Appendix D

Differential and Algebraic Topology

As we saw briefly while discussing obstructions to G-structures, the cohomology is the object that captures topological aspects of manifolds and bundles by isolating "closed" objects that are defined modulo "exact" terms. Characteristic classes, such as Stiefel-Whitney, Chern, and Pontryagin, are all particular elements of some of such cohomology groups. The latter come in several different flavors. In this appendix we will take a quick overview of (co)homology and homotopy, which connect to many chapters in the second half of the volume.

In this note, we will be content with being illustrative instead of being comprehensive, and many of such illustrations would involve some of the simplest manifolds one encounter, say $\mathbb{S}^n$ and even $\mathbb{R}^n$, which nevertheless would teach much about the machineries involved. We close with an overview of homotopy which also plays important roles in studying non-perturbative saddles of the gauge theory path integrals. Much of what follows are inspired by the classic text of Bott and Tu, *Differential Forms in Algebraic Topology*, and meant to wet the appetite in favor of the more complete treatment there.

D.1 Differential Complexes

D.1.1 de Rham Cohomology

A differential complex of $\mathcal{M}$ is a sequence of $\wedge^k \mathscr{T}^*[\mathcal{M}]$ and the usual exterior differential operator d,

$$\xrightarrow{d} \Omega^{k-1}[\mathcal{M}] \xrightarrow{d} \Omega^k[\mathcal{M}] \xrightarrow{d} \Omega^{k+1}[\mathcal{M}] \xrightarrow{d} \tag{D.1.1}$$

where $\Omega^k[\mathcal{M}]$ is the space of differential k-forms on $\mathcal{M}$, on which d acts naturally. For each $0 \leq k \leq d$, we may find a subsets of which are closed and exact, respectively,

$$\begin{aligned} Z^k &\equiv \{w^{(k)} \in \Omega^k[\mathcal{M}] \mid dw^{(k)} = 0\} \,, \\ B^k &\equiv \{dw^{(k-1)} \mid w^{(k-1)} \in \Omega^{k-1}[\mathcal{M}]\} \end{aligned} \tag{D.1.2}$$

The first is a kernel of d while the latter is image under d. The quotient between the two,

$$H^k_{\rm dR}(\mathcal{M}) = Z^k/B^k \tag{D.1.3}$$

defines the de Rham cohomology.

For a trivial example, take $k = 0$, for which $B^{(0)}$ is empty while $Z^{(0)}$ is a set of locally constant functions. As such,

$$H^0_{\rm dR}(\mathcal{M}) = \mathbb{R}^{\#} \tag{D.1.4}$$

where # is the number of disconnected components of $\mathcal{M}$. We will assume connected $\mathcal{M}$, so

$$H^0_{\rm dR}(\mathcal{M}) = \mathbb{R} \tag{D.1.5}$$

What about the other end, H^d? For d-dimensional manifold, d-forms are automatically d-closed since higher differential forms do not exist. For a compact and connected $\mathcal{M}_d$, we would have $H^d_{\rm dR}(\mathcal{M}) = \mathbb{R}$ as well, to be eventually justified by the Poincaré duality below.

For this, a useful notion is the pairing, between $H^p_{\rm dR}$ and $H^{d-p}_{\rm dR}$ such as

$$([w^{(p)}],[w^{(d-p)}]) \quad \longmapsto \quad \int_{\mathcal{M}} w^{(p)}\wedge w^{(d-p)} \;\in\; \mathbb{R} \tag{D.1.6}$$

where $w^{(p)}$ is a closed p-form representative of $[w^{(p)}]\in H^p$. The closed nature of the cohomology representatives guarantee that the pairing is well-defined on the cohomology. For this, we need the orientability. If such a pairing is non-degenerate, this would give an isomorphism between,

$$H^k_{\rm dR}\simeq H^{d-k}_{\rm dR} \tag{D.1.7}$$

The pairing is an isomorphism for compacted and closed manifold, $\mathcal{M}_d$, and generally not for open manifolds like $\mathbb{R}^d$, however. This isomorphism is a particular manifestation of the so-called Poincaré duality.

For real de Rham cohomology on compact and closed $\mathcal{M}$, a powerful fact allows us to handle the cohomology effectively is the Hodge theory. Given an exterior differential d and a metric on $\mathcal{M}$, its Hermitian conjugate operator $d^\dagger$ may be defined via

$$\int_{\mathcal{M}}(\mathcal{V}\lrcorner\, v^{(p+1)})\wedge dw^{(p)} = \int_{\mathcal{M}}(\mathcal{V}\lrcorner\, d^\dagger v^{(p+1)})\wedge w^{(p)} \tag{D.1.8}$$

which should tell us immediately, modulo a sign, $d^\dagger \simeq *d*$ with the Hodge star operation $*$. Since $d^\dagger d^\dagger=0$, $d^\dagger$ defines its own complex, mapping $\Omega^k[\mathcal{M}]$ to $\Omega^{k-1}[\mathcal{M}]$. The Hodge theory states that, for compact and closed $\mathcal{M}$, $\Omega^k[\mathcal{M}]$ may be decomposed into three parts, such that generally

$$w^{(k)} = d\omega^{(k-1)} + d^\dagger\omega^{(k+1)} + h^{(k)} \tag{D.1.9}$$

where $h^{(k)}$ is annihilated by $dd^\dagger + d^\dagger d$, which is to say that $h^{(k)}$ is harmonic. Furthermore, the subspace spanned by $h^{(k)}$ of $\Omega^k[\mathcal{M}]$ is isomorphic to $H^k_{\rm dR}$.

In fact $\mathcal{V}$ itself offers the isomorphism,

$$[h^{(p)}]\in H^p \quad \longmapsto \quad [\mathcal{V}\lrcorner\, h^{(n-p)}]\in H^p \tag{D.1.10}$$

since the harmonicity is preserved under this Hodge dual map. This last again implies that $H^k_{\rm dR}$ and $H^{d-k}_{\rm dR}$ are isomorphic to each other for such a nice $\mathcal{M}$, since harmonicity of h mean the harmonicity of $*h = \mathcal{V} \lrcorner h$ and vice versa. For example, we find

$$H^0_{\rm dR}(\mathbb{R}^n) = \mathbb{R} \quad , \qquad H^{0<q\leq n}_{\rm dR}(\mathbb{R}^n) = \emptyset \tag{D.1.11}$$

for the top cohomology as well, with its harmonic representative being the volume form $\mathcal{V}$, which maps to a constant function upon the map $H^d_{\rm dR}$ and $H^d_{\rm dR}$ by the Hodge duality operation.

Some of simpler examples are n-sphere $\mathbb{S}^n$

$$H^0_{\rm dR}(\mathbb{S}^n) = \mathbb{R} \ , \qquad H^{0<q<n}_{\rm dR}(\mathbb{S}^n) = \emptyset \ , \qquad H^n_{\rm dR}(\mathbb{S}^n) = \mathbb{R} \tag{D.1.12}$$

which we will encounter repeatedly below, and the complex projective sphere $\mathbb{CP}^k = \mathbb{C}^{k+1}/\mathbb{C}^*$,

$$\begin{aligned} H^{2p}_{\rm dR}(\mathbb{CP}^k) = \mathbb{R} \quad &, \quad 0 \leq p \leq k \\ H^{2p+1}_{\rm dR}(\mathbb{CP}^k) = \emptyset \quad &, \quad 0 \leq p < k \end{aligned} \tag{D.1.13}$$

In particular, the harmonic representative of $H^{2p}_{\rm dR}(\mathbb{CP}^k)$ is $\wedge^p w$ where w is the Kähler 2-form we referred to earlier while discussing reduced holonomies; $U(k)$ for $\mathbb{CP}^k$.

