;C)$.

– Although the group is non-compact, the negative metric component of the ${\omega_\mu}^{ab}$ gauge field would be harmless if it is confined just as the QCD gluons. After spontaneous breaking of scale invariance, the usual Einstein-Hilbert gravity will appear as a low energy effective theory which is valid below the Planck scale.

– But spin should not be confined! How is it possible?