A useful and universal fact about the cohomology is the Künneth formula

$$H^p(\mathcal{M} \times \mathcal{N}) = \oplus_{q+q'=p} H^q(\mathcal{M}) \otimes H^{q'}(\mathcal{N}) \tag{D.1.14}$$

a trivial example of which is

$$\begin{aligned} H^p_{\rm dR}(\mathbb{R}^k \times \mathcal{M}_d) = H^p_{\rm dR}(\mathcal{M}_d) \qquad & p \leq d \\ H^p_{\rm dR}(\mathbb{R}^k \times \mathcal{M}_d) = \emptyset \ , \qquad & \text{otherwise} \end{aligned} \tag{D.1.15}$$

The local form of the characteristic classes that enter the index formula are particular elements of $H^d_{\rm dR}(\mathcal{M}_d) = \mathbb{R}$. However, the integrated values are discrete, implying that these actually belong to a more refined version of the cohomology whose elements are discretely valued. With compact and closed $\mathcal{M}$, there are multiple versions of cohomology, including the Čech cohomology to be discussed later, which can be often mapped to one another. Often we construct the cohomology from these alternate, more combinatorial definitions. In fact, these more algebraic and combinatoric approach allow us to formulate other variants such as $H^*(\mathcal{M};\mathbb{Z})$ and $H^*(\mathcal{M};\mathbb{Z}_p)$ etc, which tends to carry information not accessible by the de Rham cohomology.

D.1.2 Compact de Rham Cohomology

However, life is more complicated. Our own spacetime is nowhere near compact or closed, so in many situations we need to go beyond such basic and ideal setups. The de Rham complex and the cohomology thereof still makes sense, but the above pairing and the subsequent inner product, for example, do not even make sense unless some restrictions are demanded on Ω^k. If $\mathcal{M}$ is an open manifold of infinite span but contractible, for example, we would have

$$H^0_{\mathrm{dR}}(\mathcal{M}_d) = \mathbb{R}\ , \qquad H^d_{\mathrm{dR}}(\mathcal{M}_d) = \emptyset \tag{D.1.16}$$

with the latter different from the compact cases. Apart from whether this cohomology is of any use for physics, we need to explore what other possibilities exist.

One such is to require the differential forms that enter the de Rham complex to have compact support. That is, we start with $\Omega^k_{\mathrm{c}}[\mathcal{M}]$ is the space of differential k-forms on $\mathcal{M}$ that vanishes outside some compact subset of $\mathcal{M}$, and build the complex

$$\xrightarrow{d} \quad \Omega^{k-1}_{\mathrm{c}}[\mathcal{M}] \quad \xrightarrow{d} \quad \Omega^k_{\mathrm{c}}[\mathcal{M}] \quad \xrightarrow{d} \quad \Omega^{k+1}_{\mathrm{c}}[\mathcal{M}] \quad \xrightarrow{d} \tag{D.1.17}$$

The closed and exact subspaces Z^k_{c} and B^k_{c}, and the compact de Rham cohomology follow as

$$H^k_{\mathrm{cdR}}(\mathcal{M}) = Z^k_{\mathrm{c}}/B^k_{\mathrm{c}} \tag{D.1.18}$$

Unlike de Rham, we see that $H^0_{\mathrm{cdR}}(\mathcal{M})$ is empty since the only closed functions are constant functions, and the only constant function with compact support is zero. Conversely, we have H^d_{cdR} nontrivial,

$$H^0_{\mathrm{cdR}}(\mathcal{M}_d) = \emptyset\ , \qquad H^d_{\mathrm{cdR}}(\mathcal{M}_d) = \mathbb{R} \tag{D.1.19}$$

exactly opposite of de Rham.

This is not a coincidence, but rather a consequence of the non-degenerate pairing,

$$([w^{(p)}], [v_c^{(d-p)}]) \quad \longmapsto \quad \int_{\mathcal{M}_d} w^{(p)} \wedge v_c^{(d-p)} \ \in \ \mathbb{R} \tag{D.1.20}$$

between $H^p_{\rm dR}$ and $H^{d-p}_{\rm cdR}$. Since the latter are compactly supported, the pairing makes sense unlike the ill-fated would-be pairings between de Rham cohomology of non-compact $\mathcal{M}_d$. This gives

$$H^{0\leq q<n}_{\rm dR}(\mathbb{R}^n) = \emptyset \;\;, \qquad H^n_{\rm cdR}(\mathbb{R}^n) = \mathbb{R} \tag{D.1.21}$$

for example.

The Künneth formula is equally applicable for the compact de Rham,

$$H^p_{\rm cdR}(\mathcal{M}\times\mathcal{N}) = \oplus_{q+q'=p} H^q_{\rm cdR}(\mathcal{M}) \otimes H^{q'}_{\rm cdR}(\mathcal{N}) \tag{D.1.22}$$

a trivial example of which is

$$\begin{aligned} H^p_{\rm cdR}(\mathbb{R}^k\times\mathcal{M}) &= \emptyset \;, \qquad p<k \\ H^p_{\rm cdR}(\mathbb{R}^k\times\mathcal{M}) &= H^{p-k}_{\rm cdR}(\mathcal{M}) \;, \qquad \text{otherwise} \end{aligned} \tag{D.1.23}$$

D.1.3 L^2 Cohomology and Some Cautionary Words

For physics applications that involve open space(-time) of infinite volume, however, neither de Rham nor compact de Rham would be acceptable. Wavefunctions should obey some boundary conditions but, for example, would not be generally required to be compactly supported since the latter means that the wavefunction must vanish identically beyond some closed subset.

More natural is the square-normalizability, physically at least, which we often denote L^2. It should be quite clear that the usual pairing can be defined on the L^2 differential forms and the Poincaré duality works within the L^2 cohomology, unlike de Rham or compact de Rham. Unfortunately, the L^2 cohomology seem much less accessible, mathematically speaking.

Whether the L^2 cohomology admit an analog of the Hodge theory, whereby the cohomology counts L^2 harmonic form, is another subtle issue. It is believed that this is not true in general but a related object called "reduced L^2 cohomology" counts L^2 harmonic forms. All of these go well beyond the limited mathematical scope of this volume, unfortunately, we merely offer these cautionary words and alert readers not to rely on the standard (co)homology statements too blindly when the question involves non-compact manifolds and is of physics origin. Perhaps most importantly, there is a purported equivalence of the L^2 cohomology to the so-called intersection (co)homology which is a version of singular (co)homology that deals with spaces with singularities.

On the positive side, in some simple cases such as when the asymptotic boundary region reduces to a cylinder or a cone, a rough statements can be found in the mathematics literature how the L^2 cohomology can be inferred from other types of cohomologies. A more precise phrasing relies on the relative cohomology to which we will make a brief detour later in this section, but the main idea is that the L^2 cohomology may be regarded as the common part of the above two types of de Rham cohomologies. At least it should be clear that the intersection of the two would obey the Poincaré duality, as the L^2 condition would demand.

Finally, a less detailed count of such L^2 "harmonic" sections can be sometimes inferred from the Atiyah-Patodi-Singer index theorem which we have reviewed in the main text. This is because, at least for asymptotically cylindrical boundaries, the celebrated APS boundary condition actually equals the L^2 condition.

D.2 Exact Sequences

One universal tool for computing cohomology, homology, and also homotopy to be discussed in next section is various versions of exact sequences. An exact sequence means a set of vector spaces and maps,

$$\cdots \longrightarrow A \stackrel{f}{\longrightarrow} B \stackrel{g}{\longrightarrow} C \longrightarrow \cdots \tag{D.2.1}$$

such that, for instance, the kernel of g equals the image of f. A short exact sequence is

$$\emptyset \longrightarrow A \stackrel{f}{\longrightarrow} B \stackrel{g}{\longrightarrow} C \longrightarrow \emptyset \tag{D.2.2}$$

which says, in addition, f is injective while g is surjective. This also means $C = B/A$.

Suppose that A, B, C are each a differential complex, say

$$\cdots \stackrel{d}{\longrightarrow} A^{k-1} \stackrel{d}{\longrightarrow} A^k \stackrel{d}{\longrightarrow} A^{k+1} \stackrel{d}{\longrightarrow} \cdots \tag{D.2.3}$$

etc. If f and g construct short exact sequences

$$\emptyset \longrightarrow A^k \xrightarrow{f} B^{k'} \xrightarrow{g} C^{k''} \longrightarrow \emptyset \tag{D.2.4}$$

level by level and also commute with d's of each complexes, this induces a long exact sequence,

$$\cdots \xrightarrow{d^\#} H^k(A) \xrightarrow{f^\#} H^{k'}(B) \xrightarrow{g^\#} H^{k''}(C) \xrightarrow{d^\#} H^{k+1}(A) \xrightarrow{f^\#} \cdots \tag{D.2.5}$$

where $f^\#$ and $g^\#$ are naturally induced from f and g. That is, given a representative $w^{(k)}$ of $[w^{(k)}] \in H^k(A)$, for example, we have $f^\#([w^{(k)}]) = [f(w^{(k)})] \in H^{k'}(B)$. Since f and g commute with d, cohomology elements map to cohomology elements under $f^\#$ and $g^\#$.

How is $d^\#$ constructed? Given a representative $c^{(k'')} \in C^{k''}$ of $[c^{(k'')}] \in H^{k''}(C)$, there should be $b^{(k')} \in B^{k'}$ such that $g(b^{(k')}) = c^{(k'')}$ since g is surjective. On the other hand $g(db^{(k')}) = d(g(b^{(k')})) = dc^{(k'')} = 0$; this means $db^{(k')}$ belongs to the Kernel of g in $B^{k'+1}$, so that there must be $a^{(k+1)} \in A^{k+1}$ such that $f(a^{(k+1)}) = db^{(k)}$. Finally $f(da^{(k+1)}) = d(f(a^{(k+1)}) = ddb^{(k')} = 0$ means that, since f is injective, $a^{(k+1)}$ is closed and thus represents an element $[a^{(k+1)}]$ of $H^{k+1}(A)$. This diagram chasing induces the map,

$$d^\# \; : \; [c^{(k'')}] \longmapsto [a^{(k+1)}] \tag{D.2.6}$$

One can demonstrate that the above induced long sequence is exact, through more of the same types of gymnastics.

This long exact sequence often gives us means to construct the cohomology inductively given partial information on cohomologies. The key point is that the long exact sequences are often broken up into shorter chains, by middle entries that are null, allowing us to deduce other non-vanishing items unambiguously. We will see some simple examples of how this works below. We did not specify the type of the complexes, so this kind of construction is universally applicable whenever one can construct the short exact sequence of complexes.

This type of mechanism involving long exact sequences has many different manifestations in differential topology. It is particularly powerful for bundles, manifolds built from dividing by free isomorphism, filter complexes, etc. The same techniques also work fruitfully for homologies and homotopy, of which we will see examples later. The full arsenal of this inductive techniques are clearly beyond the scope of this note. As before, we refer readers to Bott and Tu for a more comprehensive treatment, including the Čech cohomology and homotopy to be discussed below.

D.2.1 Mayer-Vietoris Sequence

The most basic form of the long exact sequence may be induced when a manifold $\mathcal{M}$ may be composed of two charts, such that we have a sequence of inclusions,

$$\mathcal{M} = \mathcal{U} \cup \tilde{\mathcal{U}} \Leftarrow \mathcal{U} + \tilde{\mathcal{U}} \Leftarrow \mathcal{U} \cap \tilde{\mathcal{U}} \tag{D.2.7}$$

where $+$ means a disjoint union. The inclusions are denoted as $\Leftarrow$ because it involves two inclusions. The left one denotes in reality $\mathcal{U} \hookrightarrow \mathcal{M}$ and $\tilde{\mathcal{U}} \hookrightarrow \mathcal{M}$ while the right one is composed of $\mathcal{U} \cap \tilde{\mathcal{U}} \hookrightarrow \mathcal{U}$ and $\mathcal{U} \cap \tilde{\mathcal{U}} \hookrightarrow \tilde{\mathcal{U}}$. This in turn induces a short exact sequence for the de Rham complex

$$\emptyset \to \Omega^k(\mathcal{M}) \xrightarrow{\text{restrictions}} \Omega^k(\mathcal{U}) \oplus \Omega^k(\tilde{\mathcal{U}}) \xrightarrow[\text{difference}]{\text{restrictions}} \Omega^k(\mathcal{U} \cap \tilde{\mathcal{U}}) \to \emptyset \tag{D.2.8}$$

where “”restrictions” is via the pull-back of the inclusion maps.

The second map onto $\Omega^k(\mathcal{U} \cap \tilde{\mathcal{U}})$ is constructed in two steps; first, restrict elements of $\Omega^k(\mathcal{U})$ and $\Omega^k(\tilde{\mathcal{U}})$ each into $\Omega^k(\mathcal{U} \cap \tilde{\mathcal{U}})$ and then take a difference. Clearly, the two restrictions of an element of $\Omega^k(\mathcal{M})$ onto $\Omega^k(\mathcal{U})$ and $\Omega^k(\tilde{\mathcal{U}})$ would coincide upon a further restriction onto $\Omega^k(\mathcal{U} \cap \tilde{\mathcal{U}})$, so the image of the first map belongs to the kernel of the second and vice versa, which shows that the sequence is exact. The long exact sequence out of this short exact sequence

$$\cdots \to H^k_{\rm dR}(\mathcal{M}) \to H^k_{\rm dR}(\mathcal{U}) \oplus H^k_{\rm dR}(\tilde{\mathcal{U}}) \to H^k_{\rm dR}(\mathcal{U} \cap \tilde{\mathcal{U}}) \to H^{k+1}_{\rm dR}(\mathcal{M}) \to \cdot \tag{D.2.9}$$

is called the Mayer-Vietoris sequence.

de Rham of $\mathbb{S}^n$

The simplest example that can illustrate this construction is found for de Rham cohomology of $\mathcal{M} = \mathbb{S}^1$ for which we can take $\mathcal{U}$ and $\tilde{\mathcal{U}}$ to be each a segments,

$$\emptyset \;\rightarrow\; H^0_{\rm dR}(\mathbb{S}^1) \;\rightarrow H^0_{\rm dR}(\mathcal{U}) \oplus H^0_{\rm dR}(\tilde{\mathcal{U}}) \;\rightarrow\; H^0_{\rm dR}(\mathcal{U}\cap\tilde{\mathcal{U}})$$

$$\rightarrow\; H^1_{\rm dR}(\mathbb{S}^1) \;\rightarrow\; H^1_{\rm dR}(\mathcal{U}) \oplus H^1_{\rm dR}(\tilde{\mathcal{U}}) \;\rightarrow\; H^1_{\rm dR}(\mathcal{U}\cap\tilde{\mathcal{U}}) \;\rightarrow\; \emptyset \quad \text{(D.2.10)}$$

Here, since both $\mathcal{U}$ and $\tilde{\mathcal{U}}$ are topologically $\mathbb{R}$, we must have

$$\emptyset \;\rightarrow\; H^0_{\rm dR}(\mathbb{S}^1) \;\rightarrow\; \mathbb{R}\oplus\mathbb{R} \;\rightarrow \mathbb{R}\oplus\mathbb{R} \;\rightarrow\; H^1_{\rm dR}(\mathbb{S}^1) \;\rightarrow\; \emptyset \qquad \text{(D.2.11)}$$

Note how $\mathcal{U}\cap\tilde{\mathcal{U}}$ a also a disjoint union of two copies of $\mathbb{R}$, topologically speaking.

The middle map reduces $H^0_{\rm dR}(\mathcal{U})\oplus H^0_{\rm dR}(\tilde{\mathcal{U}})$ to $\mathbb{R} \subset H^0_{\rm dR}(\mathcal{U}\cap\tilde{\mathcal{U}}) = \mathbb{R}\oplus\mathbb{R}$ since the taking a difference of two constants on $\mathcal{U}$ and $\tilde{\mathcal{U}}$ gives a common constant function on the disjoint overlapping regions of $\mathcal{U}\cap\tilde{\mathcal{U}}$. As such we find

$$\emptyset \;\rightarrow\; H^0_{\rm dR}(\mathbb{S}^1) \;\rightarrow\; \mathbb{R}\oplus\mathbb{R} \;\rightarrow \mathbb{R}\oplus\mathbb{R} \;\rightarrow\; H^1_{\rm dR}(\mathbb{S}^1) \;\rightarrow\; \emptyset \qquad \text{(D.2.12)}$$

consistent with the general fact $H^0_{\rm dR}(\mathcal{M}_d) = \mathbb{R} = H^d_{\rm dR}(\mathcal{M}_d)$ for closed and connected $\mathcal{M}_d$.

Let us extend this for de Rham cohomology of $\mathbb{S}^n$ for $n \geq 2$ with $\mathcal{U}$ and $\tilde{\mathcal{U}}$ both n-disk, or topologically $\mathbb{R}^n$, and $\mathcal{U}\cap\tilde{\mathcal{U}} = \mathbb{R}\times\mathbb{S}^{n-1}$. Taking a snapshot of the Mayer-Vietoris sequence, let us take a look at the segment

$$H^{k-1}_{\rm dR}(\mathbb{R}^n)^{\oplus 2} \;\rightarrow\; H^{k-1}_{\rm dR}(\mathbb{R}\times\mathbb{S}^{n-1}) \;\rightarrow\; H^k_{\rm dR}(\mathbb{S}^n) \;\rightarrow\; H^k_{\rm dR}(\mathbb{R}^n)^{\oplus 2} \qquad \text{(D.2.13)}$$

With $1 < k$ the two ends of this segment are null, which shows

$$H^{k-1}_{\rm dR}(\mathbb{S}^{n-1}) = H^{k-1}_{\rm dR}(\mathbb{R}\times\mathbb{S}^{n-1}) = H^k_{\rm dR}(\mathbb{S}^n) \qquad \text{(D.2.14)}$$

for $1 < k \leq n$ where the first equality is due to the Künneth formula.

On the other hand, with $k = 1$, we find from

$$\emptyset \;\rightarrow\; H^0_{\rm dR}(\mathbb{S}^n) \;\rightarrow\; H^0_{\rm dR}(\mathbb{R}^n)^{\oplus 2} \;\rightarrow\; H^0_{\rm dR}(\mathbb{R}\times\mathbb{S}^{n-1}) \;\rightarrow\; H^1_{\rm dR}(\mathbb{S}^n) \;\rightarrow\; \emptyset \; \text{(D.2.15)}$$

With $H^0_{\rm dR}(\mathbb{S}^n) = H^0_{\rm dR}(\mathbb{R}^n) = H^0_{\rm dR}(\mathbb{R}\times\mathbb{S}^{n-1}) = \mathbb{R}$, this forces $H^1_{\rm dR}(\mathbb{S}^n) = \emptyset$ for $n \geq 2$. Next up is $k = 2$, for which, if $n > 2$,

$$H^1_{\rm dR}(\mathbb{R}\times\mathbb{S}^{n-1}) = \emptyset \;\to\; H^2_{\rm dR}(\mathbb{S}^n) \;\to\; H^2_{\rm dR}(\mathbb{R}^n)^{\oplus 2} = \emptyset \tag{D.2.16}$$

hence $H^2(\mathbb{S}^{n>2}) = \emptyset$. These results, combined with $H^{k-1}_{\rm dR}(\mathbb{S}^{n-1}) = H^k_{\rm dR}(\mathbb{S}^n)$ above, build up the claimed $\mathbb{S}^n$ cohomology (D.1.12) starting from those of $\mathbb{S}^1$ and $\mathbb{R}^n$'s.

D.2.2 Mayer-Vietoris for Compact de Rham

How does this work for compact de Rham complexes? For the latter, the map (D.2.7) induces an exact sequence of the opposite direction. That is, we find

$$\emptyset \;\leftarrow\; \Omega^k_c(\mathcal{M}) \;\xleftarrow{\text{sum}}\; \Omega^k_c(\mathcal{U}) \oplus \Omega^k_c(\tilde{\mathcal{U}}) \;\xleftarrow{\text{double}}\; \Omega^k_c(\mathcal{U}\cap\tilde{\mathcal{U}}) \;\leftarrow\; \emptyset \tag{D.2.17}$$

where one takes $w^{(k)} \in \Omega^k_c(\mathcal{U}\cap\tilde{\mathcal{U}})$ and assign $(w^{(k)}, -w^{(k)}) \in \Omega^k_c(\mathcal{U}) \oplus \Omega^k_c(\tilde{\mathcal{U}})$, which in turn sum to zero when mapped to $\Omega^k_c(\mathcal{M})$. Clearly both maps are possible only because w's are compactly supported and thus can be extended upon embedding to a bigger space trivially, with the same compact support.

This generates a different Mayer-Vietoris sequences where the maps are reversed level by level.

$$\cdots \;\leftarrow\; H^{k+1}_{\rm cdR}(\mathcal{U}\cap\tilde{\mathcal{U}}) \;\leftarrow\; H^k_{\rm cdR}(\mathcal{M}) \;\leftarrow\; H^k_{\rm cdR}(\mathcal{U}) \oplus H^k_{\rm cdR}(\tilde{\mathcal{U}}) \;\leftarrow\; H^k_{\rm cdR}(\mathcal{U}\cap\tilde{\mathcal{U}}) \;\leftarrow \tag{D.2.18}$$

With $\mathbb{S}^1$, we thus find the long exact sequence,

$$\begin{aligned} \emptyset \;\leftarrow\; H^1_{\rm cdR}(\mathbb{S}^1) \;\leftarrow\; H^1_{\rm cdR}(\mathcal{U}) \oplus H^1_{\rm cdR}(\tilde{\mathcal{U}}) \;\leftarrow\; H^1_{\rm cdR}(\mathcal{U}\cap\tilde{\mathcal{U}}) \\ \leftarrow\; H^0_{\rm cdR}(\mathbb{S}^1) \;\leftarrow\; H^0_{\rm cdR}(\mathcal{U}) \oplus H^0_{\rm cdR}(\tilde{\mathcal{U}}) \;\leftarrow\; H^0_{\rm cdR}(\mathcal{U}\cap\tilde{\mathcal{U}}) \;\leftarrow\; \emptyset \end{aligned} \tag{D.2.19}$$

resulting in, with $H^0_{\rm cdR}(\mathbb{R}) = \emptyset$ and $H^1_{\rm cdR}(\mathbb{R}) = \mathbb{R}$,

$$\emptyset \;\leftarrow\; H^1_{\rm cdR}(\mathbb{S}^1) = \mathbb{R} \;\leftarrow\; \mathbb{R}\oplus\mathbb{R} \;\leftarrow \mathbb{R}\oplus\mathbb{R} \;\leftarrow\; H^0_{\rm cdR}(\mathbb{S}^1) = \mathbb{R} \;\leftarrow\; \emptyset \tag{D.2.20}$$

with the middle map reducing $\mathbb{R}\oplus\mathbb{R}$ to $\mathbb{R}$. The resulting $H^q_{\rm cdR}(\mathbb{S}^1)$'s are in accord with the fact that, by definition, $H_{\rm cdR} = H_{\rm dR}$ for a compact manifold with no boundary.

Compact de Rham of $\mathbb{S}^n$

Repeating the previous exercise for $\mathbb{S}^{n\geq 2}$ now with the compact de Rham is also straightforward, although details differ. For instance, we start with

$$H^{k+1}_{\mathrm{cdR}}(\mathbb{R}^n)^{\oplus 2} \;\leftarrow\; H^{k+1}_{\mathrm{cdR}}(\mathbb{R}\times\mathbb{S}^{n-1}) \;\leftarrow\; H^{k}_{\mathrm{cdR}}(\mathbb{S}^n) \;\leftarrow\; H^{k}_{\mathrm{cdR}}(\mathbb{R}^n)^{\oplus 2} \tag{D.2.21}$$

which results in, when $k+1<n$,

$$H^{k}_{\mathrm{cdR}}(\mathbb{S}^{n-1}) = H^{k+1}_{\mathrm{cdR}}(\mathbb{R}\times\mathbb{S}^{n-1}) = H^{k}_{\mathrm{cdR}}(\mathbb{S}^n) \tag{D.2.22}$$

again using the Künneth formula for the first equality. The resulting $H^{k}_{\mathrm{cdR}}(\mathbb{S}^{n-1}) = H^{k}_{\mathrm{cdR}}(\mathbb{S}^n)$ is the precise analog of $H^{k-1}_{\mathrm{dR}}(\mathbb{S}^{n-1}) = H^{k}_{\mathrm{dR}}(\mathbb{S}^n)$ since H_{dR} and H_{cdR} are Poincaré-dual to each other. The rest follows, with $n-k=1,2$, etc, bringing us to the same (D.1.12) with $H^{q}_{\mathrm{cdR}}(\mathbb{S}^n) = H^{n-q}_{\mathrm{dR}}(\mathbb{S}^n) = H^{q}_{\mathrm{dR}}(\mathbb{S}^n)$.

D.2.3 Relative Cohomology

Starting from de Rham, we cay build another type of complexes and the related short exact sequence as follows. Suppose ψ is a map between manifolds

$$\psi \;:\; \mathcal{K} \;\rightarrow\; \mathcal{M} \tag{D.2.23}$$

Then, we consider a complex starting from

$$\Omega^k(\psi) \equiv \Omega^k(\mathcal{M}) \oplus \Omega^{k-1}(\mathcal{K}) \tag{D.2.24}$$

with

$$d(\omega^{(k)}, \mu^{(k-1)}) = (d\omega^{(k)}, \psi^*(w^{(k)}) - d\mu^{(k-1)}) \tag{D.2.25}$$

One can see immediately that the cohomology $H^k(\psi)$ of this complex would be represented by closed element $w^{(k)}$ of $C^k(\mathcal{M})$ whose pull-back to $\mathcal{K}$ is exact.

The short exact sequence involving $H^k(\psi)$ is

$$\emptyset \;\longrightarrow\; C^{k-1}(\mathcal{K}) \;\xrightarrow{f}\; C^k(\psi) \;\xrightarrow{g}\; C^k(\mathcal{M}) \;\longrightarrow\; \emptyset \tag{D.2.26}$$

with $f(\mu^{(k-1)}) = (0, \mu^{(k-1)})$ and $g(w^{(k)}, \mu^{(k-1)}) = w^{(k)}$. These maps clearly commute with d, and

$$\cdots \rightarrow H^{k-1}_{\rm dR}(\mathcal{K}) \rightarrow H^k(\psi) \rightarrow H^k_{\rm dR}(\mathcal{M}) \rightarrow H^k_{\rm dR}(\mathcal{K}) \rightarrow \cdots \qquad \text{(D.2.27)}$$

emerges as the resulting long exact sequence.

When ψ is an inclusion, i.e., $\mathcal{K} \subset \mathcal{M}$, we denote $H^k(\psi)$ also as

$$H^k(\mathcal{M}, \mathcal{K}) \qquad \text{(D.2.28)}$$

and call it the relative cohomology with

$$\cdots \rightarrow H^{k-1}_{\rm dR}(\mathcal{K}) \rightarrow H^k(\mathcal{M}, \mathcal{K}) \rightarrow H^k_{\rm dR}(\mathcal{M}) \rightarrow H^k_{\rm dR}(\mathcal{K}) \rightarrow \cdots \qquad \text{(D.2.29)}$$

as the long exact sequence. Note that, when $\mathcal{K}$ is the boundary of $\mathcal{M}$, the relative cohomology enumerates closed forms that become exact at the boundary.

$\mathbb{R}^n$: Relative Equals Compact

For example, with $\mathcal{M} = \mathbb{R}^n$ and $\mathcal{K} = \mathbb{S}^{n-1}$ its asymptotic sphere, one can read off

$$\emptyset \rightarrow H^0(\mathbb{R}^n, \mathbb{S}^{n-1}) \rightarrow H^0_{\rm dR}(\mathbb{R}^n) \rightarrow H^0_{\rm dR}(\mathbb{S}^{n-1}) \rightarrow H^1(\mathbb{R}^n, \mathbb{S}^{n-1}) \rightarrow \emptyset$$

$$\emptyset \rightarrow H^{n-1}_{\rm dR}(\mathbb{S}^{n-1}) \rightarrow H^n(\mathbb{R}^n, \mathbb{S}^{n-1}) \rightarrow H^n_{\rm dR}(\mathbb{R}^n) \rightarrow \emptyset \qquad \text{(D.2.30)}$$

from the long exact sequence above, say, for $n > 2$, thanks to $H^{0<q<n-1}(\mathbb{S}^{n-1}) = \emptyset$. With $H^0_{\rm dR}(\mathbb{R}^n) = H^0_{\rm dR}(\mathbb{S}^{n-1}) = H^{n-1}_{\rm dR}(\mathbb{S}^{n-1}) = \mathbb{R}$ and $H^n_{\rm dR}(\mathbb{R}^n) = \emptyset$, we find

$$H^{q<n}(\mathbb{R}^n, \mathbb{S}^{n-1}) = \emptyset \;, \qquad H^n(\mathbb{R}^n, \mathbb{S}^{n-1}) = \mathbb{R} \qquad \text{(D.2.31)}$$

Note that these coincide with $H^k_{\rm cdR}(\mathbb{R}^n)$.

D.3 Čech and Čech-de Rham Complexes

Although the cohomology built from the differential complex is more immediate to physics applications, more abstract notions of (co)homology is still quite relevant.

Singular (co)homology and Intersection (co)homology are some of more fundamental such. These rely on the notion "simplex" which is basically a triangulation of the manifold. Perhaps the simplest combinatorial cohomology is the Čech cohomology, which made a brief appearance earlier in the context of G-structures and obstructions. The Čech cohomology is also capable of reproducing de Rham cohomology of previous section when we start building the relevant complex with locally constant real functions, so in some sense a most distilled and minimal version of the differential complex. We will attempt to explain briefly the constructions involved and reach in the end a proof of the latter equivalence as well.

Imagine a manifold $\mathcal{M}$ equipped with a set of charts $\{\mathcal{U}_a\}$,

$$\mathcal{M} = \cup\, \mathcal{U}_a \tag{D.3.1}$$

such that the intersections of arbitrary subsets thereof

$$\mathcal{U}_a \cap \cdots \cap \mathcal{U}_{a'} \tag{D.3.2}$$

are either empty or continuously contractible to a point. Such a collection of charts $\{\mathcal{U}_a\}$ is called a good cover. This should be contrasted against the double cover that we used for the Mayer-Vietoris sequence; for $\mathbb{S}^n$, the intersection of the two charts $\mathcal{U}$ and $\tilde{\mathcal{U}}$ we used is contractible to $\mathbb{S}^{n-1}$.

A q-simplex means an ordered collection of $(q+1)$-many $\mathcal{U}_a$'s,

$$\sigma_A = (\mathcal{U}_{A_0}, \ldots, \mathcal{U}_{A_q}) \tag{D.3.3}$$

and its intersection is denoted as

$$|\sigma_A| = \cap_{i=0}^{q} \mathcal{U}_{A_i}\ . \tag{D.3.4}$$

We will also use the notation $|A|$ to denote the number of $\mathcal{U}$'s involved for σ_A. For q-simplex σ_A, we have $|A| = q+1$ by definition.

Then, we define the boundary of σ_A as the formal sum of $(q-1)$-simplexes,

$$\partial\sigma_A = \sum_{k=0}^{q} (-1)^{k+1} \partial_k \sigma_A\ , \tag{D.3.5}$$

where the $(q-1)$ simplexes on the right are

$$\partial_k \sigma_A = (\mathcal{U}_{A_0}, \ldots, \mathcal{U}_{A_{k-1}}, \mathcal{U}_{A_{k+1}}, \ldots, \mathcal{U}_{A_q}) \tag{D.3.6}$$

with $\mathcal{U}_{A_k}$ omitted. The latter's intersection $|\partial_k \sigma_A|$ is also non-empty.

D.3.1 Čech Cohomology

The Čech Cohomology valued in $\mathbb{F}$ starts with the definition of cochains that are maps from such simplexes to $\mathbb{F}$ which is an Abelian group. We will be typically using $\mathbb{F} = \mathbb{Z}_n, \mathbb{Z}, \mathbb{R}$, etc. A q-cochain is an assignment,

$$f^{(q)} \;:\; |\sigma_A| \;\longmapsto\; f^{(q)}(\sigma_A) \;\in\; \mathbb{F} \tag{D.3.7}$$

for all q-simplexes σ_A. When we encounter σ_A and σ_B that differ only by the ordering of $\mathcal{U}$'s, we demand that $f^{(q)}$ obey

$$f^{(q)}(\sigma_A) = (-1)^{\#} f^{(q)}(\sigma_B) \tag{D.3.8}$$

for any given co-chain, where $(-1)^{\#}$ is the parity of the permutation needed to match σ_A and σ_B.

We denote the collection of all such of q-cochains as

$$C^q(\mathcal{U}; \mathbb{F}) \tag{D.3.9}$$

and define the coboundary operator δ as a map from $C^{q-1}(\mathcal{U}; \mathbb{F})$ to $C^q(\mathcal{U}; \mathbb{F})$,

$$[\delta f]^{(q)}(\sigma_A) = \sum_{k=0}^{q} (-1)^k f^{(q-1)}(\partial_k \sigma_A)\Big|_{|\sigma_A|} \tag{D.3.10}$$

from which one can check easily that $\delta\delta \equiv 0$.

This gives the complex

$$\cdots \xrightarrow{\delta} C^{q-1}(\mathcal{U}; \mathbb{F}) \xrightarrow{\delta} C^q(\mathcal{U}; \mathbb{F}) \xrightarrow{\delta} C^{q+1}(\mathcal{U}; \mathbb{F}) \xrightarrow{\delta} \cdots \tag{D.3.11}$$

The kernel and the image

$$\hat{Z}^q \equiv \{f^{(q)} \in C^q(\mathcal{U};\mathbb{F}) \mid \delta(f^{(q)}) = 0\} \ ,$$
$$\hat{B}^q \equiv \{\delta(f^{(q-1)}) \mid f^{(q-1)} \in C^{q-1}(\mathcal{U};\mathbb{F})\} \quad \text{(D.3.12)}$$

define the Čech cohomology

$$\hat{H}^q(\mathcal{M};\mathbb{F}) = \hat{Z}^q/\hat{B}^q \quad \text{(D.3.13)}$$

in a by-now familiar manner. A nontrivial fact is that this cohomology is independent of the choice of the charts $\{\mathcal{U}_a\}$, and depends only on the topological data of $\mathcal{M}$. for $\mathbb{F} = \mathbb{R}$, this can be seen most clearly by establishing the equivalence between de Rham and Čech cohomologies, as we will outline later in this subsection.

$\mathbb{S}^n$, Again

As an illustration, consider $\mathbb{S}^1$ with charts $\mathcal{U}_{1,2,3}$, each of which covers roughly a third of the circle with pairwise overlapping segments. Taking $\mathbb{F} = \mathbb{Z}$, C^0 are set of 0-cochains

$$f^{(0)} \ : \ \mathcal{U}_i \longmapsto l_i \ \in \ \mathbb{Z} \quad \text{(D.3.14)}$$

while 1-cochains are the assignment,

$$f^{(1)} \ : \ \mathcal{U}_i \cap \mathcal{U}_j \longmapsto m_{i<j} \ \in \ \mathbb{Z} \quad \text{(D.3.15)}$$

with three mutually unrelated integers $m_{i<j}$, constituting $C^1 = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. $\hat{B}^0$ is clearly empty while $\hat{Z}^0$ is spanned by $f^{(0)}(\sigma_i) = f^{(0)}(\sigma_j)$, i.e., $l_1 = l_2 = l_3$, which translates to

$$\hat{H}^0(\mathbb{S}^1;\mathbb{Z}) = \mathbb{Z} \quad \text{(D.3.16)}$$

On the other hand, $\hat{Z}^1 = \hat{C}^1$, while $\hat{B}^1$ is spanned by $f^{(1)}$'s such that $m_{ij} = l_i - l_j$. This forces $m_{12} + m_{23} = m_{13}$, hence $\hat{B}^1 = \mathbb{Z} \oplus \mathbb{Z}$, leaving

$$\hat{H}^1(\mathbb{S}^1;\mathbb{Z}) = \mathbb{Z} \quad \text{(D.3.17)}$$

Of these, let us take one more example of $\mathbb{S}^2$ and $\mathbb{F} = \mathbb{Z}$ for a further illustration. A good cover can be found with $p+2 \geq 5$ $\mathcal{U}_a$'s in the following; $\mathcal{U}_N$ for the northern hemisphere falling short of the equator, $\mathcal{U}_S$ for the southern hemisphere similarly, and $\mathcal{U}_{i=1,\cdots,p}$ with $p \geq 3$ covering the equatorial strip sequentially, with $\mathcal{U}_i \cap \mathcal{U}_j$ non-empty between p-many adjacent $\mathcal{U}_i$ pairs, similarly with the $\mathbb{S}^1$ case where we used $p = 3$.

This gives,

$$\begin{aligned} C^0 &= \{l_N, l_S, l_i \mid i = 1, \cdots, p\} , \\ C^1 &= \{m_{Ni}, m_{Si}, m_{ij} \mid j = i+1 \bmod p\} , \\ C^2 &= \{k_{Nij}, k_{Sij} \mid j = i+1 \bmod p\} \end{aligned} \tag{D.3.18}$$

The images under δ are

$$\begin{aligned} \hat{B}^0 &= \emptyset , \\ \hat{B}^1 &= \{m'_{Ni} = l_N - l_i, m'_{Si} = l_S - l_i, m'_{ij} = l_i - l_j \mid j = i+1 \bmod p\} \;=\; \mathbb{Z}^{p+2} , \\ \hat{B}^2 &= \{k'_{Nij} = m_{Ni} - m_{Nj} + m_{ij}, k'_{Sij} = m_{Si} - m_{Sj} + m_{ij} \mid j = i+1 \bmod p\} \;=\; \mathbb{Z}^{2p-1} , \end{aligned} \tag{D.3.19}$$

where the last is due to the single relation, $k'_{N12} + k'_{N23} + \cdots + k'_{Np1} = k'_{S12} + k'_{S23} + \cdots + k'_{Sp1}$.

Finally, the kernels are

$$\begin{aligned} \hat{Z}^0 &= \{l_N, l_S, l_i \mid l_N = l_S = l_i\ ,\ i = 1, \cdots, p\} \;=\; \mathbb{Z}^1 , \\ \hat{Z}^1 &= \{m_{Ni}, m_{Si}, m_{ij} \mid m_{ij} = m_{Nj} - m_{Ni} = m_{Sj} - m_{Si}\ ,\ j = i+1 \bmod p\} \;=\; \mathbb{Z}^{p+2} . \\ \hat{Z}^2 &= C^2 \;=\; \mathbb{Z}^{2p} \end{aligned} \tag{D.3.20}$$

The middle can be seen from how the p-many m_{ij}'s together with m_{N1} and m_{S1} determine the rest.

Taking the quotients $\hat{H}^q = \hat{Z}^q/\hat{B}^q$ brings us to

$$\hat{H}^0(\mathbb{S}^2;\mathbb{Z}) = \mathbb{Z} , \qquad \hat{H}^1(\mathbb{S}^2;\mathbb{Z}) = \emptyset , \qquad \hat{H}^2(\mathbb{S}^2;\mathbb{Z}) = \mathbb{Z} \tag{D.3.21}$$

Extending all these to all $\mathbb{S}^n$ is straightforward, if arduous, and gives

$$\hat{H}^0(\mathbb{S}^n;\mathbb{Z}) = \mathbb{Z} , \qquad \hat{H}^{0<q<n}(\mathbb{S}^n;\mathbb{Z}) = \emptyset , \qquad \hat{H}^n(\mathbb{S}^n;\mathbb{Z}) = \mathbb{Z} \tag{D.3.22}$$

With $\mathbb{F} = \mathbb{R}$, we would have obtained $\hat{H}^q(\mathbb{S}^n) = H^q_{\rm dR}(\mathbb{S}^n)$.

D.3.2 Čech-de Rham Complex

The idea behind the Mayer-Vietoris sequence can be generalized to a good cover $\mathcal{M} = \cup \mathcal{U}_a$ of the above Čech complex, from which the so-called Čech-de Rham complex emerges where d and a generalized δ together spans a planar grid of complex. This would eventually show how de Rham cohomology and real-valued Čech cohomology equal each other.

We start by constructing a long exact sequence for differential forms, over a good cover $\{\mathcal{U}_a\}$ of $\mathcal{M}$ and simplexes $\sigma_{A=abc\cdots}$,

$$\emptyset \;\rightarrow\; \Omega^k(\mathcal{M}) \;\xrightarrow{\iota}\; \oplus_a \Omega^k(\sigma_a) \;\xrightarrow{\delta}\; \oplus_{(a,b)}\Omega^k(\sigma_{ab}) \;\xrightarrow{\delta}\; \oplus_{(a,b,c)}\Omega^k(\sigma_{abc}) \;\xrightarrow{\delta} \tag{D.3.23}$$

where $\Omega^k(\sigma_A)$ is collection of differential k-form defined over $|\sigma_A|$. The first map ι is nothing but the restriction to $\mathcal{U}_a$'s, while the subsequent δ's also restriction onto $|\sigma_A|$, i.e., a pull-back under $|\sigma_A| \hookrightarrow |\partial_j \sigma_A|$, followed by a summation over j with alternating signs

$$\left(\delta w^{(k)}\right)_{\sigma_A} = \sum_{j=0}^{|A|-1} (-1)^j w^{(k)}_{\partial_j \sigma_A}\Big|_{|\sigma_A|} \,, \qquad w^{(k)}_{\sigma_A} \in \Omega^k(\sigma_A) \tag{D.3.24}$$

The operation δ is identical to that of the Čech complex, except that objects being handled are now local differential forms instead of local constants, so $\delta\delta = 0$ follows from the same algebra as the Čech complex also.

The Čech-de Rham Complex is then the planar complex consisting of

$$\mathcal{C}^{p,k}(\mathcal{U};\Omega^*) \equiv \oplus_A^{|A|=p+1} \Omega^k(\sigma_A) \tag{D.3.25}$$

An important observation is that $\Omega^k(\mathcal{M})$ may be regarded as the kernel of the very first δ, say $\delta^{0,k}$

$$\Omega^k(\mathcal{M}) = {\rm Ker}(\delta)\Big|_{\mathcal{C}^{0,k}(\mathcal{U};\Omega^*)} \tag{D.3.26}$$

This follows from how, if $w^{(k)}_{\mathcal{U}_a} = w^{(k)}_{\mathcal{U}_b}$ over $\mathcal{U}_{ab}$, they collectively define a differential form over $\mathcal{M}$. Then, for each $k \geq 0$,

$$\emptyset \;\rightarrow\; \Omega^k(\mathcal{M}) \;\xrightarrow{\iota}\; \mathcal{C}^{0,k}(\mathcal{U};\Omega^*) \;\xrightarrow{\delta}\; \mathcal{C}^{1,k}(\mathcal{U};\Omega^*) \;\xrightarrow{\delta}\; \mathcal{C}^{2,k}(\mathcal{U};\Omega^*) \;\xrightarrow{\delta}\; \cdots \qquad \text{(D.3.27)}$$

is an exact sequence, as was stated earlier.

Let us show that this sequence is indeed exact, i.e., for any given $w \in \mathcal{C}^{p,k}(\mathcal{U};\Omega^*)$ such that $\delta w = 0$, we can find v such that $w = \delta v$. For this let us recall the partition of unity, which are set of smooth functions over the local charts such that $\sum_a u_a = 1$. Then, given a δ-closed $w \in \mathcal{C}^{p,k}(\mathcal{U};\Omega^*)$, we define an element $v \in \mathcal{C}^{p-1,k}(\mathcal{U};\Omega^*)$

$$v_{b_1\cdots b_p} \equiv \sum_a u_a w_{ab_1\cdots b_p} \qquad \text{(D.3.28)}$$

where we simplified the notation as $w_{\sigma_A} \rightarrow w_A$ etc. Taking δ on both sides,

$$\begin{aligned}(\delta v)_{b_0 b_1\cdots b_p} &= \sum_j (-1)^j \left(\sum_a u_a w_{ab_0\cdots b_{j-1}b_{j+1}\cdots b_p}\right) \\ &= \sum_a u_a w_{b_0 b_1\cdots b_p} = w_{b_0 b_1\cdots b_p} \qquad \text{(D.3.29)}\end{aligned}$$

where for the middle equality we exchanged the two summations and invoked $\delta w = 0$. For the first map ι, its kernel consists of globally defined k-forms, and is nothing but $\Omega^k(\mathcal{M})$, showing that the sequence is exact.

Given how each set are differential forms over simplexes, we also have a vertical map d whose kernel in $\mathcal{C}^{p,0}(\mathcal{U};\Omega^*)$ is nothing but the Čech complex,

$$C^p(\mathcal{U};\mathbb{R}) = \mathrm{Ker}(d)\Big|_{\mathcal{C}^{p,0}(\mathcal{U};\Omega^*)} \qquad \text{(D.3.30)}$$

One might wonder why the δ-exactness of $\mathcal{C}^{p,k}(\mathcal{U};\mathbb{R})$ does not imply the same for $C^p(\mathcal{U};\mathbb{R})$ where the δ operation remains essentially the same. If it did, we would have $\hat{H} = \emptyset$. The point is that in the above proof, the partition of the unity entered the construction of v and $\delta v = w$ crucially, which becomes unavailable for constant functions. While $w \in C^p(\mathcal{U};\mathbb{R})$ consists of local constants, v constructed with the help of the partition of unity as in (D.3.28) does not belong to $C^{p-1}(\mathcal{U};\mathbb{R})$.

With this, we also have vertical exact sequences, for each $p \geq 0$,

$$\emptyset \;\to\; C^p(\mathcal{U};\mathbb{R}) \;\xrightarrow{\iota'}\; \mathcal{C}^{p,0}(\mathcal{U};\Omega^*) \;\xrightarrow{d}\; \mathcal{C}^{p,1}(\mathcal{U};\Omega^*) \;\xrightarrow{d}\; \mathcal{C}^{p,2}(\mathcal{U};\Omega^*) \;\xrightarrow{d}\; \cdots \tag{D.3.31}$$

For $\mathcal{C}^{(p,k>1)}$, d-closedness implies d-exactness since the intersections $|\sigma_A|$ are all topologically trivial, on par with $\mathbb{R}^n$. The only exceptions are $\mathcal{C}^{(p,0)}$ whose d-closed elements are nothing but constants over the local charts $\mathcal{U}_a$, constituting precisely $C^p(\mathcal{U};\mathbb{R})$.

The combined planar complex is called the Čech-de Rham complex. Since δ's are constructed from pull-backs under $|\sigma_A| \hookrightarrow |\partial_j \sigma_A|$, followed by an alternating sum over j, $d\delta = \delta d$ follows naturally. We then introduce

$$\mathbf{d} \equiv \delta + (-1)^p d \;, \qquad \mathbf{d}^2 = \delta^2 + (-1)^p(\delta d - d\delta) + d^2 = 0 \tag{D.3.32}$$

acting on the slanted partial sums

$$\mathcal{C}^q(\mathcal{U};\Omega^*) \equiv \oplus_{p+k=q}\, \mathcal{C}^{p,k}(\mathcal{U};\Omega^*) \tag{D.3.33}$$

This defines a complex

$$\cdots \;\xrightarrow{\mathbf{d}}\; \mathcal{C}^{q-1}(\mathcal{U};\Omega^*) \;\xrightarrow{\mathbf{d}}\; \mathcal{C}^{q}(\mathcal{U};\Omega^*) \;\xrightarrow{\mathbf{d}}\; \mathcal{C}^{q+1}(\mathcal{U};\Omega^*) \;\xrightarrow{\mathbf{d}}\; \cdots \tag{D.3.34}$$

leading to yet another cohomology $\mathcal{H}^q(\mathcal{M};\Omega^*)$.

Then, a general theorem states that

$$H^q_{\rm dR}(\mathcal{M}) = \mathcal{H}^q(\mathcal{M};\Omega^*) = \hat{H}^q(\mathcal{M};\mathbb{R}) \tag{D.3.35}$$

which establishes $H^q_{\rm dR}(\mathcal{M}) = \hat{H}^q(\mathcal{M};\mathbb{R})$ in particular. This also justifies how the Čech cohomology $\hat{H}^q(\mathcal{M};\mathbb{R})$ and also $\mathcal{H}^q(\mathcal{M};\Omega^*)$ are independent of the choice of the good cover $\{\mathcal{U}_a\}$, since the definition of $H^q_{\rm dR}(\mathcal{M})$ does not rely on one.

Proof

The theorem may be proven from a simple diagram chasing. Note that a $\mathbf{d}$-closed element may be written as a formal sum of $w^{p,q-p} \in \mathcal{C}^{p,q-p}(\mathcal{U};\Omega^*)$, such that

$$\mathbf{d}(w^{k,q-k} + w^{k+1,q-k-1} + \cdots + w^{m,q-m}) = 0 \tag{D.3.36}$$

with some $0 \le k \le m \le q$ and

$$dw^{k,q-k} = 0 \ , \qquad \delta w^{m,q-m} = 0 \tag{D.3.37}$$

With these two conditions met, the rest follows from the exact nature of the horizontal and vertical sequences. Since $d\delta = \delta d$, $d\delta w^{k,q-k} = \delta d w^{k,q-k} = 0$, which guarantees, given the d-exact nature of the vertical sequences, an element $w^{k+1,q-k-1}$ that solves $\delta w^{k,q-k} = (-1)^k dw^{k+1,q-k-1}$. In turn, $\delta w^{k+1,q-k-1}$ is d-closed, so we again have $\delta w^{k+1,q-k-1} = (-1)^{k+1} dw^{k+2,q-k-2}$, etc. This ends at some point, here in this example with $\delta w^{m,q-m} = 0$, if the element is a cohomology representative.

The exact nature of the horizontal sequences means that there is $v^{m-1,q-m}$ such that $w^{m,q-m} = \delta v^{m-1,q-m}$. Subtracting $\mathbf{d}v^{m-1,q-m}$ from the above $\mathbf{d}$-closed element, we obtain a shortened representative of the same cohomology element, such that

$$\mathbf{d}(w^{k,q-k} + w^{k+1,q-k-1} + \cdots + \tilde{w}^{m-1,q-m+1}) = 0 \tag{D.3.38}$$

where the last entry is shifted from $w^{m-1,q-m+1}$ by $(-1)^m dv^{m-1,q-m}$ and we now have $\delta\tilde{w}^{m-1,q-m+1} = 0$. This way, we can remove the lower-right entries $w^{p,q-p}$ with the largest value of p step by step without changing $\mathcal{H}^q(\mathcal{M};\Omega^*)$ cohomology value. In fact, this can remove the formal sum entirely, unless we started with $k = 0$.

An entry $w^{0,q}$ sits on the left-most column of this complex, so shifting it away by δ of something is no longer possible. This tells us that (1) a nontrivial cohomology element of $\mathcal{H}^q(\mathcal{M};\Omega^*)$ must have $w^{0,q} \neq 0$ and (2) by the above successive shifts, we can manipulate its formal sum into the form

$$\mathbf{d}\tilde{w}^{0,q} = 0 \quad \Rightarrow \quad d\tilde{w}^{0,q} = 0 \ , \ \ \delta\tilde{w}^{0,q} = 0 \tag{D.3.39}$$

with its single entry. In particular, the latter two conditions, combined with the exactness of the horizontal sequences, tell us that $\tilde{w}^{0,q}$ is an image under the injection ι of a d-closed element $\hat{w}^{(q)}$ of $\Omega^*(\mathcal{M})$. It represents a nontrivial element of $\mathcal{H}^q(\mathcal{M};\Omega^*)$ if and only if $\tilde{w}^{0,q} \neq \mathbf{d}v^{0,q-1}$. The latter is the same as $\hat{w}^{(q)} \neq dv^{(q-1)}$ in $\Omega^*(\mathcal{M})$ again because $\tilde{w}^{0,q}$ belongs to the leftmost column of the complex and δ-closed. This shows $H^q_{\rm dR}(\mathcal{M}) = \mathcal{H}^q(\mathcal{M};\Omega^*)$.

Note that the key fact for the proof is that the horizontal sequences are all exact. For the proof of the other equality, $\mathcal{H}^q(\mathcal{M};\Omega^*) = \hat{H}^q(\mathcal{M};\mathbb{R})$, it suffices to exchange the roles of δ and $(-1)^p d$, whereby the vertical exact sequences allow us to remove upper-left corners ($w^{p,q-p}$ with the smallest value of p) of a **d**-closed formal sum one by one. The end result is a δ-closed $\tilde{w}^{q,0}$ that is also d-closed. As above, it is an image under ι' of a representative of $\hat{H}^q(\mathcal{M};\mathbb{R})$.

One can see that the proof is quite general and does not rely on any details of δ and d. Given a planar complex where the horizontal and the vertical sequences are all exact when each is supplemented by the kernels of either the leftmost or the bottom maps, one can show that the three cohomologies, those two from the supplementary leftmost and bottom kernels, respectively, and the last from the planar complex itself turned into a linear one, all coincide.

D.4 Homotopy